Graphs

Definition. *A* **graph** *G is a pair (V, E), where V is a finite set and E is a collection of unordered pairs in $V \times V$. The elements of V are referred to as the vertices of the graph and the elements of E are referred to as the edges.*

A graph is an abstract structure, but immediately we associate a representation to any graph by drawing the vertices as points and the edges as lines.
Example Define a graph G_1 with $V = \{1, 2, 3, 4, 5\}$ and $E = (1, 2)$ $(1, 3)$ $(1, 4)$ $(2, 4)$ $(3, 4)$ $(1, 5)$ $(4, 6)$. We represent G with the picture in Figure 1.

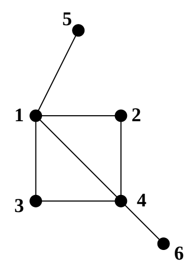

Figure 1: A graph with 5 vertices and 7 edges.

We could have repeated elements in E. For example, we could include an edge twice. In this case, we say that G has *multiple edges*. We could also have an edge between just one vertex. In this case, we call the edge a *loop*
Example In Figure 2, $(3, 3)$ is an edge and $(1, 5)$ has been included twice.
A *simple* graph is a graph with no loops and no multiple edges.

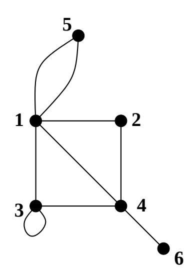

Figure 2: A graph with loops and multiple edges

In our definition of a graph, edges are unordered pairs. So the edge $(1, 4)$ is the same thing as the edge $(4, 1)$. This kind of graph is referred to as *undirected*. A *directed* graph is a graph whose edges are ordered pairs. In the representation of points and lines, we show this by using arrows. For an edge (v_1, v_2), we draw an arrow beginning at v_1 and ending at v_2.

Example Figure 3 shows a directed graph with $V = \{1, 2, 3, 4, 5, 6\}$ and $E = (1, 2)\ (1, 3)\ (4, 1)\ (4, 2)\ (3, 4)\ (1, 5)\ (6, 4)$.

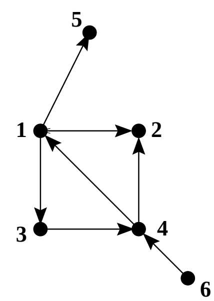

Figure 3: A directed graph.

A *subgraph* $H = (V', E')$ of a graph $G = (V, E)$ is a pair $V' \subseteq V$ and $E' \subseteq E$. We say that H is an *induced* subgraph of G if all the edges between the vertices in V' from E are in E'.

Example Figure 4 shows two subgraphs of G_1. The first subgraph is an induced subgraph. All edges between the vertices $2, 3, 4$, and 6 that are in G_1 are also in this graph. The second subgraph is not an induced subgraph because the edges $(2, 4)$ and $(1, 4)$ are missing.

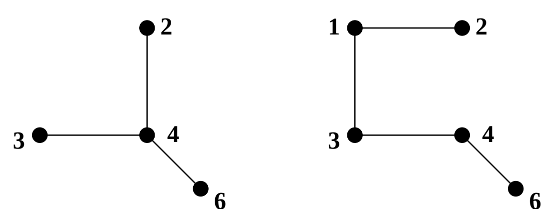

Figure 4: Subgraphs of G_1.

We say that two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are the same or *isomorphic* if there is a one to one and onto map f from V_1 to V_2 such that $(f(v_i), f(v_j))$ is an edge in G_2 if and only if (v_i, v_j) is an edge in G_1.

Example The two graphs of Figure 5 are isomorphic. In this example, we can simply take the map $f(1) = 1$, $f(2) = 2, \ldots$, $f(6) = 6$. In general, if we have a graph whose vertices are labeled and we can find a map such that $f(i) = i$ then we say that the graphs are isomorphic as labeled graphs.

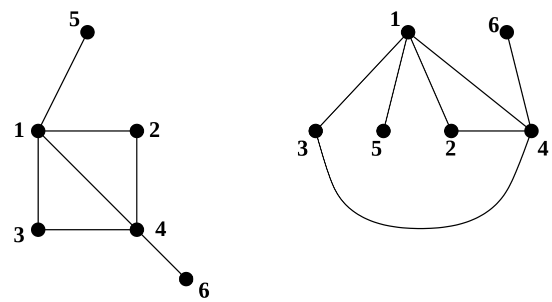

Figure 5: Two isomorphic graphs.

We can change the labeling on one of the graphs so that they will no longer be isomorphic as labeled graphs (they are still isomorphic as unlabeled graphs).
Example In Figure 6 the labels 2 and 5 have been changed, but the graphs are still isomorphic.

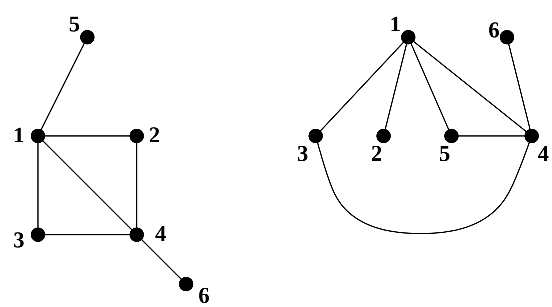

Figure 6: Two isomorphic graphs with different labellings.

Two vertices v_1, v_2 are *adjacent* or *neighbors* if $(v_1,v_2) \in E$. Two edges are *adjacent* if they share a common vertex. A *path* from v_1 to v_n in a graph is a sequence of adjacent edges such that v_1 is in the first edge of the sequence and v_2 is in the last edge of the sequence.

Example In Figure 7 the path $(5,1)(1,3)(3,4)(4,2)$ from 5 to 2 is highlighted.

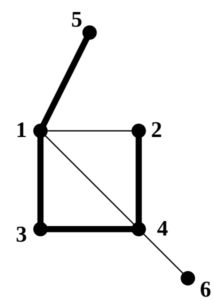

Figure 7: A path from vertex 5 to vertex 2.

We say a graph is *connected* if for every pair of vertices there exists a path between them. All the graphs we have seen so far are connected. Figure 8 shows a graph that is not connected. A disconnected graph consists of multiple connected pieces called *components*.

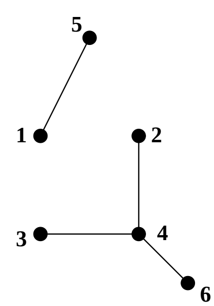

Figure 8: A disconnected graph with 2 connected components.