Consider the Markov chain depicted above. The only recurrent states are 1 and 5, each trivially recurrent since it can only go to itself, and hence each forms its own recurrent class. (The stationary distributions have $p_1 = 1$ and $p_5 = 1$.)

Assume the Markov chain is in state 0 just before the first step, some questions:

(a) What is the probability of being in state 7 after n steps?
The unique path is a first step from 0 to 7, then $n - 1$ steps from 7 to itself, hence the probability is $.5(.4)^{n-1}$.

(b) What is the probability of reaching state 2 for the first time on the kth step?
The unique path for this is a first step from 0 to 7, then $k - 2$ steps from 7 to itself, then a step from 7 to 2, hence the probability is $(.5)(.4)^{k-2}(.2)$

(c) What is the probability of never reaching state 1 ?
The only way to avoid state 1 is to follow the path from $0 \to 7 \to 4$. From state 7 it's twice as likely to go to state 4 as to state 2, therefore $2/3$ of the time one will eventually end up in the direction of state 4 from state 7. Since the probability of going from state 0 to state 7 is $1/2$, the overall probability of going from state 0 to state 4, and never reaching state 1, is $(1/2) \cdot (2/3) = 1/3$.
[Another way to reproduce the above is to note that the overall probability to reach state 2 from state 7 is to sum over paths that return k times to state 7 (and use $\sum_{n=0}^{\infty} r^n = 1/(1 - r)$): $p_{72} \sum_{k=0}^{\infty} p_{77}^k = \frac{p_{72}}{1-p_{77}} = .2/.6 = 1/3$. Similarly, the probability to eventually reach state 4 from state 7 is $p_{74} \sum_{k=0}^{\infty} p_{77}^k = \frac{p_{74}}{1-p_{77}} = .4/.6 = 2/3$.]

(d) What is the expected number of steps until hitting state 5?
If the probability of remaining at a state is p, then the probability of remaining for exactly n steps is $p^n(1 - p)$, and the expectation value for the number of steps is

$$E[n] = \sum_{n=0}^{\infty} n\, p^n (1-p) = (1-p)p\frac{\partial}{\partial p} \sum_{n=0}^{\infty} p^n = (1-p)p\frac{\partial}{\partial p}\frac{1}{1-p} = (1-p)p\frac{1}{(1-p)^2} = \frac{p}{1-p} \ .$$

Thus the expected number of steps to remain at states 7,4,6 respectively are $.4/.6 = 2/3$, $.2/.8 = 1/4$, $.7/.3 = 7/3$, and the expected number of steps until hitting state 5 is thus $4 + 2/3 + 1/4 + 7/3 = 7\ 1/4$. [Note that the expected duration in a state, i.e., the average number of consecutive times in the state, is equal to one more than the expected number of steps that return to the state, hence is given by $1 + p/(1 - p) = 1/(1 - p)$.]

(e) What is the probability of eventually hitting state 5?

This is the same probability of $1/3$ as (c), since any path that doesn't reach 1 eventually hits 5. More specifically, the probability of going from state 0 to state 7 is $1/2$, the probability of eventually going from $7 \to 4$ is $2/3$, and from state 4 one eventually hits state 5, so the overall probability of going from state 0 to state 5 is $p(0 \to 5) = (1/2) \cdot (2/3) = 1/3$.

(f) What is the probability of being in state 4 after two steps, given that one is in state 5 after 8 steps?

This is given by the joint probability of being in state 4 after two steps *and* being in state 5 after 8 steps, divided by the overall probability of being in state 5 after 8 steps

$$p(\text{state 4 after 2 steps} \mid \text{state 5 after 8 steps}) = \frac{p(\text{state 4 after 2 steps , state 5 after 8 steps})}{p(\text{state 5 after 8 steps})} .$$

The probability of being in state 5 after 8 steps is

$$p(\text{state 5 after 8 steps}) = p_{07} \, p_{74} \, p_{46} \, p_{65} \sum_{\substack{\{i,j,k,\ell \mid \\ i+j+k+\ell=4\}}} p_{77}^i \, p_{44}^j \, p_{66}^k \, p_{55}^\ell = .5 \cdot .4 \cdot .8 \cdot .3 \cdot (6.6759) \approx .32 ,$$

where the sum is over all ways of returning to states 7,4,6, and 5 for an overall total of four times. (Note in this case that $p_{55} = 1$ so it doesn't matter when state 5 is reached or how many times one returns to it.) There are many ways to calculate the above sum. One way is to consider the product

$$(1 + .6x + .6^2 x^2 + .6^3 x^3 + .6^4 x^4)(1 + .2x + .2^2 x^2 + .2^3 x^3 + .2^4 x^4)(1 + .7x + .7^2 x^2 + .7^3 x^3 + .7^4 x^4)$$

$$= 1 + 1.5x + 1.57x^2 + 1.419x^3 + 1.1869x^4 + \ldots$$

The powers of x count the sum of the number of steps back to states 7,4,6, so returning to those states four or fewer times corresponds to $1 + 1.5 + 1.57 + 1.419 + 1.1869 = 6.6759$.

The joint probability of being in state 4 after two steps and being in state 5 after 8 steps is given by the same sum as above except with $i = 0$, since there can be no transitions from state 7 back to itself if state 4 is reached in just 2 steps. This gives

$$p(\text{state 4 after 2 steps , state 5 after 8 steps}) =$$

$$= p_{07} \, p_{74} \, p_{46} \, p_{65} \sum_{\{j,k,\ell \mid j+k+\ell=4\}} p_{44}^j \, p_{66}^k \, p_{55}^\ell = .5 \cdot .4 \cdot .8 \cdot .3 \cdot (3.3825) \approx .162 .$$

In this case the sum was given by considering the product

$$(1 + .2x + .2^2 x^2 + .2^3 x^3 + .2^4 x^4)(1 + .7x + .7^2 x^2 + .7^3 x^3 + .7^4 x^4)$$

$$= 1 + .9x + .67x^2 + .477x^3 + .3355x^4 + \ldots$$

where the power of x counts the combined number of steps back to 4 and 6, and for a total of four or fewer times corresponds to the sum $1 + .9 + .67 + .477 + .3355 = 3.3825$.

The result for the conditional probability is

$$p(\text{state 4 after 2 steps} \mid \text{state 5 after 8 steps}) = \frac{.5 \cdot .4 \cdot .8 \cdot .3 \cdot (3.3825)}{.5 \cdot .4 \cdot .8 \cdot .3 \cdot (6.6759)} \approx .506 .$$