Stationary distributions

Next we will consider the long term behavior of a Markov chain. Namely, do the $r_{ij}(n)$ converge as we take the limit over n? tending toward infinity. The $r_{ij}(n)$ do converge for an ergodic Markov chain. In this case, we write the limit as π_j and call π the *stationary* distribution of the chain. Notice that this distribution does not depend on the start state. We can think of π_j as the long term frequency of being in state j.

In terms of the transition matrix, convergence of the $r_{ij}(n)$ translates into the rows of T^n all converging to π.(why?)

The key property to notice about π is that $\pi = \pi T$. Consider the following system of equations:

$$\begin{aligned} \begin{pmatrix} \pi \\ \vdots \\ \pi \end{pmatrix} &= \lim_{n\to\infty} T^{n+1} \\ &= (\lim_{n\to\infty} T^n)T \\ &= \begin{pmatrix} \pi \\ \vdots \\ \pi \end{pmatrix} T \end{aligned}$$

In order to find π for a given chain, we will use the recursion $\pi_j = \Sigma_i \pi_i p_{ij}$, and formula $\Sigma_j \pi_j = 1$. π is the unique solution to these equations.

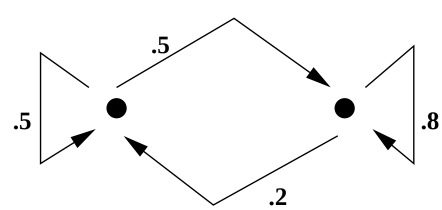

Figure 1: An ergodic Markov chain with two states

Example Figure 1 shows an ergodic Markov chain with two states. We can use $\pi_j = \Sigma_i \pi_i p_{ij}$, and $\Sigma_j \pi_j = 1$ to solve for the stationary distribution. In this case we get: $\pi_1 = 2/7$ and $\pi_2 = 5/7$. (check this!)

A distribution π is said to satisfy *detailed balance* if the following equation holds: $\pi_j p_{ji} = \pi_i p_{ij}$. In this case, the Markov chain is said to be *reversible*. Detailed balance implies that $p_{ij} > 0$ iff $p_{ji} > 0$.

A distribution is symmetric if $p_{ij} = p_{ji}$ for all i, j.