Chapter 6

Games

In the opening chapter of the book, we emphasized that the "connectedness" of a complex social, natural, or technological system really means two things: first, an underlying structure of interconnecting links; and second, an interdependence in the behaviors of the individuals who inhabit the system, so that the outcome for any one depends at least implicitly on the combined behaviors of all. The first issue – network structure – was addressed in the first part of the book using graph theory. In this second part of the book, we study interconnectedness at the level of behavior, developing basic models for this in the language of *game theory*.

Game theory is designed to address situations in which the outcome of a person's decision depends not just on how they choose among several options, but also on the choices made by the people they are interacting with. Game-theoretic ideas arise in many contexts. Some contexts are literally games; for example, choosing how to target a soccer penalty kick and choosing how to defend against it can be modeled using game theory. Other settings are not usually called games, but can be analyzed with the same tools. Examples include the pricing of a new product when other firms have similar new products; deciding how to bid in an auction; choosing a route on the Internet or through a transportation networks; deciding whether to adopt an aggressive or a passive stance in international relations; or choosing whether to use performance-enhancing drugs in a professional sport. In these examples, each decision-maker's outcome depends on the decisions made by others. This introduces a strategic element that game theory is designed to analyze.

As we will see later in Chapter 7, game-theoretic ideas are also relevant to settings where no one is overtly making decisions. Evolutionary biology provides perhaps the most striking example. A basic principle is that mutations are more likely to succeed in a population when they improve the fitness of the organisms that carry the mutation. But often, this fitness cannot be assessed in isolation; rather, it depends on what all the other (non-mutant)

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organisms are doing, and how the mutant's behavior interacts with the non-mutants' behaviors. In such situations, reasoning about the success or failure of the mutation involves game-theoretic definitions, and in fact very closely resembles the process of reasoning about decisions that intelligent actors make. Similar kinds of reasoning have been applied to the success or failure of new cultural practices and conventions — it depends on the existing patterns of behavior into which they are introduced. This indicates that the ideas of game theory are broader than just a model of how people reason about their interactions with others; game theory more generally addresses the question of which behaviors tend to sustain themselves when carried out in a larger population.

Game-theoretic ideas will appear in many places throughout the book. Chapters 8 and 9 describe two initial and fundamental applications: to network traffic, where travel time depends on the routing decisions of others; and to auctions, where the success of a bidder depends on how the other bidders behave. There will be many further examples later in the book, including the ways in which prices are set in markets and the ways in which people choose to adopt new ideas in situations where adoption decisions are affected by what others are doing.

As a first step, then, we begin with a discussion of the basic ideas behind game theory. For now, this will involve descriptions of situations in which people interact with one another, initially without an accompanying graph structure. Once these ideas are in place, we will bring graphs back into the picture in subsequent chapters, and begin to consider how structure and behavior can be studied simultaneously.

6.1 What is a Game?

Game theory is concerned with situations in which decision-makers interact with one another, and in which the happiness of each participant with the outcome depends not just on his or her own decisions but on the decisions made by everyone. To help make the definitions concrete, it's useful to start with an example.

A First Example. Suppose that you're a college student, and you have two large pieces of work due the next day: an exam, and a presentation. You need to decide whether to study for the exam, or to prepare for the presentation. For simplicity, and to make the example as clean as possible, we'll impose a few assumptions. First, we'll assume you can either study for the exam or prepare for the presentation, but not both. Second, we'll assume you have an accurate estimate of the expected grade you'll get under the outcomes of different decisions.

The outcome of the exam is easy to predict: if you study, then your expected grade is a 92, while if you don't study, then your expected grade is an 80.

The presentation is a bit more complicated to think about. For the presentation, you're doing it jointly with a partner. If both you and your partner prepare for the presentation, then the presentation will go extremely well, and your expected joint grade is a 100. If just one of you prepares (and the other doesn't), you'll get an expected joint grade of 92; and if neither of you prepares, your expected joint grade is 84.

The challenge in reasoning about this is that your partner also has the same exam the next day, and we'll assume that he has the same expected outcome for it: 92 if he studies, and 80 if he doesn't. He also has to choose between studying for the exam and preparing for the presentation. We'll assume that neither of you is able to contact the other, so you can't jointly discuss what to do; each of you needs to make a decision independently, knowing that the other will also be making a decision.

Both of you are interested in maximizing the average grade you get, and we can use the discussion above to work out how this average grade is determined by way the two of you invest your efforts:

- If both of you prepare for the presentation, you'll both get 100 on the presentation and 80 on the exam, for an average of 90.
- If both of you study for the exam, you'll both get 92 on the exam and 84 on the presentation, for an average of 88.
- If one of you studies for the exam while the other prepares for the presentation, the result is as follows.
  - The one who prepares for the presentation gets a 92 on the presentation but only an 80 on the exam, for an average of 86.
  - On the other hand, the one who studies for the exam still gets a 92 on the presentation — since it's a joint grade, this person benefits from the fact that one of the two of you prepared for it. This person also get a 92 on the exam, through studying, and so gets an average of 92.

There's a simple tabular way to summarize all these outcomes, as follows. We represent your two choices — to prepare for the presentation, or to study for the exam — as the rows of a 2×2 table. We represent your partner's two choices as the columns. So each box in this table represents a decision by each of you. In each box, we record the average grade you each receive: first yours, then your partner's. Writing all this down, we have the table shown in Figure 6.1.

This describes the set-up of the situation; now you need to figure out what to do: prepare for the presentation, or study for the exam? Clearly, your average grade depends not just on which of these two options you choose, but also on what your partner decides. Therefore, as part of your decision, you have to reason about what your partner is likely to do. Thinking

		Your Partner: *Presentation*	Your Partner: *Exam*
You	*Presentation*	90, 90	86, 92
	Exam	92, 86	88, 88

Figure 6.1: Exam or Presentation?

about the strategic consequences of your own actions, where you need to consider the effect of decisions by others, is precisely the kind of reasoning that game theory is designed to facilitate. So before moving on to the actual outcome of this exam-or-presentation scenario, it is useful to introduce some of the basic definitions of game theory, and then continue the discussion in this language.

Basic Ingredients of a Game. The situation we've just described is an example of a *game*. For our purposes, a game is any situation with the following three aspects.

(i) There is a set of participants, whom we call the *players*. In our example, you and your partner are the two players.

(ii) Each player has a set of options for how to behave; we will refer to these as the player's possible *strategies*. In the example, you and your partner each have two possible strategies: to prepare for the presentation, or to study for the exam.

(iii) For each choice of strategies, each player receives a *payoff* that can depend on the strategies selected by everyone. The payoffs will generally be numbers, with each player preferring larger payoffs to smaller payoffs. In our current example, the payoff to each player is the average grade he or she gets on the exam and the presentation. We will generally write the payoffs in a *payoff matrix* as in Figure 6.1.

Our interest is in reasoning about how players will behave in a given game. For now we focus on games with only two players, but the ideas apply equally well to games with any number of players. Also, we will focus on simple, one-shot games: games in which the players simultaneously and independently choose their actions, and they do so only once. In Section 6.10 at the end of this chapter, we discuss how to reinterpret the theory to deal with dynamic games, in which actions can be played sequentially over time.

6.2 Reasoning about Behavior in a Game

Once we write down the description of a game, consisting of the players, the strategies, and the payoffs, we can ask how the players are likely to behave — that is, how they will go about selecting strategies.

Underlying Assumptions. In order to make this question tractable, we will make a few assumptions. First, we assume everything that a player cares about is summarized in the player's payoffs. In the Exam-or-Presentation Game described in Section 6.1, this means that the two players are solely concerned with maximizing their own average grade. However, nothing in the framework of game theory requires that players care only about personal rewards. For example, a player who is altruistic may care about both his or her own benefits, and the other player's benefit. If so, then the payoffs should reflect this; once the payoffs have been defined, they should constitute a complete description of each player's happiness with each of the possible outcomes of the game.

We also assume that each player knows everything about the structure of the game. To begin with, this means that each player knows his or her own list of possible strategies. It seems reasonable in many settings to assume that each player also knows who the other player is (in a two-player game), the strategies available to this other player, and what his or her payoff will be for any choice of strategies. In the Exam-or-Presentation Game, this corresponds to the assumption that you realize you and your partner are each faced with the choice of studying for the exam or preparing for the presentation, and you have an accurate estimate of the expected outcome under different courses of action. There is considerable research on how to analyze games in which the players have much less knowledge about the underlying structure, and in fact John Harsanyi shared the 1994 Nobel Prize in Economics for his work on games with incomplete information [206].

Finally, we suppose that each individual chooses a strategy to maximize her own payoff, given her beliefs about the strategy used by the other player. This model of individual behavior, which is usually called *rationality*, actually combines two ideas. The first idea is that each player wants to maximize her own payoff. Since the individual's payoff is defined to be whatever the individual cares about, this hypothesis seems reasonable. The second idea is that each player actually succeeds in selecting the optimal strategy. In simple settings, and for games played by experienced players, this too seems reasonable. In complex games, or for games played by inexperienced players, it is surely less reasonable. It is interesting to consider players who make mistakes and learn from the play of the game. There is an extensive literature which analyzes problems of this sort [174], but we will not consider these issues here.

Reasoning about Behavior in the Exam-or-Presentation Game. Let's go back to the Exam-or-Presentation Game and ask how we should expect you and your partner — the two players in the game — to behave.

We first focus on this from your point of view. (The reasoning for your partner will be symmetric, since the game looks the same from his point of view.) It would be easier to decide what to do if you could predict what your partner would do, but to begin with, let's consider what you should do for each possible choice of strategy by your partner.

- First, if you knew your partner was going to study for the exam, then you would get a payoff of 88 by also studying, and a payoff of only 86 by preparing for the presentation. So in this case, you should study for the exam.

- On the other hand, if you knew that your partner was going to prepare for the presentation, then you'd get a payoff of 90 by also preparing for the presentation, but a payoff of 92 by studying for the exam. So in this case too, you should study for the exam.

This approach of considering each of your partner's options separately turns out to be a very useful way of analyzing the present situation: it reveals that no matter what your partner does, you should study for the exam.

When there is a strategy that is the best choice regardless of what the other player does, we will refer to it as a *strictly dominant strategy*. When a player has a strictly dominant strategy, we should expect that they will definitely play it. In the Exam-or-Presentation Game, studying for the exam is also a strictly dominant strategy for your partner (by the same reasoning), and so we should expect that the outcome will be for both of you to study, each getting an average grade of 88.

So this game has a very clean analysis, and it's easy to see how to end up with a prediction for the outcome. Despite this, there's something striking about the conclusion. If you and your partner could somehow agree that you would both prepare for the presentation, you would each get an average grade of 90 — in other words, you would each be better off. But despite the fact that you both understand this, this payoff of 90 cannot be achieved by rational play. The reasoning above makes it clear why not: even if you were to personally commit to preparing for the presentation — hoping to achieve the outcome where you both get 90 — and even if your partner knew you were doing this, your partner would still have an incentive to study for the exam so as to achieve a still-higher payoff of 92 for himself.

This result depends on our assumption that the payoffs truly reflect everything each player values in the outcome — in this case, that you and your partner only care about maximizing your own average grade. If, for example, you cared about the grade that your partner received as well, then the payoffs in this game would look different, and the outcome could be different. Similarly, if you cared about the fact that your partner will be angry at you for not preparing for the joint presentation, then this too should be incorporated into the payoffs, again potentially affecting the results. But with the payoffs as they are, we are left with the interesting situation where there is an outcome that is better for both of you — an average grade of 90 each — and yet it cannot be achieved by rational play of the game.

A Related Story: The Prisoner's Dilemma. The outcome of the Exam-or-Presentation Game is closely related to one of the most famous examples in the development of game theory, the *Prisoner's Dilemma*. Here is how this example works.

Suppose that two suspects have been apprehended by the police and are being interrogated in separate rooms. The police strongly suspect that these two individuals are responsible for a robbery, but there is not enough evidence to convict either of them of the robbery. However, they both resisted arrest and can be charged with that lesser crime, which would carry a one-year sentence. Each of the suspects is told the following story. "If you confess, and your partner doesn't confess, then you will be released and your partner will be charged with the crime. Your confession will be sufficient to convict him of the robbery and he will be sent to prison for 10 years. If you both confess, then we don't need either of you to testify against the other, and you will both be convicted of the robbery. (Although in this case your sentence will be less — 4 years only — because of your guilty plea.) Finally, if neither of you confesses, then we can't convict either of you of the robbery, so we will charge each of you with resisting arrest. Your partner is being offered the same deal. Do you want to confess?"

To formalize this story as a game we need to identify the players, the possible strategies, and the payoffs. The two suspects are the players, and each has to choose between two possible strategies — *Confess* (C) or *Not-Confess* (NC). Finally, the payoffs can be summarized from the story above as in Figure 6.2. (Note that the payoffs are all 0 or less, since there are no good outcomes for the suspects, only different gradations of bad outcomes.)

		Suspect 2	
		NC	C
Suspect 1	NC	$-1, -1$	$-10, 0$
	C	$0, -10$	$-4, -4$

Figure 6.2: Prisoner's Dilemma

As in the Exam-or-Presentation Game, we can consider how one of the suspects — say Suspect 1 — should reason about his options.

- If Suspect 2 were going to confess, then Suspect 1 would receive a payoff of -4 by confessing and a payoff of -10 by not confessing. So in this case, Suspect 1 should confess.

- If Suspect 2 were not going to confess, then Suspect 1 would receive a payoff of 0 by confessing and a payoff of -1 by not confessing. So in this case too, Suspect 1 should confess.

So confessing is a strictly dominant strategy — it is the best choice regardless of what the other player chooses. As a result, we should expect both suspects to confess, each getting a payoff of -4.

We therefore have the same striking phenomenon as in the Exam-or-Presentation Game: there is an outcome that the suspects know to be better for both of them — in which they both choose not to confess — but under rational play of the game there is no way for them to achieve this outcome. Instead, they end up with an outcome that is worse for both of them. And here too, it is important that the payoffs reflect everything about the outcome of the game; if, for example, the suspects could credibly threaten each other with retribution for confessing, thereby making confessing a less desirable option, then this would affect the payoffs and potentially the outcome.

Interpretations of the Prisoner's Dilemma. The Prisoner's Dilemma has been the subject of a huge amount of literature since its introduction in the early 1950s [338, 341], since it serves as a highly streamlined depiction of the difficulty in establishing cooperation in the face of individual self-interest. While no model this simple can precisely capture complex scenarios in the real world, the Prisoner's Dilemma has been used as an interpretive framework for many different real-world situations.

For example, the use of performance-enhancing drugs in professional sports has been modeled as a case of the Prisoner's Dilemma game [208, 362]. Here the athletes are the players, and the two possible strategies are to use performance-enhancing drugs or not. If you use drugs while your opponent doesn't, you'll get an advantage in the competition, but you'll suffer long-term harm (and may get caught). If we consider a sport where it is difficult to detect the use of such drugs, and we assume athletes in such a sport view the downside as a smaller factor than the benefits in competition, we can capture the situation with numerical payoffs that might look as follows. (The numbers are arbitrary here; we are only interested in their relative sizes.)

		Athlete 2	
		Don't Use Drugs	*Use Drugs*
Athlete 1	*Don't Use Drugs*	$3, 3$	$1, 4$
	Use Drugs	$4, 1$	$2, 2$

Figure 6.3: Performance-Enhancing Drugs

Here, the best outcome (with a payoff of 4) is to use drugs when your opponent doesn't, since then you maximize your chances of winning. However, the payoff to both using drugs (2) is worse than the payoff to both not using drugs (3), since in both cases you're evenly matched, but in the former case you're also causing harm to yourself. We can now see that using drugs is a strictly dominant strategy, and so we have a situation where the players use drugs even though they understand that there's a better outcome for both of them.

More generally, situations of this type are often referred to as *arms races*, in which two competitors use an increasingly dangerous arsenal of weapons simply to remain evenly matched. In the example above, the performance-enhancing drugs play the role of the weapons, but the Prisoner's Dilemma has also been used to interpret literal arms races between opposing nations, where the weapons correspond to the nations' military arsenals.

To wrap up our discussion of the Prisoner's Dilemma, we should note that it only arises when the payoffs are aligned in a certain way — as we will see in the remainder of the chapter, there are many situations where the structure of the game and the resulting behavior looks very different. Indeed, even simple changes to a game can change it from an instance of the Prisoner's Dilemma to something more benign. For example, returning to the Exam-or-Presentation Game, suppose that we keep everything the same as before, except that we make the final exam much easier, so that you'll get a 100 on it if you study, and a 96 if you don't. Then we can check that the payoff matrix now becomes

		Your Partner	
		Presentation	*Exam*
You	*Presentation*	98, 98	94, 96
	Exam	96, 94	92, 92

Figure 6.4: Exam-or-Presentation Game with an easier exam.

Furthermore, we can check that with these new payoffs, preparing for the presentation now becomes a strictly dominant strategy; so we can expect that both players will play this strategy, and both will benefit from this decision. The downsides of the previous scenario no longer appear: like other dangerous phenomena, the Prisoner's Dilemma only manifests itself when the conditions are right.

6.3 Best Responses and Dominant Strategies

In reasoning about the games in the previous section, we used two fundamental concepts that will be central to our discussion of game theory. As such, it is useful to define them carefully here, and then to look further at some of their implications.

The first concept is the idea of a *best response*: it is the best choice of one player, given a belief about what the other player will do. For instance, in the Exam-or-Presentation Game, we determined your best choice in response to each possible choice of your partner.

We can make this precise with a bit of notation, as follows. If S is a strategy chosen by Player 1, and T is a strategy chosen by Player 2, then there is an entry in the payoff matrix corresponding to the pair of chosen strategies (S, T). We will write $P_1(S, T)$ to denote the payoff to Player 1 as a result of this pair of strategies, and $P_2(S, T)$ to denote the payoff to Player 2 as a result of this pair of strategies. Now, we say that a strategy S for Player 1 is a *best response* to a strategy T for Player 2 if S produces at least as good a payoff as any other strategy paired with T:

$$P_1(S,T) \geq P_1(S',T)$$

for all other strategies S' of Player 1. Naturally, there is a completely symmetric definition for Player 2, which we won't write down here. (In what follows, we'll present the definitions from Player 1's point of view, but there are direct analogues for Player 2 in each case.)

Notice that this definition allows for multiple different strategies of Player 1 to be tied as the best response to strategy T. This can make it difficult to predict which of these multiple different strategies Player 1 will use. We can emphasize that one choice is uniquely the best by saying that a strategy S of Player 1 is a *strict best response* to a strategy T for Player 2 if S produces a strictly higher payoff than any other strategy paired with T:

$$P_1(S,T) > P_1(S',T)$$

for all other strategies S' of Player 1. When a player has a strict best response to T, this is clearly the strategy she should play when faced with T.

The second concept, which was central to our analysis in the previous section, is that of a strictly dominant strategy. We can formulate its definition in terms of best responses as follows.

- We say that a *dominant strategy* for Player 1 is a strategy that is a best response to every strategy of Player 2.

- We say that a *strictly dominant strategy* for Player 1 is a strategy that is a strict best response to every strategy of Player 2.

In the previous section, we made the observation that if a player has a strictly dominant strategy, then we can expect him or her to use it. The notion of a dominant strategy is slightly weaker, since it can be tied as the best option against some opposing strategies. As a result, a player could potentially have multiple dominant strategies, in which case it may not be obvious which one should be played.

The analysis of the Prisoner's Dilemma was facilitated by the fact that both players had strictly dominant strategies, and so it was easy to reason about what was likely to happen. But most settings won't be this clear-cut, and we now begin to look at games which lack strictly dominant strategies.

A Game in Which Only One Player Has a Strictly Dominant Strategy. As a first step, let's consider a setting in which one player has a strictly dominant strategy and the other one doesn't. As a concrete example, we consider the following story.

Suppose there are two firms that are each planning to produce and market a new product; these two products will directly compete with each other. Let's imagine that the population of consumers can be cleanly divided into two market segments: people who would only buy a low-priced version of the product, and people who would only buy an upscale version. Let's also assume that the profit any firm makes on a sale of either a low price or an upscale product is the same. So to keep track of profits it's good enough to keep track of sales. Each firm wants to maximize its profit, or equivalently its sales, and in order to do this it has to decide whether its new product will be low-priced or upscale.

So this game has two players — Firm 1 and Firm 2 — and each has two possible strategies: to produce a low-priced product or an upscale one. To determine the payoffs, here is how the firms expect the sales to work out.

- First, people who would prefer a low-priced version account for 60% of the population, and people who would prefer an upscale version account for 40% of the population.

- Firm 1 is the much more popular brand, and so when the two firms directly compete in a market segment, Firm 1 gets 80% of the sales and Firm 2 gets 20% of the sales. (If a firm is the only one to produce a product for a given market segment, it gets all the sales.)

Based on this, we can determine payoffs for different choices of strategies as follows.

- If the two firms market to different market segments, they each get all the sales in that segment. So the one that targets the low-priced segment gets a payoff .60 and the one that targets the upscale segment gets .40.

- If both firms target the low-priced segment, then Firm 1 gets 80% of it, for a payoff of .48, and Firm 2 gets 20% of it, for a payoff of .12.

- Analogously, if both firms target the upscale segment, then Firm 1 gets a payoff of $(.8)(.4) = .32$ and Firm 2 gets a payoff of $(.2)(.4) = .08$.

This can be summarized in the following payoff matrix.

		Firm 2	
		Low-Priced	*Upscale*
Firm 1	*Low-Priced*	.48, .12	.60, .40
	Upscale	.40, .60	.32, .08

Figure 6.5: Marketing Strategy

Notice that in this game, Firm 1 has a strictly dominant strategy: for Firm 1, *Low-Priced* is a strict best response to each strategy of Firm 2. On the other hand, Firm 2 does not have a dominant strategy: *Low-Priced* is its best response when Firm 1 plays *Upscale*, and *Upscale* is its best response when Firm 1 plays *Low-Priced*.

Still, it is not hard to make a prediction about the outcome of this game. Since Firm 1 has a strictly dominant strategy in *Low-Priced*, we can expect it will play it. Now, what should Firm 2 do? If Firm 2 knows Firm 1's payoffs, and knows that Firm 1 wants to maximize profits, then Firm 2 can confidently predict that Firm 1 will play *Low-Priced*. Then, since *Upscale* is the strict best response by Firm 2 to *Low-Priced*, we can predict that Firm 2 will play *Upscale*. So our overall prediction of play in this marketing game is *Low-Priced* by Firm 1 and *Upscale* by Firm 2, resulting in payoffs of .60 and .40 respectively.

Note that although we're describing the reasoning in two steps — first the strictly dominant strategy of Firm 1, and then the best response of Firm 2 — this is still in the context of a game where the players move simultaneously: both firms are developing their marketing strategies concurrently and in secret. It is simply that the reasoning about strategies naturally follows this two-step logic, resulting in a prediction about how the simultaneous play will occur. It's also interesting to note the intuitive message of this prediction. Firm 1 is so strong that it can proceed without regard to Firm 2's decision; given this, Firm 2's best strategy is to stay safely out of the way of Firm 1.

Finally, we should also note how the Marketing Strategy Game makes use of the knowledge we assume players have about the game being played and about each other. In particular, we assume that each player knows the entire payoff matrix. And in reasoning about this specific game, it is important that Firm 2 knows that Firm 1 wants to maximize profits, and that Firm 2 knows that Firm 1 knows its own profits. In general, we will assume that the players have *common knowledge* of the game: they know the structure of the game, they know that each of them know the structure of the game, they know that each of them know that each of them know, and so on. While we will not need the full technical content of common knowledge in anything we do here, it is an underlying assumption and a topic of research in the game theory literature [28]. As mentioned earlier, it is still possible to analyze games in situations where common knowledge does not hold, but the analysis becomes more complex [206]. It's also worth noting that the assumption of common knowledge is a bit stronger than we need for reasoning about simple games such as the Prisoner's Dilemma, where strictly dominant strategies for each player imply a particular course of action regardless of what the other player is doing.

6.4 Nash Equilibrium

When neither player in a two-player game has a strictly dominant strategy, we need some other way of predicting what is likely to happen. In this section, we develop methods for doing this; the result will be a useful framework for analyzing games in general.

An Example: A Three-Client Game. To frame the question, it helps to think about a simple example of a game that lacks strictly dominant strategies. Like our previous example, it will be a marketing game played between two firms; however, it has a slightly more intricate set-up. Suppose there are two firms that each hope to do business with one of three large clients, A, B, and C. Each firm has three possible strategies: whether to approach A, B, or C. The results of their two decisions will work out as follows.

- If the two firms approach the same client, then the client will give half its business to each.
- Firm 1 is too small to attract business on its own, so if it approaches one client while Firm 2 approaches a different one, then Firm 1 gets a payoff of 0.
- If Firm 2 approaches client B or C on its own, it will get their full business. However, A is a larger client, and will only do business with the firms if both approach A.
- Because A is a larger client, doing business with it is worth 8 (and hence 4 to each firm if it's split), while doing business with B or C is worth 2 (and hence 1 to each firm if it's split).

From this description, we can work out the following payoff matrix.

		Firm 2 A	Firm 2 B	Firm 2 C
Firm 1	A	$4,4$	$0,2$	$0,2$
Firm 1	B	$0,0$	$1,1$	$0,2$
Firm 1	C	$0,0$	$0,2$	$1,1$

Figure 6.6: Three-Client Game

If we study how the payoffs in this game work, we see that neither firm has a dominant strategy. Indeed, each strategy by each firm is a strict best response to some strategy by the other firm. For Firm 1, A is a strict best response to strategy A by Firm 2, B is a strict best response to B, and C is a strict best response to C. For Firm 2, A is a strict best response to strategy A by Firm 1, C is a strict best response to B, and B is a strict best response to C. So how should we reason about the outcome of play in this game?

Defining Nash Equilibrium. In 1950, John Nash proposed a simple but powerful principle for reasoning about behavior in general games [308, 309], and its underlying premise is the following: even when there are no dominant strategies, we should expect players to use strategies that are best responses to each other. This is no longer a concept that can be derived purely from rationality on the part of the players; instead, it is an *equilibrium* concept. The idea is that if the players chose strategies that are best responses to each other, then no player has an incentive to deviate to an alternative strategy — so the system is in a kind of equilibrium state, with no force pushing it toward a different outcome.

More precisely, suppose that Player 1 chooses a strategy S and Player 2 chooses a strategy T. We say that this pair of strategies (S, T) is a *Nash equilibrium* if S is a best response to T, and T is a best response to S. Nash shared the 1994 Nobel Prize in Economics for his development and analysis of this idea.

To understand the idea of Nash equilibrium, we should first ask why a pair of strategies that are not best responses to each other would not constitute an equilibrium. The answer is that the players cannot both believe that these strategies will be actually used in the game, as they know that at least one player would have an incentive to deviate to another strategy. So Nash equilibrium can be thought of as an equilibrium in beliefs. If each player believes that the other player will actually play a strategy that is part of a Nash equilibrium, then she is willing to play her part of the Nash equilibrium.

Let's consider the Three-Client Game from the perspective of Nash equilibrium. If Firm 1 chooses A and Firm 2 chooses A, then we can check that Firm 1 is playing a best response to Firm 2's strategy, and Firm 2 is playing a best response to Firm 1's strategy. Hence, the pair of strategies (A, A) forms a Nash equilibrium. Moreover, we can check that this is the only Nash equilibrium. No other pair of strategies are best responses to each other.[1]

This discussion also suggests two ways to find Nash equilibria. The first is to simply check all pairs of strategies, and ask for each one of them whether the individual strategies are best responses to each other. The second is to compute each player's best response(s) to each strategy of the other player, and then find strategies that are mutual best responses.

6.5 Multiple Equilibria: Coordination Games

For a game with a single Nash equilibrium, such as the Three-Client Game in the previous section, it seems reasonable to predict that the players will play the strategies in this equilibrium: under any other play of the game, at least one player will not be using a best response to what the other is doing. Some natural games, however, can have more than one Nash equilibrium, and in this case it becomes difficult to predict how rational players will actually behave in the game. We consider some fundamental examples of this problem here.

A Coordination Game. A simple but central example is the following *Coordination Game*, which we can motivate through the following story. Suppose you and a partner are

[1] In this discussion, each player only has three available strategies: A, B, or C. Later in this we will introduce the possibility of more complex strategies in which players can randomize over their available options. With this more complex formulation of possible strategies, we will find additional equilibria for the Three-Client Game.

each preparing slides for a joint project presentation; you can't reach your partner by phone, and need to start working on the slides now. You have to decide whether to prepare your half of the slides in PowerPoint or in Apple's Keynote software. Either would be fine, but it will be much easier to merge your slides together with your partner's if you use the same software.

So we have a game in which you and your partner are the two players, choosing PowerPoint or choosing Keynote form the two strategies, and the payoffs are as shown in Figure 6.7.

		Your Partner	
		PowerPoint	*Keynote*
You	*PowerPoint*	1, 1	0, 0
	Keynote	0, 0	1, 1

Figure 6.7: Coordination Game

This is called a *Coordination Game* because the two players' shared goal is really to coordinate on the same strategy. There are many settings in which coordination games arise. For example, two manufacturing companies that work together extensively need to decide whether to configure their machinery in metric units of measurement or English units of measurement; two platoons in the same army need to decide whether to attack an enemy's left flank or right flank; two people trying to find each other in a crowded mall need to decide whether to wait at the north end of the mall or at the south end. In each case, either choice can be fine, provided that both participants make the same choice.

The underlying difficulty is that the game has two Nash equilibria — i.e., *(PowerPoint,PowerPoint)* and *(Keynote,Keynote)* in our example from Figure 6.7. If the players fail to coordinate on one of the Nash equilibria, perhaps because one player expects PowerPoint to be played and the other expects Keynote, then they receive low payoffs. So what do the players do?

This remains a subject of considerable discussion and research, but some proposals have received attention in the literature. Thomas Schelling [359] introduced the idea of a *focal point* as a way to resolve this difficulty. He noted that in some games there are natural reasons (possibly outside the payoff structure of the game) that cause the players to focus on one of the Nash equilibria. For example, suppose two drivers are approaching each other at night on an undivided country road. Each driver has to decide whether to move over to the left or the right. If the drivers coordinate — making the same choice of side — then they pass each other, but if they fail to coordinate, then they get a severely low payoff due to the resulting collision. Fortunately, social convention can help the drivers decide what to do in this case: if this game is being played in the U.S., convention strongly suggests that they should move to the right, while if the game is being played in England, convention strongly suggests that they should move to the left. In other words, social conventions, while often arbitrary, can sometimes be useful in helping people coordinate among multiple equilibria.

Variants on the Basic Coordination Game. One can enrich the structure of our basic Coordination Game to capture a number of related issues surrounding the problem of multiple equilibria. To take a simple extension of our previous example, suppose that both you and your project partner each prefer Keynote to PowerPoint. You still want to coordinate, but you now view the two alternatives as unequal. This gives us the payoff matrix for an *Unbalanced Coordination Game*, shown in Figure 6.8.

		Your Partner	
		PowerPoint	*Keynote*
You	*PowerPoint*	$1,1$	$0,0$
	Keynote	$0,0$	$2,2$

Figure 6.8: Unbalanced Coordination Game

Notice that *(PowerPoint,PowerPoint)* and *(Keynote,Keynote)* are still both Nash equilibria for this game, despite the fact that one of them gives higher payoffs to both players. (The point is that if you believe your partner will choose PowerPoint, you still should choose PowerPoint as well.) Here, Schelling's theory of focal points suggests that we can use a feature *intrinsic* to the game — rather than an arbitrary social convention — to make a prediction about which equilibrium will be chosen by the players. That is, we can predict that when the players have to choose, they will select strategies so as to reach the equilibrium that gives higher payoffs to both of them. (To take another example, consider the two people trying to meet at a crowded mall. If the north end of the mall has a bookstore they both like, while the south end consists of a loading dock, the natural focal point would be the equilibrium in which they both choose the north end.)

Things get more complicated if you and partner don't agree on which software you prefer, as shown in the payoff matrix of Figure 6.9.

		Your Partner	
		PowerPoint	*Keynote*
You	*PowerPoint*	$1,2$	$0,0$
	Keynote	$0,0$	$2,1$

Figure 6.9: Battle of the Sexes

In this case, the two equilibria still correspond to the two different ways of coordinating, but your payoff is higher in the *(Keynote,Keynote)* equilibrium, while your partner's payoff is higher in the *(PowerPoint,PowerPoint)* equilibrium. This game is traditionally called the *Battle of the Sexes*, because of the following motivating story. A husband and wife want to see a movie together, and they need to choose between a romantic comedy and an action movie. They want to coordinate on their choice, but the *(Romance,Romance)* equilibrium gives a higher payoff to one of them while the *(Action,Action)* equilibrium gives a higher payoff to the other.

In Battle of the Sexes, it can be hard to predict the equilibrium that will be played using either the payoff structure or some purely external social convention. Rather, it helps to know something about conventions that exist between the two players themselves, suggesting how they resolve disagreements when they prefer different ways of coordinating.

It's worth mentioning one final variation on the basic Coordination Game, which has attracted attention in recent years. This is the *Stag Hunt Game* [369]; the name is motivated by the following story from writings of Rousseau. Suppose that two people are out hunting; if they work together, they can catch a stag (which would be the highest-payoff outcome), but on their own each can catch a hare. The tricky part is that if one hunter tries to catch a stag on his own, he will get nothing, while the other one can still catch a hare. Thus, the hunters are the two players, their strategies are *Hunt Stag* and *Hunt Hare*, and the payoffs are as shown in Figure 6.10.

		Hunter 2	
		Hunt Stag	*Hunt Hare*
Hunter 1	*Hunt Stag*	$4, 4$	$0, 3$
	Hunt Hare	$3, 0$	$3, 3$

Figure 6.10: Stag Hunt

This is quite similar to the Unbalanced Coordination Game, except that if the two players miscoordinate, the one who was trying for the higher-payoff outcome gets penalized more than the one who was trying for the lower-payoff outcome. (In fact, the one trying for the lower-payoff outcome doesn't get penalized at all.) As a result, the challenge in reasoning about which equilibrium will be chosen is based on the trade-off between the high payoff of one and the low downside of miscoordination from the other.

It has been argued that the Stag Hunt Game captures some of the intuitive challenges that are also raised by the Prisoner's Dilemma. The structures are clearly different, since the Prisoner's Dilemma has strictly dominant strategies; both, however, have the property that players can benefit if they cooperate with each other, but risk suffering if they try cooperating while their partner doesn't. Another way to see some of the similarities between the two games is to notice that if we go back to the original Exam-or-Presentation Game and make one small change, then we end up changing it from an instance of Prisoner's Dilemma to something closely resembling Stag Hunt. Specifically, suppose that we keep the grade outcomes the same as in Section 6.1, except that we require both you and your partner to prepare for the presentation in order to have any chance of a better grade. That is, if you both prepare, you both get a 100 on the presentation, but if at most one of you prepares, you both get the base grade of 84. With this change, the payoffs for the Exam-or-Presentation Game become what is shown in Figure 6.11.

		Your Partner	
		Presentation	*Exam*
You	*Presentation*	$90, 90$	$82, 88$
	Exam	$88, 82$	$88, 88$

Figure 6.11: Exam-or-Presentation Game (Stag Hunt version)

We now have a structure that closely resembles the Stag Hunt Game: coordinating on *(Presentation,Presentation)* or *(Exam,Exam)* are both equilibria, but if you attempt to go for the higher-payoff equilibrium, you risk getting a low grade if your partner opts to study for the exam.

6.6 Multiple Equilibria: The Hawk-Dove Game

Multiple Nash equilibria also arise in a different but equally fundamental kind of game, in which the players engage in a kind of "anti-coordination" activity. Probably the most basic form of such a game is the *Hawk-Dove Game*, which is motivated by the following story.

Suppose two animals are engaged in a contest to decide how a piece of food will be divided between them. Each animal can choose to behave aggressively (the *Hawk* strategy) or passively (the *Dove* strategy). If the two animals both behave passively, they divide the food evenly, and each get a payoff of 3. If one behaves aggressively while the other behaves passively, then the aggressor gets most of the food, obtaining a payoff of 5, while the passive one only gets a payoff of 1. But if both animals behave aggressively, then they destroy the food (and possibly injure each other), each getting a payoff of 0. Thus we have the payoff matrix in Figure 6.12.

		Animal 2	
		D	H
Animal 1	D	$3, 3$	$1, 5$
	H	$5, 1$	$0, 0$

Figure 6.12: Hawk-Dove Game

This game has two Nash equilibria: (D, H) and (H, D). Without knowing more about the animals we cannot predict which of these equilibria will be played. We can predict that (D, D) and (H, H) will not be played, since the individual strategies in these pairs are not best responses to each other. So as in the coordination games we looked at earlier, the concept of Nash equilibrium helps to narrow down the set of reasonable predictions, but it does not provide a unique prediction.

The Hawk-Dove game has been studied in many contexts. For example, suppose we substitute two countries for the two animals, and suppose that the countries are simultaneously choosing whether to be aggressive or passive in their foreign policy. Each country hopes to gain through being aggressive, but if both act aggressively they risk actually going to war, which would be disastrous for both. So in equilibrium, we can expect that one will be aggressive and one will be passive, but we can't predict who will follow which strategy. Again we would need to know more about the countries to predict which equilibrium will be played.

Hawk-Dove is another example of a game that can arise from a small change to the payoffs in the Exam-or-Presentation Game. Let's again recall the set-up from the opening section, and now vary things so that if neither you nor your partner prepares for the presentation, you will get a very low joint grade of 60. (If one or both of you prepare, the grades for the presentation are the same as before.) If we compute the average grades you get for different choices of strategies in this version of the game, we have the payoffs in Figure 6.13.

		Your Partner	
		Presentation	*Exam*
You	*Presentation*	90, 90	86, 92
	Exam	92, 86	76, 76

Figure 6.13: Exam or Presentation? (Hawk-Dove version)

In this version of the game, there are two equilibria: *(Presentation, Exam)* and *(Exam, Presentation)*. Essentially, one of you must behave passively and prepare for the presentation, while the other achieves the higher payoff by studying for the exam. If you both try to avoid the role of the passive player, you end up with very low payoffs, but we cannot predict from the structure of the game alone who will play this passive role.

The Hawk-Dove game is also known by a number of other names in the game theory literature. For example, it is frequently referred to as the game of *Chicken*, to evoke the image of two teenagers racing their cars toward each other, daring each other to be the one to swerve out of the way. The two strategies here are *Swerve* and *Don't Swerve*: the one who swerves first suffers humiliation from his friends, but if neither swerves, then both suffer an actual collision.

6.7 Mixed Strategies

In the previous two sections, we have been discussing games whose conceptual complexity comes from the existence of multiple equilibria. However, there are also games which have no Nash equilibria at all. For such games, we will make predictions about players' behavior by enlarging the set of strategies to include the possibility of randomization; once players are allowed to behave randomly, one of John Nash's main results establishes that equilibria always exist [308, 309].

Probably the simplest class of games to expose this phenomenon are what might be called "attack-defense" games. In such games, one player behaves as the attacker, while the other behaves as the defender. The attacker can use one of two strategies — let's call them A and B — while the defender's two strategies are "defend against A" or "defend against B." If the defender defends against the attack the attacker is using, then the defender gets the higher payoff; but if the defender defends against the wrong attack, then the attacker gets the higher payoff.

Matching Pennies. A simple attack-defense game is called *Matching Pennies*, and is based on a game in which two people each hold a penny, and simultaneously choose whether to show heads (H) or tails (T) on their penny. Player 1 loses his penny to player 2 if they match, and wins player 2's penny if they don't match. This produces a payoff matrix as shown in Figure 6.14.

		Player 2: H	Player 2: T
Player 1	H	$-1, +1$	$+1, -1$
	T	$+1, -1$	$-1, +1$

Figure 6.14: Matching Pennies

Matching pennies is a simple example of a large class of interesting games with the property that the payoffs of the players sum to zero in every outcome. Such games are called *zero-sum games*, and many attack-defense games — and more generally, games where the players' interests are in direct conflict — have this structure. Games like Matching Pennies have in fact been used as metaphorical descriptions of decisions made in combat; for example, the Allied landing in Europe on June 6, 1944 — one of the pivotal moments in World War II — involved a decision by the Allies whether to cross the English Channel at Normandy or at Calais, and a corresponding decision by the German army whether to mass its defensive forces at Normandy or Calais. This has an attack-defense structure that closely resembles the Matching Pennies game [122].

The first thing to notice about Matching Pennies is that there is no pair of strategies that are best responses to each other. To see this, observe that for any pair of strategies, one of the players gets a payoff of -1, and this player would improve his or her payoff to $+1$ by switching strategies. So for any pair of strategies, one of the players wants to switch what they're doing.[2]

[2]Incidentally, although it's not crucial for the discussion here, it's interesting to note that the Three-Client Game used as an example in Section 6.4 can be viewed intuitively as a kind of hybrid of the Matching

This means that if we treat each player as simply having the two strategies H or T, then there is no Nash equilibrium for this game. This not so surprising if we consider how Matching Pennies works. Nash equilibrium requires that even given knowledge of each other's strategies, neither player would have an incentive to switch to an alternate strategy. But in Matching Pennies, if Player 1 knows that Player 2 is going to play a particular choice of H or T, then Player 1 can exploit this by choosing the opposite and receiving a payoff of $+1$. Analogous reasoning holds for Player 2.

When we think intuitively about how games of this type are played in real life, we see that players generally try to make it difficult for their opponents to predict what they will play. This suggests that in our modeling of a game like Matching Pennies, we shouldn't treat the strategies as simply H or T, but as ways of randomizing one's behavior between H and T. We now see how to build this into a model for the play of this kind of game.

Mixed Strategies. The simplest way to introduce randomized behavior is to say that each player is not actually choosing H or T directly, but rather is choosing a *probability* with which she will play H. So in this model, the possible strategies for Player 1 are numbers p between 0 and 1; a given number p means that Player 1 is committing to play H with probability p, and T with probability $1 - p$. Similarly, the possible strategies for Player 2 are numbers q between 0 and 1, representing the probability that Player 2 will play H.

Since a game consists of a set of players, strategies, and payoffs, we should notice that by allowing randomization, we have actually changed the game. It no longer consists of two strategies by each player, but instead a set of strategies corresponding to the interval of numbers between 0 and 1. We will refer to these as *mixed strategies*, since they involve "mixing" between the options H and T. Notice that the set of mixed strategies still include the original two options of committing to definitely play H or T; these two choices correspond to selecting probabilities of 1 or 0 respectively, and we will refer to them as the two *pure strategies* in the game. To make things more informal notationally, we will sometimes refer to the choice of $p = 1$ by Player 1 equivalently as the "pure strategy H", and similarly for $p = 0$ and $q = 1$ or 0.

Payoffs from Mixed Strategies. With this new set of strategies, we also need to determine the new set of payoffs. The subtlety in defining payoffs is that they are now random quantities: each player will get $+1$ with some probability, and will get -1 with the remaining probability. When payoffs were numbers it was obvious how to rank them: bigger was better. Now that payoffs are random, it is not immediately obvious how to rank them: we

Pennies Game and the Stag Hunt Game. If we look just at how the two players evaluate the options of approaching Clients B and C, we have Matching Pennies: Firm 1 wants to match, while Firm 2 wants to not match. However, if they coordinate on approaching Client A, then they both get even higher payoffs – analogously to the two hunters coordinating to hunt stag.

want a principled way to say that one random outcome is better than another.

To think about this issue, let's start by considering Matching Pennies from Player 1's point of view, and focus first on how she evaluates her two pure strategies of definitely playing H or definitely playing T. Suppose that Player 2 chooses the strategy q; that is, he commits to playing H with probability q and T with probability $1-q$. Then if Player 1 chooses pure strategy H, she receives a payoff of -1 with probability q (since the two pennies match with probability q, in which event she loses), and she receives a payoff of $+1$ with probability $1-q$ (since the two pennies don't match with probability $1-q$). Alternatively, if Player 1 chooses pure strategy T, she receives $+1$ with probability q, and -1 with probability $(1-q)$. So even if Player 1 uses a pure strategy, her payoffs can still be random due to the randomization employed by Player 2. How should we decide which of H or T is more appealing to Player 1 in this case?

In order to rank random payoffs numerically, we will attach a number to each distribution that represents how attractive this distribution is to the player. Once we have done this, we can then rank outcomes according to their associated number. The number we will use for this purpose is the *expected value* of the payoff. So for example, if Player 1 chooses the pure strategy H while Player 2 chooses a probability of q, as above, then the expected payoff to Player 1 is

$$(-1)(q) + (1)(1-q) = 1 - 2q.$$

Similarly, if Player 1 chooses the pure strategy T while Player 2 chooses a probability of q, then the expected payoff to Player 1 is

$$(1)(q) + (-1)(1-q) = 2q - 1.$$

We will assume players are seeking to maximize the expected payoff they get from a choice of mixed strategies. Although the expectation is a natural quantity, it is a subtle question whether maximizing expectation is a reasonable modeling assumption about the behavior of players. By now, however, there is a well-established foundation for the assumption that players rank distributions over payoffs according to their expected values [284, 358, 392], and so we will follow it here.

We have now defined the mixed-strategy version of the Matching Pennies game: strategies are probabilities of playing H, and payoffs are the expectations of the payoffs from the four pure outcomes $(H, H), (H, T), (T, H)$, and (T, T). We can now ask whether there is a Nash equilibrium for this richer version of the game.

Equilibrium with Mixed Strategies. We define a Nash equilibrium for the mixed-strategy version just as we did for the pure-strategy version: it is a pair of strategies (now probabilities) so that each is a best response to the other.

First, let's observe that no pure strategy can be part of a Nash equilibrium. This is equivalent to the reasoning we did at the outset of this section. Suppose, for example, that the pure strategy H (i.e. probability $p = 1$) by Player 1 were part of a Nash equilibrium. Then Player 2's unique best response would be the pure strategy H as well (since Player 2 gets $+1$ whenever he matches). But H by Player 1 is not a best response to H by Player 2, so in fact this couldn't be a Nash equilibrium. Analogous reasoning applies to the other possible pure strategies by the two players. So we reach the natural conclusion that in any Nash equilibrium, both players must be using probabilities that are strictly between 0 and 1.

Next, let's ask what Player 1's best response should be to the strategy q used by Player 2. Above, we determined that the expected payoff to Player 1 from the pure strategy H in this case is

$$1 - 2q,$$

while the expected payoff to Player 1 from the pure strategy T is

$$2q - 1.$$

Now, here's the key point: if $1 - 2q \neq 2q - 1$, then one of the pure strategies H or T is in fact the unique best response by Player 1 to a play of q by Player 2. This is simply because one of $1 - 2q$ or $2q - 1$ is larger in this case, and so there is no point for Player 1 to put any probability on her weaker pure strategy. But we already established that pure strategies cannot be part of any Nash equilibrium for Matching Pennies, and because pure strategies are the best responses whenever $1 - 2q \neq 2q - 1$, probabilities that make these two expectations unequal cannot be part of a Nash equilibrium either.

So we've concluded that in any Nash equilibrium for the mixed-strategy version of Matching Pennies, we must have

$$1 - 2q = 2q - 1,$$

or in other words, $q = 1/2$. The situation is symmetric when we consider things from Player 2's point of view, and evaluate the payoffs from a play of probability p by Player 1. We conclude from this that in any Nash equilibrium, we must also have $p = 1/2$.

Thus, the pair of strategies $p = 1/2$ and $q = 1/2$ is the only possibility for a Nash equilibrium. We can check that this pair of strategies in fact do form best responses to each other. As a result, this is the unique Nash equilibrium for the mixed-strategy version of Matching Pennies.

Interpreting the Mixed-Strategy Equilibrium for Matching Pennies. Having derived the Nash equilibrium for this game, it's useful to think about what it means, and how we can apply this reasoning to games in general.

First, let's picture a concrete setting in which two people actual sit down to play Matching Pennies, and each of them actually commits to behaving randomly according to probabilities p and q respectively. If Player 1 believes that Player 2 will play H strictly more than half the time, then she should definitely play T — in which case Player 2 should not be playing H more than half the time. The symmetric reasoning applies if Player 1 believes that Player 2 will play T strictly more than half the time. In neither case would we have a Nash equilibrium. So the point is that the choice of $q = 1/2$ by Player 2 makes Player 1 *indifferent* between playing H or T: the strategy $q = 1/2$ is effectively "non-exploitable" by Player 1. This was in fact our original intuition for introducing randomization: each player wants their behavior to unpredictable to the other, so that their behavior can't be taken advantage of. We should note that the fact that both probabilities turned out to be $1/2$ is a result of the highly symmetric structure of Matching Pennies; as we will see in subsequent examples in the next section, when the payoffs are less symmetric, the Nash equilibrium can consist of unequal probabilities.

This notion of indifference is a general principle behind the computation of mixed-strategy equilibria, in two-player, two-strategy games when there are no equilibria involving pure strategies: each player should randomize so as to make the other player indifferent between their two alternatives. This way, neither player's behavior can be exploited by a pure strategy, and the two choices of probabilities are best responses to each other. And although we won't pursue the details of it here, a generalization of this principle applies to games with any finite number of players and any finite number of strategies: Nash's main mathematical result accompanying his definition of equilibrium was to prove that *every* such game has at least one mixed-strategy equilibrium [308, 309].

It's also worth thinking about how to interpret mixed-strategy equilibria in real-world situations. There are in fact several possible interpretations that are appropriate in different situations:

- Sometimes, particularly when the participants are genuinely playing a sport or game, the players may be actively randomizing their actions [106, 332, 399]: a tennis player may be randomly deciding whether to serve the ball up the center or out to the side of the court; a card-player may be randomly deciding whether to bluff or not; two children may be randomizing among rock, paper, and scissors in the perennial elementary-school contest of the same name. We will look at examples of this in the next section.

- Sometimes the mixed strategies are better viewed as proportions within a population. Suppose for example that two species of animals, in the process of foraging for food, regularly engage in one-on-one attack-defense games with the structure of Matching Pennies. Here, a single member of the first species always plays the role of attacker, and a single member of the second species always plays the role of defender.

Let's suppose that each individual animal is genetically hard-wired to always play H or always play T; and suppose further that the population of each species consists half of animals hard-wired to play H, and half of animals hard-wired to play T. Then with this population mixture, H-animals in each species do exactly as well on average, over many random interactions, as T-animals. Hence the population as a whole is in a kind of mixed equilibrium, even though each individual is playing a pure strategy. This story suggests an important link with evolutionary biology, which has in fact been developed through a long line of research [370, 371]; this topic will be our focus in Chapter 7.

- Maybe the most subtle interpretation is based on recalling, from Section 6.4, that Nash equilibrium is often best thought of as an equilibrium in beliefs. If each player believes that her partner will play according to a particular Nash equilibrium, then she too will want to play according to it. In the case of Matching Pennies, with its unique mixed equilibrium, this means that it is enough for you to expect that when you meet an arbitrary person, they will play their side of Matching Pennies with a probability of 1/2. In this case, playing a probability of 1/2 makes sense for you too, and hence this choice of probabilities is self-reinforcing — it is in equilibrium — across the entire population.

6.8 Mixed Strategies: Examples and Empirical Analysis

Because mixed-strategy equilibrium is a subtle concept, it's useful to think about it through further examples. We will focus on two main examples, both drawn from the realm of sports, and both with attack-defense structures. The first is stylized and partly metaphorical, while the second represents a striking empirical test of whether people in high-stakes situations actually follow the predictions of mixed-strategy equilibrium. We conclude the section with a general discussion of how to identify all the equilibria of a two-player, two-strategy game.

The Run-Pass Game. First, let's consider a streamlined version of the problem faced by two American football teams as they plan their next play in a football game. The offense can choose either to run or to pass, and the defense can choose either to defend against the run or to defend against the pass. Here is how the payoffs work.

- If the defense correctly matches the offense's play, then the offense gains 0 yards.

- If the offense runs while the defense defends against the pass, the offense gains 5 yards.

- If the offense passes while the defense defends against the run, the offense gains 10 yards.

Hence we have the payoff matrix shown in Figure 6.15.

		Defense	
		Defend Pass	*Defend Run*
Offense	*Pass*	$0, 0$	$10, -10$
	Run	$5, -5$	$0, 0$

Figure 6.15: Run-Pass Game

(If you don't know the rules of American football, you can follow the discussion simply by taking the payoff matrix as self-contained. Intuitively, the point is simply that we have an attack-defense game with two players named "offense" and "defense" respectively, and where the attacker has a stronger option (pass) and a weaker option (run).)

Just as in Matching Pennies, it's easy to check that there is no Nash equilibrium where either player uses a pure strategy: both have to make their behavior unpredictable by randomizing. So let's work out a mixed-strategy equilibrium for this game: let p be the probability that the offense passes, and let q be the probability that the defense defends against the pass. (We know from Nash's result that at least one mixed-strategy equilibrium must exist, but not what the actual values of p and q should be.)

We use the principle that a mixed equilibrium arises when the probabilities used by each player makes his opponent indifferent between his two options.

- First, suppose the defense chooses a probability of q for defending against the pass. Then the expected payoff to the offense from passing is

  $$(0)(q) + (10)(1 - q) = 10 - 10q,$$

  while the expected payoff to the offense from running is

  $$(5)(q) + (0)(1 - q) = 5q.$$

  To make the offense indifferent between its two strategies, we need to set $10 - 10q = 5q$, and hence $q = 2/3$.

- Next, suppose the offense chooses a probability of p for passing. Then the expected payoff to the defense from defending against the pass is

  $$(0)(p) + (-5)(1 - p) = 5p - 5,$$

  with the expected payoff to the defense from defending against the run is

  $$(-10)(p) + (0)(1 - p) = -10p.$$

To make the defense indifferent between its two strategies, we need to set $5p-5 = -10p$, and hence $p = 1/3$.

Thus, the only possible probability values that can appear in a mixed-strategy equilibrium are $p = 1/3$ for the offense, and $q = 2/3$ for the defense, and this in fact forms an equilibrium. Notice also that the expected payoff to the offense with these probabilities is $10/3$, and the corresponding expected payoff to the defense is $-10/3$. Also, in contrast to Matching Pennies, notice that because of the asymmetric structure of the payoffs here, the probabilities that appear in the mixed-strategy equilibrium are unbalanced as well.

Strategic Interpretation of the Run-Pass Game. There are several things to notice about this equilibrium. First, the strategic implications of the equilibrium probabilities are intriguing and a bit subtle. Specifically, although passing is the offense's more powerful weapon, it uses it less than half the time: it places only probability $p = 1/3$ on passing. This initially seems counter-intuitive: why not spend more time using your more powerful option? But the calculation that gave us the equilibrium probabilities also supplies the answer to this question. If the offense placed any higher probability on passing, then the defense's best response would be to always defend against the pass, and the offense would actually do worse in expectation.

We can see how this works by trying a larger value for p, like $p = 1/2$. In this case, the defense will always defend against the pass, and so the offense's expected payoff will be $5/2$, since it gains 5 half the time and 0 the other half the time:

$$(1/2)(0) + (1/2)(5) = 5/2.$$

Above, we saw that with the equilibrium probabilities, the offense has an expected payoff of $10/3 > 5/2$. Moreover, because $p = 1/3$ makes the defense indifferent between its two strategies, an offense that uses $p = 1/3$ is guaranteed to get $10/3 > 5/2$ no matter what the defense does.

One way to think about the real power of passing as a strategy is to notice that in equilibrium, the defense is defending against the pass $2/3$ of the time, even though the offense is using it only $1/3$ of the time. So somehow the *threat* of passing is helping the offense, even though it uses it relatively rarely.

This example clearly over-simplifies the strategic issues at work in American football: there are many more than just two strategies, and teams are concerned with more than just their yardage on the very next play. Nevertheless, this type of analysis has been applied quantitatively to statistics from American football, verifying some of the main qualitative conclusions at a broad level — that teams generally run more than they pass, and that the expected yardage gained per play from running is close to the expected yardage gained per play from passing for most teams [81, 83, 350].

The Penalty-Kick Game. The complexity of American football makes it hard to cast it truly accurately as a two-person, two-strategy game. We now focus on a different setting, also from professional sports, where such a formalization can be done much more exactly — the modeling of penalty kicks in soccer as a two-player game.

In 2002, Ignacio Palacios-Huerta undertook a large study of penalty kicks from the perspective of game theory [332], and we focus on his analysis here. As he observed, penalty kicks capture the ingredients of two-player, two-strategy games remarkably faithfully. The kicker can aim the ball to the left or the right of the goal, and the goalie can dive to either the left or right as well.[3] The ball moves to the goal fast enough that the decisions of the kicker and goalie are effectively being made simultaneously; and based on these decisions the kicker is likely to score or not. Indeed, the structure of the game is very much like Matching Pennies: if the goalie dives in the direction where the ball is aimed, he has a good chance of blocking it; if the goalie dives in the wrong direction, it is very likely to go in the goal.

Based on an analysis of roughly 1400 penalty kicks in professional soccer, Palacios-Huerta determined the empirical probability of scoring for each of the four basic outcomes: whether the kicker aims left or right, and whether the goalie dives left or right. This led to a payoff matrix as shown in Figure 6.16.

		Goalie	
		L	R
Kicker	L	$0.58, -0.58$	$0.95, -0.95$
	R	$0.93, -0.93$	$0.70, -0.70$

Figure 6.16: The Penalty-Kick Games (from empirical data [332]).

There are a few contrasts to note in relation to the basic Matching Pennies Game. First, a kicker has a reasonably good chance of scoring even when the goalie dives in the correct direction (although a correct choice by the goalie still greatly reduces this probability). Second, kickers are generally right-footed, and so their chance of scoring is not completely symmetric between aiming left and aiming right.[4]

Despite these caveats, the basic premise of Matching Pennies is still present here: there is no equilibrium in pure strategies, and so we need to consider how players should randomize their behavior in playing this game. Using the principle of indifference as in previous examples, we see that if q is the probability that a goalie chooses L, we need to set q so as to make the kicker indifferent between his two options:

$$(.58)(q) + (.95)(1-q) = (.93)(q) + (.70)(1-q).$$

[3] Kicks up the center, and decisions by the goalie to remain in the center, are very rare, and can be ignored in a simple version of the analysis.

[4] For purposes of the analysis, we take all the left-footed kickers in the data and apply a left-right reflection to all their actions, so that R always denotes the "natural side" for each kicker.

Solving for q, we get $q = .42$. We can do the analogous calculation to obtain the value of p that makes the goalie indifferent, obtaining $p = .39$.

The striking punchline to this study is that in the dataset of real penalty kicks, the goalies dive left a .42 fraction of the time (matching the prediction to two decimal places), and the kickers aim left a .40 fraction of the time (coming within .01 of the prediction). It is particularly nice to find the theory's predictions borne out in a setting such as professional soccer, since the two-player game under study is being played by experts, and the outcome is important enough to the participants that they are investing significant attention to their choice of strategies.

Finding all Nash Equilibria. To conclude our discussion of mixed-strategy equilibria, we consider the general question of how to find all Nash equilibria of a two-player, two-strategy game.

First, it is important to note that a game may have both pure-strategy and mixed-strategy equilibria. As a result, one should first check all four pure outcomes (given by pairs of pure strategies) to see which, if any, form equilibria. Then, to check whether there are any mixed-strategy equilibria, we need to see whether there are mixing probabilities p and q that are best responses to each other. If there is a mixed-strategy equilibrium, then we can determine Player 2's strategy (q) from the requirement that Player 1 randomizes. Player 1 will only randomize if his pure strategies have equal expected payoff. This equality of expected payoffs for Player 1 gives us one equation which we can solve to determine q. The same process gives an equation to solve for determining Player 2's strategy p. If both of the obtained values p and q are strictly between 0 and 1, and are thus legitimate mixed strategies, then we have a mixed-strategy equilibrium.

Thus far, our examples of mixed-strategy equilibria have been restricted to games with an attack-defense structure, and so we have not seen an example exhibiting both pure and mixed equilibria. However, it is not hard to find such examples: in particular, Coordination and Hawk-Dove games with two pure equilibria will have a third mixed equilibrium in which each player randomizes. As an example, let's consider the Unbalanced Coordination Game from Section 6.5:

		Your Partner	
		PowerPoint	*Keynote*
You	*PowerPoint*	$1,1$	$0,0$
	Keynote	$0,0$	$2,2$

Figure 6.17: Unbalanced Coordination Game

Suppose that you place a probability of p strictly between 0 and 1 on PowerPoint, and your partner places a probability of q strictly between 0 and 1 on PowerPoint. Then you'll be indifferent between PowerPoint and Keynote if

$$(1)(q) + (0)(1 - q) = (0)(q) + (2)(1 - q),$$

or in other words, if $q = 2/3$. Since the situation is symmetric from your partner's point of view, we also get $p = 2/3$. Thus, in addition to the two pure equilibria, we also get an equilibrium in which each of you chooses PowerPoint with probability 2/3. Note that unlike the two pure equilibria, this mixed equilibrium comes with a positive probability that the two of you will miscoordinate; but this is still an equilibrium, since if you truly believe that your partner is choosing PowerPoint with probability 2/3 and Keynote with probability 1/3, then you'll be indifferent between the two options, and will get the same expected payoff however you choose.

6.9 Pareto-Optimality and Social Optimality

In a Nash equilibrium, each player's strategy is a best response to the other player's strategies. In other words, the players are optimizing individually. But this doesn't mean that, as a group, the players will necessarily reach an outcome that is in any sense good. The Exam-or-Presentation Game from the opening section, and related games like the Prisoner's Dilemma, serve as examples of this. (We redraw the payoff matrix for the basic Exam-or-Presentation Game in Figure 6.18.)

		Your Partner	
		Presentation	*Exam*
You	*Presentation*	$90, 90$	$86, 92$
	Exam	$92, 86$	$88, 88$

Figure 6.18: Exam or Presentation?

It is interesting to classify outcomes in a game not just by their strategic or equilibrium properties, but also by whether they are "good for society." In order to reason about this latter issue, we first need a way of making it precise. There are two useful candidates for such a definition, as we now discuss.

Pareto-Optimality. The first definition is *Pareto-optimality*, named after the Italian economist Vilfredo Pareto who worked in the late 1800's and early 1900's.

> *A choice of strategies — one by each player — is* Pareto-optimal *if there is no other choice of strategies in which all players receive payoffs at least as high, and at least one player receives a strictly higher payoff.*

To see the intuitive appeal of Pareto-optimality, let's consider a choice of strategies that is *not* Pareto-optimal. In this case, there's an alternate choice of strategies that makes at least one player better off without harming any player. In basically any reasonable sense, this alternate choice is superior than what's currently being played. If the players could jointly agree on what to do, and make this agreement binding, then surely they would prefer to move to this superior choice of strategies.

The motivation here relies crucially on the idea that the players can construct a binding agreement to actually play the superior pair choice of strategies: if this alternate choice is not a Nash equilibrium, then absent a binding agreement, at least one player would want to switch to a different strategy. As an illustration of why this is a crucial point, consider the outcomes in the Exam-or-Presentation Game. The outcome in which you and your partner both study for the exam is not Pareto-optimal, since the outcome in which you both prepare for the presentation is strictly better for both of you. This is the central difficulty at the heart of this example, now phrased in terms of Pareto-optimality. It shows that even though you and your partner realize there is a superior solution, there is no way to maintain it without a binding agreement between the two of you.

In this example, the two outcomes in which exactly one of you prepares for the presentation are also Pareto-optimal. In this case, although one of you is doing badly, there is no alternate choice of strategies in which *everyone* is doing at least as well. So in fact, the Exam-or-Presentation Game – and the Prisoner's Dilemma — are examples of games in which the *only* outcome that is not Pareto-optimal is the one corresponding to the unique Nash equilibrium.

Social Optimality. A stronger condition that is even simpler to state is *social optimality.*

> *A choice of strategies — one by each player — is a* social welfare maximizer *(or* socially optimal*) if it maximizes the sum of the players' payoffs.*

In the Exam-or-Presentation Game, the social optimum is achieved by the outcome in which both you and your partner prepare for the presentation, which produces a combined payoff of $90 + 90 = 180$. Of course, this definition is only appropriate to the extent that it makes sense to add the payoffs of different players together — it's not always clear that we can meaningfully combine my satisfaction with an outcome and your satisfaction by simply adding them up.

Outcomes that are socially optimal must also be Pareto-optimal: if such an outcome weren't Pareto-optimal, there would be a different outcome in which all payoffs were at least as large, and one was larger — and this would be an outcome with a larger sum of payoffs. On the other hand, a Pareto-optimal outcome need not be socially optimal. For example, the Exam-or-Presentation Game has three outcomes that are Pareto-optimal, but only one of these is the social optimum.

Finally, of course, it's not the case that Nash equilibria are at odds with goal of social optimality in every game. For example, in the version of the Exam-or-Presentation Game with an easier exam, yielding the payoff matrix that we saw earlier in Figure 6.4, the unique Nash equilibrium is also the unique social optimum.

6.10 Advanced Material: Dominated Strategies and Dynamic Games

In this final section, we consider two further issues that arise in the analysis of games. First, we study the role of *dominated strategies* in reasoning about behavior in a game, and find that the analysis of dominated strategies can provide a way to make predictions about play based on rationality, even when no player has a dominant strategy. Second, we discuss how to reinterpret the strategies and payoffs in a game to deal with situations in which play actually occurs sequentially through time.

Before doing this, however, we begin with a formal definition for games that have more than two players.

A. Multi-Player Games

A (multi-player) game consists, as in the two-player case, of a set of players, a set of strategies for each player, and a payoff to each player for each possible outcome.

Specifically, suppose that a game has n players named $1, 2, \ldots, n$. Each player has a set of possible strategies. An *outcome* (or *joint strategy*) of the game is a choice of a strategy for each player. Finally, each player i has a *payoff function* P_i that maps outcomes of the game to a numerical payoff for i: that is, for each outcome consisting of strategies $(S_1, S_2, \ldots, S_n)$, there is a payoff $P_i(S_1, S_2, \ldots, S_n)$ to player i.

Now, we can say that a strategy S_i is a *best response* by Player i to a choice of strategies $(S_1, S_2, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n)$ by all the other players if

$$P_i(S_1, S_2, \ldots, S_{i-1}, S_i, S_{i+1}, \ldots, S_n) \geq P_i(S_1, S_2, \ldots, S_{i-1}, S_i', S_{i+1}, \ldots, S_n)$$

for all other possible strategies S_i' available to player i.

Finally, an outcome consisting of strategies $(S_1, S_2, \ldots, S_n)$ is a *Nash equilibrium* if each strategy it contains is a best response to all the others.

B. Dominated Strategies and their Role in Strategic Reasoning

In Sections 6.2 and 6.3, we discussed (strictly) dominant strategies — strategies that are a (strict) best response to every possible choice of strategies by the other players. Clearly if a player has a strictly dominant strategy then this is the strategy she should employ. But we

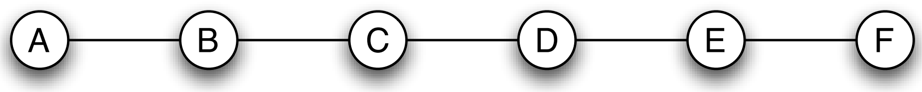

Figure 6.19: In the Facility Location Game, each player has strictly dominated strategies but no dominant strategy.

also saw that even for two-player, two-strategy games, it is common to have no dominant strategies. This holds even more strongly for larger games: although dominant and strictly dominant strategies can exist in games with many players and many strategies, they are rare.

However, even if a player does not have a dominant strategy, she may still have strategies that are dominated by other strategies. In this section, we consider the role that such dominated strategies play in reasoning about behavior in games.

We begin with a formal definition: a strategy is *strictly dominated* if there is some other strategy available to the same player that produces a strictly higher payoff in response to *every* choice of strategies by the other players. In the notation we've just developed, strategy S_i for player i is strictly dominated if there is another strategy S_i' for player i such that

$$P_i(S_1, S_2, \ldots, S_{i-1}, S_i', S_{i+1}, \ldots, S_n) > P_i(S_1, S_2, \ldots, S_{i-1}, S_i, S_{i+1}, \ldots, S_n)$$

for all choices of strategies $(S_1, S_2, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n)$ by the other players.

Now, in the two-player, two-strategy games we've been considering thus far, a strategy is strictly dominated precisely when the other strategy available to the same player is strictly dominant. In this context, it wouldn't make sense to study strictly dominated strategies as a separate concept on their own. However, if a player has many strategies, then it's possible for a strategy to be strictly dominated without any strategy being dominant. In such cases, we will find that strictly dominated strategies can play a very useful role in reasoning about play in a game. In particular, we will see that there are cases in which there are no dominant strategies, but where the outcome of the game can still be uniquely predicted using the structure of the dominated strategies. In this way, reasoning based on dominated strategies forms an intriguing intermediate approach between dominant strategies and Nash equilibrium: on the one hand, it can be more powerful than reasoning based solely on dominant strategies; but on the other hand, it still relies only on the premise that players seek to maximize payoffs, and doesn't require the introduction of an equilibrium notion.

To see how this works, it's useful to introduce the approach in the context of a basic example.

Example: The Facility Location Game. Our example is a game in which two firms compete through their choice of locations. Suppose that two firms are each planning to open a store in one of six towns located along six consecutive exits on a highway. We can represent the arrangement of these towns using a six-node graph as in Figure 6.19.

Now, based on leasing agreements, Firm 1 has the option of opening its store in any of towns A, C, or E, while Firm 2 has the option of opening its store in any of towns B, D, or F. These decisions will be executed simultaneously. Once the two stores are opened, customers from the towns will go to the store that is closer to them. So for example, if Firm 1 open its store in town C and Firm 2 opens its store in town B, then the store in town B will attract customers from A and B, while the store in town C will attract customers from C, D, E, and F. If we assume that the towns contain an equal number of customers, and that payoffs are directly proportional to the number of customers, this would result in a payoff of 4 for Firm 1 and 2 for Firm 2, since Firm 1 claims customers from 4 towns while Firm 2 claims customers from the remaining 2 towns. Reasoning in this way about the number of towns claimed by each store, based on proximity to their locations, we get the payoff matrix shown in Figure 6.20.

		Firm 2		
		B	D	F
	A	$1,5$	$2,4$	$3,3$
Firm 1	C	$4,2$	$3,3$	$4,2$
	E	$3,3$	$2,4$	$5,1$

Figure 6.20: Facility Location Game

We refer to this as a *Facility Location Game*. The competitive location of facilities is a topic that has been the subject of considerable study in operations research and other areas [134]. Moreover, closely related models have been used when the entities being "located" are not stores along a one-dimensional highway but the positions of political candidates along a one-dimensional ideological spectrum — here too, choosing a certain position relative to one's electoral opponent can attract certain voters while alienating others [345]. We will return to issues related to political competition, though in a slightly different direction, in Chapter 23.

We can verify that neither player has a dominant strategy in this game: for example, if Firm 1 locates at node A, then the strict best response of Firm 2 is B, while if Firm 1 locates at node E, then the strict best response of Firm 2 is D. The situation is symmetric if we interchange the roles of the two firms (and read the graph from the other direction).

Dominated Strategies in the Facility Location Game. We can make progress in reasoning about the behavior of the two players in the Facility Location Game by thinking about their dominated strategies. First, notice that A is a strictly dominated strategy for Firm 1: in any situation where Firm 1 has the option of choosing A, it would receive a strictly higher payoff by choosing C. Similarly, F is a strictly dominated strategy for Firm 2: in any situation where Firm 1 has the option of choosing F, it would receive a strictly higher payoff by choosing D.

It is never in a player's interest to use a strictly dominated strategy, since it should always be replaced by a strategy that does better. So Firm 1 isn't going to use strategy A. Moreover, since Firm 2 knows the structure of the game, including Firm 1's payoffs, Firm 2 knows that Firm 1 won't use strategy A. It can be effectively eliminated from the game. The same reasoning shows that F can be eliminated from the game.

We now have a smaller instance of the Facility Location Game, involving only the four nodes B, C, D, and E, and the payoff matrix shown in Figure 6.21.

		Firm 2	
		B	D
Firm 1	C	$4, 2$	$3, 3$
	E	$3, 3$	$2, 4$

Figure 6.21: Smaller Facility Location Game

Now something interesting happens. The strategies B and E weren't previously strictly dominated: they were useful in case the other player used A or F respectively. But with A and F eliminated, the strategies B and E now *are* strictly dominated — so by the same reasoning, both players know they won't be used, and so we can eliminate them from the game. This gives us the even smaller game shown in Figure 6.22.

		Firm 2
		D
Firm 1	C	$3, 3$

Figure 6.22: Even smaller Facility Location Game

At this point, there is a very clear prediction for the play of the game: Firm 1 will play C, and Firm 2 will play D. And the reasoning that led to this is clear: after repeatedly removing strategies that were (or became) strictly dominated, we were left with only a single plausible option for each player.

The process that led us to this reduced game is called the *iterative deletion of strictly dominated strategies*, and we will shortly describe it in its full generality. Before doing this, however, it's worth making some observations about the example of the Facility Location Game.

First, the pair of strategies (C, D) is indeed the unique Nash equilibrium in the game, and when we discuss the iterated deletion of strictly dominated strategies in general, we will see that it is an effective way to search for Nash equilibria. But beyond this, it is also an effective way to *justify* the Nash equilibria that one finds. When we first introduced Nash equilibrium, we observed that it couldn't be derived purely from an assumption of rationality on the part of the players; rather, we had to assume further that play of the game would be found at an equilibrium from which neither player had an incentive to deviate. On the other hand, when a unique Nash equilibrium emerges from the iterated deletion of strictly dominated strategies, it is in fact a prediction made purely based on the assumptions of the players' rationality and their knowledge of the game, since all the steps that led to it were based simply on removing strategies that were strictly inferior to others from the perspective of payoff-maximization.

A final observation is that iterated deletion can in principle be carried out for a very large number of steps, and the Facility Location Game illustrates this. Suppose that instead of a path of length six, we had a path of length 1000, with the options for the two firms still strictly alternating along this path (constituting 500 possible strategies for each player). Then it would be still be the case that only the outer two nodes would be strictly dominated; after their removal, we'd have a path of length 998 in which the two new outer nodes had now become strictly dominated. We can continue removing nodes in this way, and after 499 steps of such reasoning, we'll have a game in which only the 500^{th} and 501^{st} nodes have survived as strategies. This is the unique Nash equilibrium for the game, and this unique prediction can be justified by a very long sequence of deletions of dominated strategies.

It's also interesting how this prediction is intuitively natural, and one that is often seen in real life: two competing stores staking out positions next to each other near the center of the population, or two political candidates gravitating toward the ideological middle ground as they compete for voters in a general election. In each case, this move toward the center is the unique way to maximize the territory that you can claim at the expense of your competitor.

Iterated Deletion of Dominated Strategies: The General Principle. In general, for a game with an arbitrary number of players, the process of *iterated deletion of strictly dominated strategies* proceeds as follows.

- We start with any n-player game, find all the strictly dominated strategies, and delete them.

- We then consider the reduced game in which these strategies have been removed. In this reduced game there may be strategies that are now strictly dominated, despite not having been strictly dominated in the full game. We find these strategies and delete them.

- We continue this process, repeatedly finding and removing strictly dominated strategies until none can be found.

An important general fact is that the set of Nash equilibria of the original game coincides with the set of Nash equilibria for the final reduced game, consisting only of strategies that survive iterated deletion. To prove this fact, it is enough to show that the set of Nash equilibria does not change when we perform one round of deleting strictly dominated strategies; if this is true, then we have established that the Nash equilibria continue to remain unchanged through an arbitrary finite sequence of deletions.

To prove that the set of Nash equilibria remains the same through one round of deletion, we need to show two things. First, any Nash equilibrium of the original game is a Nash equilibrium of the reduced game. To see this, note that otherwise there would be a Nash equilibrium of the original game involving a strategy S that was deleted. But in this case, S is strictly dominated by some other strategy S'. Hence S cannot be part of a Nash equilibrium of the original game: it is not a best response to the strategies of the other players, since the strategy S' that dominates it is a better response. This establishes that no Nash equilibrium of the original game can be removed by the deletion process. Second, we need to show that any Nash equilibrium of the reduced game is also a Nash equilibrium of the original game. In order for this not to be the case, there would have to be a Nash equilibrium $E = (S_1, S_2, \ldots, S_n)$ of the reduced game, and a strategy S_i' that was deleted from the original game, such that player i has an incentive to deviate from its strategy S_i in E to the strategy S_i'. But strategy S_i' was deleted because it was strictly dominated by at least one other strategy; we can therefore find a strategy S_i'' that strictly dominated it and was not deleted. Then player i also has an incentive to deviate from S_i to S_i'', and S_i'' is still present in the reduced game, contradicting our assumption that E is a Nash equilibrium of the reduced game.

This establishes that the game we end up with, after iterated deletion of strictly dominated strategies, still has all the Nash equilibria of the original game. Hence, this process can be a powerful way to restrict the search for Nash equilibria. Moreover, although we described the process as operating in rounds, with all currently strictly dominated strategies being removed in each round, this is not essential. One can show that eliminating strictly dominated strategies in any order will result in the same set of surviving strategies.

Weakly Dominated Strategies. It is also natural to ask about notions that are slightly weaker than our definition of strictly dominated strategies. One fundamental definition in this spirit is that of a weakly dominated strategy. We say that a strategy is *weakly dominated* if there is another strategy that does at least as well no matter what the other players do, and does strictly better against some joint strategy of the other players. In our notation from earlier, we say that a strategy S_i for player i is weakly dominated if there is another strategy S_i' for player i such that

$$P_i(S_1, S_2, \ldots, S_{i-1}, S_i', S_{i+1}, \ldots, S_n) \geq P_i(S_1, S_2, \ldots, S_{i-1}, S_i, S_{i+1}, \ldots, S_n)$$

for all choices of strategies $(S_1, S_2, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n)$ by the other players, and

$$P_i(S_1, S_2, \ldots, S_{i-1}, S'_i, S_{i+1}, \ldots, S_n) > P_i(S_1, S_2, \ldots, S_{i-1}, S_i, S_{i+1}, \ldots, S_n)$$

for at least one choice of strategies $(S_1, S_2, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n)$ by the other players.

For strictly dominated strategies, the argument for deleting them was compelling: they are never best responses. For weakly dominated strategies, the issue is more subtle. Such strategies could be best responses to some joint strategy by the other players. So a rational player could play a weakly dominated strategy, and in fact Nash equilibria can involve weakly dominated strategies.

There are simple examples that make this clear even in two-player, two-strategy games. Consider for example a version of the Stag Hunt Game in which the payoff from successfully catching a stag is the same as the payoff from catching a hare:

		Hunter 2	
		Hunt Stag	*Hunt Hare*
Hunter 1	*Hunt Stag*	$3, 3$	$0, 3$
	Hunt Hare	$3, 0$	$3, 3$

Figure 6.23: Stag Hunt: A version with a weakly dominated strategy

In this case, *Hunt Stag* is a weakly dominated strategy, since each player always does at least as well, and sometimes strictly better, by playing *Hunt Hare*. Nevertheless, the outcome in which both players choose *Hunt Stag* is a Nash equilibrium, since each is playing a best response to the other's strategy. Thus, deleting weakly dominated strategies is not in general a safe thing to do, if one wants to preserve the essential structure of the game: such deletion operations can destroy Nash equilibria.

Of course, it might seem reasonable to suppose that a player should not play an equilibrium involving a weakly dominated strategy (such as (*Hunt Stag, Hunt Stag*)) if he had any uncertainty about what the other players would do — after all, why not use an alternate strategy that is at least as good in every eventuality? But Nash equilibrium does not take into account this idea of uncertainty about the behavior of others, and hence has no way to rule out such outcomes. In the next chapter, we will discuss an alternate equilibrium concept known as evolutionary stability that in fact does eliminate weakly dominated strategies in a principled way. The relationship between Nash equilibrium, evolutionary stability and weakly dominated strategies is considered in the exercises at the end of this part of the book.

C. Dynamic Games

Our focus in this chapter has been on games in which all players choose their strategies simultaneously, and then receive payoffs based on this joint decision. Of course, actual

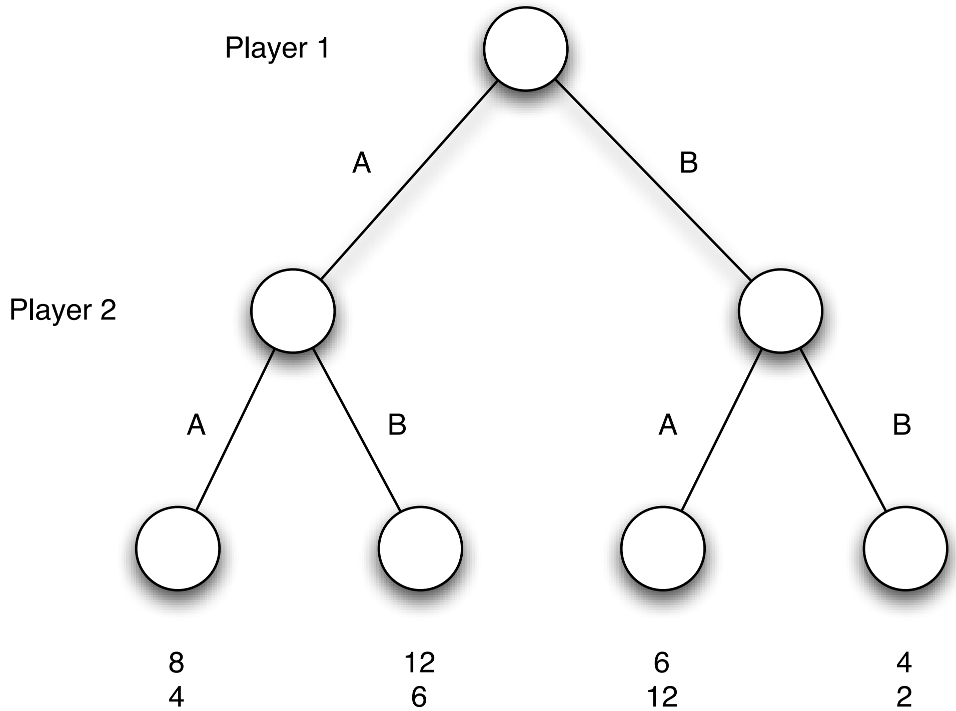

Figure 6.24: A simple game in extensive form.

simultaneity is not crucial for the model, but it has been central to our discussions so far that each player is choosing a strategy without knowledge of the actual choices made by the other players.

Many games, however, are played over time: some player or set of players moves first, other players observe the choice(s) made, and then they respond, perhaps according to a predetermined order of governing who moves when. Such games are called *dynamic games*, and there are many basic examples: board games and card games in which players alternate turns; negotiations, which usually involve a sequence of offers and counter-offers; and bidding in an auction or pricing competing goods where participants must make decisions over time. Here we'll discuss an adaptation of the theory of games that incorporates this dynamic aspect.

Normal and Extensive Forms of a Game. To begin with, specifying a dynamic game is going to require a new kind of notation. Thus far, we've worked with something called the *normal-form* representation of a game; this specifies the list of players, their possible strategies, and the payoffs arising from every possible (simultaneous) choice of strategies by the players. (For two-player games, the payoff matrices we've seen in this chapter encode the normal-form representation of a game in a compact way.)

To describe a dynamic game, we're going to need a richer representation; we need to be able to specify who moves when, what each player knows at any opportunity they have to move, what they can do when it is their turn to move, and what the payoffs are at the end of the game. We refer to this specification as the *extensive-form* representation of the game.

Let's start with a very simple example of a dynamic game, so that we can discuss what its extensive-form representation looks like. As we'll see, this game is simple enough that it avoids some of the subtleties that arise in the analysis of dynamic games, but it is useful as a first illustration, and we'll proceed to a more complex second example afterward.

In our first example, we imagine that there are two firms — Firm 1 and Firm 2 — each of whom is trying to decide whether to focus its advertising and marketing on two possible regions, named A or B. Firm 1 gets to choose first. If Firm 2 follows Firm 1 into the same region, then Firm 1's "first-mover advantage" gives it 2/3 of the profit obtainable from the market in that region, while Firm 2 will only get 1/3. If Firm 2 moves into the other region, then each firm gets all the profit obtainable in their respective region. Finally, Region A has twice as large a market as Region B: the total profit obtainable in region A is equal to 12, while in Region B, it's 6.

We write the extensive-form representation as a "game tree," depicted in Figure 6.24. This tree is designed to be read downward from the top. The top node represents Firm 1's initial move, and the two edges descending from this node represent its two options A or B. Based on which branch is taken, this leads to a node representing Firm 2's subsequent move. Firm 2 can then also choose option A or B, again represented by edges descending from the node. This leads to a terminal node representing the end of play in the game; each terminal node is labeled with the payoffs to the two players.

Thus, a specific play — determined by a sequence of choices by Firm 1 and Firm 2 — corresponds to a path from the top node in the tree down to some terminal node. First Firm 1 chooses A or B, then Firm 2 chooses A or B, and then the two players receive their payoffs. In a more general model of dynamic games, each node could contain an annotation saying what information about the previous moves is known to the player currently making a move; however, for our purposes here, we will focus on the case in which each player knows the complete history of past moves when they go to make their current move.

Reasoning about Behavior in a Dynamic Game. As with simultaneous-move games, we'd like to make predictions for what players will do in dynamic games. One way is to reason from the game tree. In our current example, we can start by considering how Firm 2 will behave after each of the two possible opening moves by Firm 1. If Firm 1 chooses A, then Firm 2 maximizes its payoff by choosing B, while if Firm 1 chooses B, then Firm 2 maximizes its payoff by choosing A. Now let's consider Firm 1's opening move, given what we've just concluded about Firm 2's subsequent behavior. If Firm 1 chooses A, then it expects Firm 2 to choose B, yielding a payoff of 12 for Firm 1. If Firm 1 chooses B, then it expects Firm 2 to choose A, yielding a payoff of 6 for Firm 1. Since we expect the firms to try maximizing their payoffs, we predict that Firm 1 should choose A, after which Firm 2 should choose B.

This is a useful way to analyze dynamic games. We start one step above the terminal nodes, where the last player to move has complete control over the outcome of the payoffs. This lets us predict what the last player will do in all cases. Having established this, we then move one more level up the game tree, using these predictions to reason about what the player one move earlier will do. We continue in this way up the tree, eventually making predictions for play all the way up to the top node.

A different style of analysis exploits an interesting connection between normal and extensive forms, allowing us to write a normal-form representation for a dynamic game as follows. Suppose that, before the game is played, each player makes up a plan for how to play the entire game, covering every possible eventuality. This plan will serve as the player's strategy. One way to think about such strategies, and a useful way to be sure that they include a complete description of every possibility, is to imagine that each player has to provide all of the information needed to write a computer program which will actually play the game in their place.

For the game in Figure 6.24, Firm 1 only has two possible strategies: A or B. Since Firm 2 moves after observing what Firm 1 did, and Firm 2 has two possible choices for each of the two options by Firm 1, Firm 2 has four possible plans for playing the game. They can be written as contingencies, specifying what Firm 2 will do in response to each possible move by Firm 1:

$$(A \text{ if } A,\ A \text{ if } B),\ (A \text{ if } A,\ B \text{ if } B),\ (B \text{ if } A,\ A \text{ if } B), \text{ and } (B \text{ if } A,\ B \text{ if } B),$$

or in abbreviated form as

$$(AA,\ AB),\ (AA,\ BB),\ (BA,\ AB), \text{ and } (BA,\ BB),$$

If each player chooses a complete plan for playing the game as its strategy, then we can determine the payoffs directly from this pair of chosen strategies via a payoff matrix.

		Firm 2			
		AA, AB	AA, BB	BA, AB	BA, BB
Firm 1	A	$8, 4$	$8, 4$	$12, 6$	$12, 6$
	B	$6, 12$	$4, 2$	$6, 12$	$4, 2$

Figure 6.25: Conversion to normal form.

Because the plans describe everything about how a player will behave, we have managed to describe this dynamic game in normal form: each player chooses a strategy (consisting of a complete plan) in advance, and from this joint choice of strategies, we can determine payoffs. We will see later that there are some important subtleties in using this interpretation of the underlying dynamic game, and in particular the translation from extensive to normal form will sometimes not preserve the full structure implicit in the game. But the translation is a useful tool for analysis, and the subtle lack of fidelity that can arise in the translation is in itself a revealing notion to develop and explore.

With this in mind, we first finish our simple example, where the translation will work perfectly, and then move on to a second example where the complications begin to arise. For the normal-form payoff matrix corresponding to our first example, the payoff matrix has eight cells, while the extensive-form representation only has four terminal nodes with payoffs. This occurs because each terminal node can be reached with two different pairs of strategies, with each pair forming a cell of the payoff matrix. Both pairs of strategies dictate the same actions in the path of the game tree which actually occurs, but describe different hypothetical actions in other unrealized paths. For example, the payoffs in the entries for $(A,(AA,AB))$ and for $(A,(AA,BB))$ are the same because both strategy combinations lead to the same terminal node. In both cases Firm 2 chooses A in response to what Firm 1 actually does; Firm 2's plan for what to do in the event Firm 1 chose B is not realized by the actual play.

Now, using the normal-form representation, we can quickly see that for Firm 1, strategy A is strictly dominant. Firm 2 does not have a strictly dominant strategy, but it should play a best response to Firm 1, which would be either (BA,AB) or (BA,BB). Notice that this prediction of play by Firm 1 and Firm 2 based on the normal-form representation is the same as our prediction based on direct analysis of the game tree, where we reasoned upward from the terminal nodes: Firm 1 will play A, and in response Firm 2 will play B.

A More Complex Example: The Market Entry Game. In our first dynamic game, reasoning based on the extensive and normal form representations led to essentially identical conclusions. As games get larger, extensive forms are representationally more streamlined than normal forms for dynamic games, but if this were the only distinction, it would be hard to argue that dynamic games truly add much to the overall theory of games. In fact, however, the dynamic aspect leads to new subtleties, and this can be exposed by considering a case in which the translation from extensive form to normal form ends up obscuring some of the structure that is implicit in the dynamic game.

For this, we consider a second example of a dynamic game, also played between two competing firms. We call this the Market Entry Game, and it's motivated by the following scenario. Consider a region where Firm 2 is currently the only serious participant in a given line of business, and Firm 1 is considering whether to enter the market.

- The first move in this game is made by Firm 1, who must decide whether to stay out

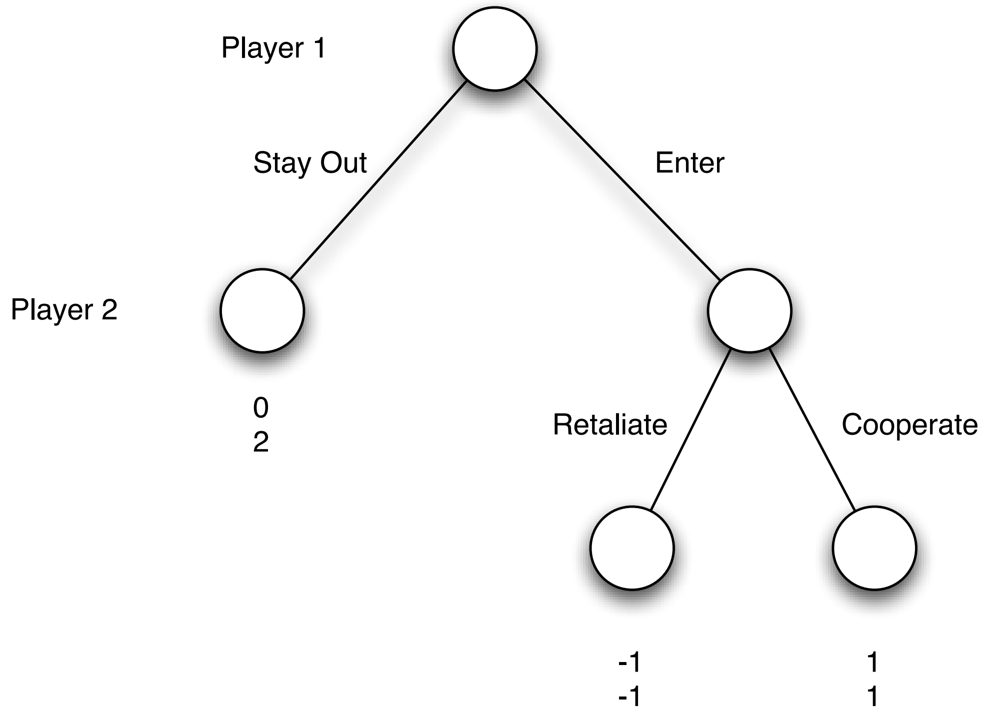

Figure 6.26: Extensive-form representation of the Market Entry Game.

of the market or enter it.

- If Firm 1 chooses to stay out, then the game ends, with Firm 1 getting a payoff of 0 and Firm 2 keeping the payoff from the entire market.
- If Firm 1 chooses to enter, then the game continues to a second move by Firm 2, who must choose whether to cooperate and divide the market evenly with Firm 1, or retaliate and engage in a price war.
  - If Firm 2 cooperates, then each firm gets a payoff corresponding to half the market.
  - If Firm 2 retaliates, then each firm gets a negative payoff.

Choosing numerical payoffs to fill in this story, we can write the extensive-form representation of the Market Entry Game as in Figure 6.26.

Subtle Distinctions Between Extensive and Normal Form Representations. Let's take the two ways we developed to analyze our previous dynamic game and apply them here. First, we can work our way up the game tree starting at the terminal nodes, as follows. If Firm 1 chooses to enter the market, then Firm 2 achieves a higher payoff by cooperating than by retaliating, so we should predict cooperation in the event the game reaches this point. Given this, when Firm 1 goes to make its first move, it can expect a payoff of 0 by staying out, and a payoff of 1 by entering, so it should choose to enter the market. We can therefore predict that Firm 1 will enter the market, and then Firm 2 will cooperate.

Now let's consider the normal-form representation. Firm 1's possible plans for playing the game are just to choose Stay Out (S) or Enter (E). Firm 2's possible plans are to choose retaliation in the event of entry, or cooperation in the event of entry. We'll denote these two plans by R and C respectively. This gives us the payoff matrix in Figure 6.27.

		Firm 2	
		R	C
Firm 1	S	$0, 2$	$0, 2$
	E	$-1, -1$	$1, 1$

Figure 6.27: Normal Form of the Market Entry Game

Here's the surprise: when we look at this game in normal form, we discover two distinct (pure-strategy) Nash equilibria: (E, C), and (S, R). The first of these corresponds to the prediction for play that we obtained by analyzing the extensive-form representation. What does the second one correspond to?

To answer this, it helps to recall our view of the normal-form representation as capturing the idea that each player commits in advance to a computer program that will play the game in its place. Viewed this way, the equilibrium (S, R) corresponds to an outcome in which Firm 2 commits in advance to a computer program that will automatically retaliate in the event that Firm 1 enters the market. Firm 1, meanwhile, commits to a program that stays out of the market. Given this pair of choices, neither firm has an incentive to change the computer program they're using: for example, if Firm 1 were to switch to a program that entered the market, it would trigger retaliation by the program that Firm 2 is using.

This contrast between the prediction from the extensive and normal forms highlights some important points. First, it shows that the premise behind our translation from extensive to normal form — that each player commits ahead of time to a complete plan for playing the game — is not really equivalent to our initial premise in defining dynamic games — namely, that each player makes an optimal decision at each intermediate point in the game, based on what has already happened up to that point. Firm 2's decision to retaliate on entry highlights this clearly. If Firm 2 can truly pre-commit to this plan, then the equilibrium (S, R) makes sense, since Firm 1 will not want to provoke the retaliation that is encoded in Firm 2's plan. But if we take the dynamic game as originally defined in extensive form, then pre-commitment to a plan is not part of the model: rather, Firm 2 only gets to evaluate its decision to cooperate or retaliate once Firm 1 has already entered the market, and at that point its payoff is better if it cooperates. Given this, Firm 1 can predict that it is safe to enter.

In game theory, the standard model for dynamic games in extensive form assumes that players will seek to maximize their payoff at any intermediate stage of play that can be reached in the game. In this interpretation, there is a unique prediction for play in our Market Entry Game, corresponding to the equilibrium (E, C) in normal form. However, the issues surrounding the other equilibrium (S, R) are not simply notational or representational; they are deeper than this. For any given scenario, it is really a question of what we believe is being modeled by the underlying dynamic game in extensive form. It is a question of whether we are in a setting where a player can irrevocably pre-commit to a certain plan, to the extent that other players will believe the commitment as a credible threat — or not.

Further, the Market Entry Game shows how the ability to commit to a particular course of action — when possible — can in fact be a valuable thing for an individual player, even if that course of action would be bad for everyone if it were actually carried out. In particular, if Firm 2 could make Firm 1 believe that it really would retaliate in the event of entry, then Firm 1 would choose to stay out, resulting in a higher payoff for Firm 2. In practice, this suggests particular courses of action that Firm 2 could take before the game even starts. For example, suppose that before Firm 1 had decided whether to enter the market, Firm 2 were to publically advertise an offer to beat any competitor's price by 10%. This would be a safe thing to do as long as Firm 2 is the only serious participant in the market, but it becomes dangerous to both firms if Firm 1 actually enters. The fact that the plan has been publically announced means that it would be very costly (reputationally, and possibly legally) for Firm 2 to back away from it. In this way, the announcement can serve as a way of switching the underlying model from one in which Firm 2's threat to retaliate is not credible to one in which Firm 2 can actually pre-commit to a plan for retaliation.

Relationship to Weakly Dominated Strategies. In discussing these distinctions, it is also interesting to note the role that weakly dominated strategies play here. Notice that in the normal-form representation in Figure 6.27, the strategy R for Firm 2 is weakly dominated, and for a simple reason: it yields the same payoff if Firm 1 chooses S (since then Firm 2 doesn't actually get to move), and it yields a lower payoff if Firm 1 chooses E. So our translation from extensive form to normal form for dynamic games provides another reason to be careful about predictions of play in a normal-form game that rely on weakly dominated strategies: if the structure in fact arises from a dynamic game in extensive form, then information about the dynamic game that is lost in the translation to normal form could potentially be sufficient to eliminate such equilibria.

However, we can't simply fix up the translation by eliminating weakly dominated strategies. We saw earlier that iterated deletion of strictly dominated strategies can be done in any order: all orders yield the same final result. But this is not true for the iterated deletion of weakly dominated strategies. To see this, suppose we vary the Market Entry Game slightly so that the payoff from the joint strategy (E, C) is $(0, 0)$. (In this version, both firms know they will fail to gain a positive payoff even if Firm 2 cooperates on entry, although they still don't do as badly as when Firm 2 retaliates.) R is a weakly dominated strategy as before, but now so is E. (E and S produce the same payoff for Firm 1 when Firm 2 chooses C, and S produces a strictly higher payoff when Firm 2 chooses R.)

In this version of the game, there are now three (pure-strategy) Nash equilibria: (S, C), (E, C), and (S, R). If we first eliminate the weakly dominated strategy R, then we are left with (S, C) and (E, C) as equilibria. Alternately, if we first eliminate the weakly dominated strategy E, then we are left with (S, C) and (S, R) as equilibria. In both cases, no further elimination of weakly dominated strategies is possible, so the order of deletion affects the final set of equilibria. We can ask which of these equilibria actually make sense as predictions of play in this game. If this normal form actually arose from the dynamic version of the Market Entry Game, then C is still the only reasonable strategy for Firm 2, while Firm 1 could now play either S or E.

Final Comments. The analysis framework we developed for most of this chapter is based on games in normal form. One approach to analyzing dynamic games in extensive form is to first find all Nash equilibria of the translation to normal form, treating each of these as a candidate prediction of play in the dynamic game, and then go back to the extensive-form version to see which of these make sense as actual predictions.

There is an alternate theory that works directly with the extensive-form representation. The simplest technique used in this theory is the style of analysis we employed to analyze an extensive-form representation from the terminal nodes upward. But there are more complex components to the theory as well, allowing for richer structure such as the possibility that players at any given point have only partial information about the history of play up to that point. While we will not go further into this theory here, it is developed in a number of books on game theory and microeconomic theory [260, 284, 331, 392].

6.11 Exercises

1. Say whether the following claim is true or false, and provide a brief (1-3 sentence) explanation for your answer.

   Claim: If player A in a two-person game has a dominant strategy s_A, then there is a pure strategy Nash equilibrium in which player A plays s_A and player B plays a best response to s_A.

2. Consider the following statement:

> *In a Nash equilibrium of a two-player game each player is playing an optimal strategy, so the two player's strategies are social-welfare maximizing.*

Is this statement correct or incorrect? If you think it is correct, give a brief (1-3 sentence) explanation for why. If you think it is incorrect, give an example of a game discussed in Chapter 6 that shows it to be incorrect (you do not need to spell out all the details of the game, provided you make it clear what you are referring to), together with a brief (1-3 sentence) explanation.

3. Find all pure strategy Nash equilibria in the game below. In the payoff matrix below the rows correspond to player A's strategies and the columns correspond to player B's strategies. The first entry in each box is player A's payoff and the second entry is player B's payoff.

		Player B	
		L	R
Player A	U	$1,2$	$3,2$
	D	$2,4$	$0,2$

4. Consider the two-player game with players, strategies and payoffs described in the following game matrix.

		Player B		
		L	M	R
	t	$0,3$	$6,2$	$1,1$
Player A	m	$2,3$	$0,1$	$7,0$
	b	$5,3$	$4,2$	$3,1$

Figure 6.28: Payoff Matrix

(a) Does either player have a dominant strategy? Explain briefly (1-3 sentences).

(b) Find all pure strategy Nash equilibria for this game.

5. Consider the following two-player game in which each player has three strategies.

		Player B		
		L	M	R
	U	$1,1$	$2,3$	$1,6$
Player A	M	$3,4$	$5,5$	$2,2$
	D	$1,10$	$4,7$	$0,4$

Find all the (pure strategy) Nash equilibria for this game.

6. In this question we will consider several two-player games. In each payoff matrix below the rows correspond to player A's strategies and the columns correspond to player B's strategies. The first entry in each box is player A's payoff and the second entry is player B's payoff.

   (a) Find all pure (non-randomized) strategy Nash equilibria for the game described by the payoff matrix below.

   Player B

Player A	L	R
U	$2, 15$	$4, 20$
D	$6, 6$	$10, 8$

   (b) Find all pure (non-randomized) strategy Nash equilibria for the game described by the payoff matrix below.

   Player B

Player A	L	R
U	$3, 5$	$4, 3$
D	$2, 1$	$1, 6$

   (c) Find *all* Nash equilibria for the game described by the payoff matrix below.

   Player B

Player A	L	R
U	$1, 1$	$4, 2$
D	$3, 3$	$2, 2$

   [Hint: This game has a both pure strategy equilibria and a mixed strategy equilibrium. To find the mixed strategy equilibrium let the probability that player A uses strategy U be p and the probability that player B uses strategy L be q. As we learned in our analysis of matching pennies, if a player uses a mixed strategy (one that is not really just some pure strategy played with probability one) then the player must be indifferent between two pure strategies. That is the strategies must have equal expected payoffs. So, for example, if p is not 0 or 1 then it must be the case that $q+4(1-q) = 3q+2(1-q)$ as these are the expected payoffs to player A from U and D when player B uses probability q.]

7. In this question we will consider several two-player games. In each payoff matrix below the rows correspond to player A's strategies and the columns correspond to player B's strategies. The first entry in each box is player A's payoff and the second entry is player B's payoff.

		Player B	
		L	R
Player A	U	$1,1$	$3,2$
	D	$0,3$	$4,4$

(a) Find all Nash equilibria for the game described by the payoff matrix below.

(b) Find all Nash equilibria for the game described by the payoff matrix below (include an explanation for your answer).

		Player B	
		L	R
Player A	U	$5,6$	$0,10$
	D	$4,4$	$2,2$

[Hint: This game has a mixed strategy equilibrium. To find the equilibrium let the probability that player A uses strategy U be p and the probability that player B uses strategy L be q. As we learned in our analysis of matching pennies, if a player uses a mixed strategy (one that is not really just some pure strategy played with probability one) then the player must be indifferent between two pure strategies. That is, the strategies must have equal expected payoffs. So, for example, if p is not 0 or 1 then it must be the case that $5q + 0(1-q) = 4q + 2(1-q)$ as these are the expected payoffs to player A from U and D when player B uses probability q.]

8. Consider the two-player game described by the payoff matrix below.

		Player B	
		L	R
Player A	U	$1,1$	$0,0$
	D	$0,0$	$4,4$

(a) Find all pure-strategy Nash equilibria for this game.

(b) This game also has a mixed-strategy Nash equilibrium; find the probabilities the players use in this equilibrium, together with an explanation for your answer.

(c) Keeping in mind Schelling's focal point idea from Chapter 6, what equilibrium do you think is the best prediction of how the game will be played? Explain.

9. For each of the following two player games find all Nash equilibria. In each payoff matrix below the rows correspond to player A's strategies and the columns correspond to player B's strategies. The first entry in each box is player A's payoff and the second entry is player B's payoff.

(a)

		Player B: L	Player B: R
Player A	U	$8,4$	$5,5$
	D	$3,3$	$4,8$

(b)

		Player B: L	Player B: R
Player A	U	$0,0$	$-1,1$
	D	$-1,1$	$2,-2$

10. In the payoff matrix below the rows correspond to player A's strategies and the columns correspond to player B's strategies. The first entry in each box is player A's payoff and the second entry is player B's payoff.

		Player B: L	Player B: R
Player A	U	$3,3$	$1,2$
	D	$2,1$	$3,0$

(a) Find all pure strategy Nash equilibria of this game.

(b) Notice from the payoff matrix above that Player A's payoff from the pair of strategies (U, L) is 3. Can you change player A's payoff from this pair of strategies to some non-negative number in such a way that the resulting game has *no* pure-strategy Nash equilibrium? Give a brief (1-3 sentence) explanation for your answer.

(Note that in answering this question, you should only change Player A's payoff for this one pair of strategies (U, L). In particular, leave the rest of the structure of the game unchanged: the players, their strategies, the payoff from strategies other than (U, L), and B's payoff from (U, L).)

(c) Now let's go back to the original payoff matrix from part (a) and ask an analogous question about player B. So we're back to the payoff matrix in which players A and B each get a payoff of 3 from the pair of strategies (U, L).

Can you change player B's payoff from the pair of strategies (U, L) to some non-negative number in such a way that the resulting game has *no* pure-strategy Nash equilibrium? Give a brief (1-3 sentence) explanation for your answer.

(Again, in answering this question, you should only change Player B's payoff for this one pair of strategies (U, L). In particular, leave the rest of the structure of the game unchanged: the players, their strategies, the payoff from strategies other than (U, L), and A's payoff from (U, L).)

11. In the text we've discussed dominant strategies and noted that if a player has dominant strategy we would expect it to be used. The opposite of a dominant strategy is a strategy that is dominated. The definition of dominated is:

    A strategy s_i^* is *dominated* if player i has another strategy s_i' with the property that player i's payoff is greater from s_i' than from s_i^* no matter what the other players in the game do.

    We do not expect a player to use a strategy that is dominated and this can help in finding Nash equilibria. Here is an example of this idea. In this game, M is a dominated strategy (it is dominated by R) and player B will not use it.

		Player B		
		L	M	R
Player A	U	$2,4$	$2,1$	$3,2$
	D	$1,2$	$3,3$	$2,4$

    So in analyzing the game we can delete M and look at the remaining game

		Player B	
		L	R
Player A	U	$2,4$	$3,2$
	D	$1,2$	$2,4$

    Now player A has a dominant strategy (U) and it is easy to see that the Nash equilibrium of the 2-by-2 game is (U,L). You can check the original game to see that (U,L) is a Nash equilibrium. Of course, using this procedure requires that we know that a dominated strategy cannot be used in Nash equilibrium.[5]

    Consider any two player game which has at least one (pure strategy) Nash equilibrium. Explain why the strategies used in an equilibrium of this game will not be dominated strategies.

12. In Chapter 6 we discussed dominant strategies and noted that if a player has a dominant strategy we would expect it to be used. The opposite of a dominant strategy is a strategy that is dominated. There are several possible notions of what it means for a strategy to be dominated. In this problem we will focus on weak domination.

    A strategy s_i^* is *weakly dominated* if player i has another strategy s_i' with the property that:

[5]This is actually true for any number of players. It would also help to know that if we iteratively remove dominated strategies (in any order) and analyze the reduced games we still find the Nash equilibria of the original game. This is also true, but it is a bit more complicated.

(a) No matter what the other player does, player i's payoff from s_i' is at least as large as the payoff from s_i^*, and

(b) There is some strategy for the other player so that player i's payoff from s_i' is strictly greater than the payoff from s_i^*.

(a) It seems unlikely that a player would use a weakly dominated strategy, but these strategies can occur in a Nash equilibrium. Find all pure (non-randomized) Nash equilibria for the game below. Do any of them use weakly dominated strategies?

		Player B	
		L	R
Player A	U	$1,1$	$1,1$
	D	$0,0$	$2,1$

(b) One way to reason about the weakly dominated strategies that you should have found in answering the question above is to consider the following sequential game. Suppose that actually the players move sequentially, but the player to move second does not know what the player moving first chose. Player A moves first, and if he chooses U, then player B's choice does not matter. Effectively the game is over if A chooses U as no matter what B does the payoff is $(1,1)$. If player A chooses D, then player B's move matters, and the payoff is $(0,0)$ if B chooses L or $(2,1)$ if B chooses R. [Note that as B does not observe A's move the simultaneous move game with payoff matrix above is equivalent to this sequential move game.]

In this game how would you expect the players to behave? Explain your reasoning. [The players are not allowed to change the game. They play it once just as it is given above. You may reason from the payoff matrix or the story behind the game, but if you use the story remember that B does not observe A's move until after the game is over.]

13. Here we consider a game with three players, named 1, 2 and 3. To define the game we need to specify the sets of strategies available to each player; also, when each of the three players chooses a strategy, this gives a triple of strategies, and we need to specify the payoff each player receives from any possible triple of strategies played. Let's suppose that player 1's strategy set is $\{U, D\}$, players 2's strategy set is $\{L, R\}$ and player 3's strategy set is $\{l, r\}$.

One way to specify the payoffs would be to write down every possible triple of strategies, and the payoffs for each. A different but equivalent way to interpret triples of strategies, which makes it easier to specify the payoffs, is to imagine that player 3 chooses which of two distinct two-player games players 1 and 2 will play. If 3 chooses l then the payoff matrix is

Payoff Matrix l:

		Player B: L	Player B: R
Player A	U	$4, 4, 4$	$0, 0, 1$
	D	$0, 2, 1$	$2, 1, 0$

where the first entry in each cell is the payoff to player 1, the second entry is the payoff to player 2 and the third entry is the payoff to player 3.

If 3 chooses r then the payoff matrix is

Payoff Matrix r:

		Player B: L	Player B: R
Player A	U	$2, 0, 0$	$1, 1, 1$
	D	$1, 1, 1$	$2, 2, 2$

So, for example, if player 1 chooses U, player 2 chooses R and player 3 chooses r the payoffs are 1 for each player.

(a) First suppose the players all move simultaneously. That is, players 1 and 2 do not observe which game player 3 has selected until after they each chose a strategy. Find all of the (pure strategy) Nash equilibria for this game.

(b) Now suppose that player 3 gets to move first and that players 1 and 2 observe player 3's move before they decide how to play. That is, if player 3 chooses the strategy r then players 1 and 2 play the game defined by payoff matrix r and they both know that they are playing this game. Similarly, if player 3 chooses the strategy l then players 1 and 2 play the game defined by payoff matrix l and they both know that they are playing this game.

Let's also suppose that if players 1 and 2 play the game defined by payoff matrix r they play a (pure strategy) Nash equilibrium for that game; and similarly, if players 1 and 2 play the game defined by payoff matrix l they play a (pure strategy) Nash equilibrium for that game. Finally, let's suppose that player 3 understands that this is how players 1 and 2 will behave.

What do you expect player 3 to do and why? What triple of strategies would you expect to see played? Is this list of strategies a Nash equilibrium of the simultaneous move game between the three players?