Problem set # 5

1. Find the autocorrelation function of the following processes.
   (a) \( X_t = \alpha X_{t-1} + \varepsilon_t \) where \( |\rho| < 1 \) and \( \varepsilon_t \sim i.i.d. \ (0, \sigma^2_{\varepsilon}) \)
   (b) \( Y_t = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} \) where \( \varepsilon_t \sim i.i.d. \ (0, \sigma^2_{\varepsilon}) \)
   (c) \( Z_t = \rho Z_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1} \) where \( |\rho| < 1 \) and \( \varepsilon_t \sim i.i.d. \ (0, \sigma^2_{\varepsilon}) \)

2. Suppose that we have the following regression model;
   \[ Y_t = \beta_1 + \beta_2 X_t + \beta_3 Y_{t-1} + u_t \]
   where \( X_t \) is non-stochastic. Further, we have the following error structure;
   \[ u_t = \rho u_{t-1} + \varepsilon_t \]
   where \( |\rho| < 1 \) and \( \varepsilon_t \sim i.i.d. \ (0, \sigma^2_{\varepsilon}) \).
   (a) Prove that \( \text{plim} \hat{\beta}_3 \neq \beta_3 \)
   where \( \hat{\beta}_3 \) is the OLS estimator of \( \beta_3 \).
   (b) How can you obtain a consistent estimator for \( \beta_3 \)?

3. Consider estimation of \( \sigma^2 \) in the generalized linear regression model. There is a fundamental ambiguity in regard to this parameter as it is merely a scaling of \( E(\varepsilon \varepsilon') = \sigma^2 \Omega \). Since both components are unknown, \( \sigma^2 \) cannot be estimable until some scaling of \( \Omega \) is assumed to remove the indeterminacy. The most convenient assumption is that \( \text{tr} (\Omega) = N \)
   The classical regression model in which \( \Omega = I \) is one such case, so this provides a useful benchmark. Now, consider the estimator \( s^2 = e'e / (N - k) \), where \( e \) is the vector of the OLS residuals.
   (a) Prove that
   \[
   E(s^2) = \frac{N \sigma^2}{(N - k)} - \frac{\sigma^2 \text{tr} \left[ \left( \frac{X' X}{N - k} \right)^{-1} \left( \frac{X' \Omega X}{N - k} \right) \right]}{(N - k)}
   \]
(b) Prove that if
\[ \text{plim} \frac{X'X}{N} = Q \] where Q is positive definite
\[ \text{plim} \frac{X'\varepsilon}{N} = 0 \]
\[ \text{plim} \frac{X'\Omega X}{N} = L \] where L is positive definite

then,
\[ \lim_{N \to \infty} E(s^2) = \sigma^2 \]

(c) To consider the issue of consistency, prove that
\[ \text{plims}^2 = \text{plim} \left( \frac{1}{N-k} \sum_{i=1}^{N} \varepsilon_i^2 \right) \]

Under what conditions is plims$^2 = \sigma^2$?

4. What are the cases in which the seemingly unrelated regression estimator (SURE) is equivalent to the OLS estimator? Prove your claim.

5. a) Expand the rational lag model:
\[ y_t = \frac{0.6+2L}{1-0.6L+0.3L^2} x_t + \varepsilon_t \]

What are the coefficients on $x_{t-1}, x_{t-2}, x_{t-3}$ and $x_{t-4}$?

b) Suppose that the model in part a) were specified as
\[ y_t = \alpha + \frac{\beta+\gamma L}{1-\delta L-\eta L^2} x_t + \varepsilon_t \]

How can the parameters be estimated? Is OLS consistent?