Solutions for PS # 2

1. (a) $X^2 \sim \chi^2(1)$
   (b) $N(0,1)$
   (c) $\chi^2(1)$ (by the continuous mapping theorem)

2.

$$Z'Z = \begin{pmatrix} 760 & 30 & 1300 \\ 30 & 31 & 0 \\ 1300 & 0 & 2480 \end{pmatrix}$$

From the above matrix, we can know that

$n = 31, \sum x_i = 0, \sum y_i = 30, \sum x_i^2 = 2480, \sum y_i^2 = 760, \sum x_i y_i = 1300$

Therefore, we have sample means for $x$ and $y$ as follows.

$x = 0, y = 30/31$

i) sample size: 31

ii) the mean of $X$: $0 \div 31 = 0$

iii) We know that in the two variable case $R^2$ is the square of the correlation between $X$ and $Y$. Therefore, we’re going to use the following formula

$$r = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum (x_i - \bar{x})^2 \sum (y_i - \bar{y})^2}}$$

$$= \frac{\sum x_i y_i - n \bar{x} \bar{y}}{\sqrt{\sum x_i^2 - n \bar{x}^2} \sqrt{\sum y_i^2 - n \bar{y}^2}}$$

$$= \frac{1300 - 31 \times 0 \times 30/31}{\sqrt{2480 - 31 \times 0^2} \sqrt{760 - 31 \times (30/31)^2}}$$

$$= 0.9655$$

$$R^2 = r^2 = 0.9323$$

iv) OLS estimator
\[
\hat{\beta} = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2} = \frac{\sum x_i y_i - n \bar{x} \bar{y}}{\sum x_i^2 - n \bar{x}^2
\]
\[
= \frac{1300 - 31 \times 0 	imes 30/31}{2480 - 31 \times 0^2}
\]
\[
= 0.5242
\]

v) F-statistics for \(\alpha = 0, \beta = 0\)

For this, we have to calculate \(S^2\)
\(\left(= \frac{e' e}{n-k} = \frac{e' e}{29}\right)\)

\[e' e = \sum(y_i - \bar{y})^2 - \beta^2 \sum(x_i - \bar{x})^2 = \sum y_i^2 - n \bar{y}^2 - 0.5242^2(\sum x_i^2 - n \bar{x}^2)
\]
\[
= 760 - 31 \times (30/31)^2 - 0.5242^2(2480 - 31 \times 0^2)
\]
\[
= 49.50
\]
\[S^2 = 49.50/29 = 1.71
\]

For F-statistics, we want to use the following formula (Refer to the lecture note 5. However, * implies restricted model here). Note that \(e'^* e^* = \sum y_i^2\). (Since there is no explanatory variable left under the null hypothesis)

\[F = \frac{(e'^* e^* - e' e) / (trM^* - trM)}{e' e / trM}
\]
\[
= \frac{(760 - 49.50)/2}{1.71}
\]
\[
= 207.7485
\]

vi) \(Ex = \alpha^* + \beta^* y\)

\(R^2\) is same as before.

\[
\hat{\beta}^* = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(y_i - \bar{y})^2} = \frac{\sum x_i y_i - n \bar{x} \bar{y}}{\sum y_i^2 - n \bar{y}^2}
\]
\[
= \frac{1300 - 31 \times 0 	imes 30/31}{760 - 31 \times (30/31)^2}
\]
\[
= 1.78
\]
\[ \bar{e} \bar{e} = \sum (x_i - \bar{x})^2 - \hat{\beta}^* \sum (y_i - \bar{y})^2 = \sum x_i^2 - nx^2 - 1.78^2(\sum y_i^2 - n\bar{y}^2) \]
\[ = 2480 - 31 \times 0^2 - 1.78^2(760 - 31 \times (30/31)^2) \]
\[ = 164.00 \]
\[ \bar{S}^2 = 164.00/29 = 5.66 \]
\[ s.e.(\hat{\beta}^*) = \sqrt{5.66 \times \left( \frac{1}{ \sum (y_i^2 - n\bar{y}^2) } \right)^{-1}} = \sqrt{5.66 \times \left( \frac{1}{ \sqrt{760 - 31 \times (30/31)^2} } \right)^{-1}} \]
\[ = 0.0880 \]
\[ t = \frac{\hat{\beta}^* / s.e.(\hat{\beta}^*)}{1.78/0.0880} = 20.23 \]

(a) \( y = X_1 \beta_1 + X_2 \beta_2 + \varepsilon \Rightarrow \hat{\beta}_2 = (X'_2 M_1 X_2)^{-1} X'_2 M_1 y \)
(b) \( P_1 y = X_2 \beta_2 + \varepsilon \Rightarrow \hat{\beta}_2 = (X'_2 X_2)^{-1} X'_2 P_1 y \)
(c) \( P_1 y = P_1 X_2 \beta_2 + \varepsilon \Rightarrow \hat{\beta}_2 = (X'_2 P_1 X_2)^{-1} X'_2 P_1 y \)
(d) \( M_1 y = X_2 \beta_2 + \varepsilon \Rightarrow \hat{\beta}_2 = (X'_2 X_2)^{-1} X'_2 M_1 y \)
(e) \( y = M_1 X_2 \beta_2 + \varepsilon \Rightarrow \hat{\beta}_2 = (X'_2 M_1 X_2)^{-1} X'_2 M_1 y \)
(f) \( M_1 y = M_1 X_2 \beta_2 + \varepsilon \Rightarrow \hat{\beta}_2 = (X'_2 M_1 X_2)^{-1} X'_2 M_1 y \)
(g) \( M_1 y = X_1 \beta_1 + M_1 X_2 \beta_2 + \varepsilon \Rightarrow \hat{\beta}_2 = (X'_2 M_1 X_2)^{-1} X'_2 M_1 y \)
(h) \( M_1 y = M_1 X_1 \beta_1 + M_1 X_2 \beta_2 + \varepsilon \Rightarrow \hat{\beta}_2 = (X'_2 M_1 X_2)^{-1} X'_2 M_1 y \)

Therefore, e,f,g,h give the same results.

3. Unbiasedness implies that \( c_1 + c_2 = 1 \), since \( E(b) = (c_1 + c_2)\beta \).

Therefore, the problem consists of minimizing \( V ar(b) = c_1^2 v_1 + c_2^2 v_2 \) subject to \( c_1 + c_2 = 1 \).

Second order conditions will be met, since the objective function is quadratic and the restriction is linear and from the first order conditions we can conclude that \( c_1 = \frac{v_2}{v_1 + v_2} \) and \( c_2 = \frac{v_1}{v_1 + v_2} \).

(a) \( E[b \mid X] = \beta + (X'X)^{-1} X' E[\varepsilon \mid X] = \beta \) since \( E[\varepsilon \mid X] = 0 \).

Therefore, \( E[b] = E(E[b \mid X] \mid X) = E(\beta \mid X) = \beta \).

Hence, if the regressors are stochastic, as long as they are uncorrelated with the error term (and the all other assumptions hold), the OLS estimator of \( \beta \) is still unbiased.

(b) \( V ar(b) = E_X [V ar(b \mid X)] + V ar_X E[b \mid X] \)

Note that \( V ar(b \mid X) = \sigma^2 (X'X)^{-1} \) (random variable, since \( X \) is random)

and \( E[b \mid X] = \beta \) (constant)

Therefore, \( V ar(b) = E_X [\sigma^2 (X'X)^{-1}] + V ar_X [\beta] = \sigma^2 E[(X'X)^{-1}] + 0 = \sigma^2 E[(X'X)^{-1}] \)
\[ X_n = 3 - \frac{1}{n^2} \]
\[ Y_n = \sqrt{n} \frac{Z_n}{\sigma} \]

where \( \overline{Z_n} = \frac{1}{n} \sum_{i=1}^{n} Z_i \) and \( Z_i \)'s are i.i.d. with mean zero and variance \( \sigma^2 \).

Note that \( X_n - 3 = o_p(1) \) (\( p \lim X_n = 3 \)) and \( Y_n \xrightarrow{d} N(0,1) \).

(a) \( X_n + Y_n \xrightarrow{d} N(3,1) \)
(b) \( X_n Y_n \xrightarrow{d} N(0,3^2) \)
(c) \( Y_n^2 \xrightarrow{d} \chi^2(1) \)

(a) Prove that \( \hat{\beta} \xrightarrow{p} \beta \). (Using Law of Large Numbers)

\[
\hat{\beta} = \beta + \left( \frac{X'X}{N} \right)^{-1} \left( \frac{X'\varepsilon}{N} \right)
\]

\[
p \lim \hat{\beta} = \beta + p \lim \left( \frac{X'X}{N} \right)^{-1} p \lim \left( \frac{X'\varepsilon}{N} \right)
\]

\[
= \beta + Q^{-1} \cdot 0
\]

\[
= \beta
\]

(b) Find the asymptotic distribution of \( \sqrt{N}(\hat{\beta} - \beta) \).

\[
\sqrt{N}(\hat{\beta} - \beta) = \left( \frac{X'X}{N} \right)^{-1} \left( \frac{X'\varepsilon}{\sqrt{N}} \right)
\]

Note that \( p \lim \left( \frac{X'X}{N} \right)^{-1} = Q^{-1} \) by Law of Large Numbers and Continuous Mapping Theorem and that \( \left( \frac{X'\varepsilon}{\sqrt{N}} \right) \xrightarrow{d} N(0, \sigma^2 Q) \) by Central Limit Theorem.

Then, we have the following limiting distribution.

\[
\sqrt{N}(\hat{\beta} - \beta) = \left( \frac{X'X}{N} \right)^{-1} \left( \frac{X'\varepsilon}{\sqrt{N}} \right)
\]

\[
\xrightarrow{d} N(0, \sigma^2 Q^{-1} Q Q^{-1})
\]

\[
= N(0, \sigma^2 Q^{-1})
\]
(c) Prove that \( \rho \lim S^2 = \sigma^2 \) where \( S^2 = \frac{\epsilon' \epsilon}{N-K} \)

\[
S^2 = \frac{\epsilon' \epsilon}{N-K} = \frac{\epsilon'M\epsilon}{N-K}
\]

\[
= \frac{\epsilon'(I - X(X'X)^{-1}X')\epsilon}{N-K}
\]

\[
= \frac{N}{N-K} \left[ \frac{\epsilon' \epsilon}{N} - \frac{\epsilon'X(X'X)^{-1}X'\epsilon}{N} \right]
\]

Now, apply the Law of Large Numbers.

\[
\rho \lim S^2 = \rho \lim \frac{N}{N-K} \left[ \rho \lim \frac{\epsilon' \epsilon}{N} - \rho \lim \frac{\epsilon'X}{N} \cdot \rho \lim \left( \frac{X'X}{N} \right)^{-1} \cdot \rho \lim \frac{X'\epsilon}{N} \right]
\]

\[
= \frac{1}{N-K} \cdot \left[ \rho \lim \frac{\epsilon' \epsilon}{N} - \rho \lim \frac{\epsilon'X}{N} \cdot \rho \lim \left( \frac{X'X}{N} \right)^{-1} \cdot \rho \lim \frac{X'\epsilon}{N} \right]
\]

\[
= \sigma^2
\]