The Two-Sector Model

1. Find cost functions for the following production functions:
   (a) Cobb-Douglas.
   (b) CES, \( f(k, l) = (ak^r + bl^r)^{1/r} \).
   (c) linear, \( f(k, l) = ak + bl \).
   (d) Leontief, \( f(k, l) = \min\{ak, bl\} \).
   (e) von Thünen’s production function, \( f(k, l) = (1 - e^{-ak})(1 - e^{-bl}) \).

2. Suppose in a two-country world, countries A and B, that \( f_A \) and \( f_B \) are the usual neoclassical production functions of capital and labor, with Inada conditions.
   (a) Show that all revenues (national product) are distributed to the factors.
   (b) Suppose that \( p_A \) increases. Under what conditions on \( f_A \) and \( f_B \) will the capital share of national product increase?

3. For a country with an endowment in the interior of the cone of diversification, derive and prove a result on the effects of a small increase in the quantity of a factor on output.

4. Here is another two-sector model. Sector 1 produces investment goods (capital goods). Sector 2 produces consumption goods. Each sector is characterized by a neoclassical production function (strictly concave, \( C^2 \), Inada conditions at 0) with constant returns to scale. Write \( Y_i = F_i(K_i, L_i) \) for output in sector \( i \) as a function \( F_i \) of capital \( K_i \) and labor \( L_i \) employed in sector \( i \).
   (a) Rewrite these relationships in terms of the output/labor and capital/labor ratio: \( y_i = f_i(k_i) \) where \( y_i = Y_i/L_i \), etc. What properties do the foregoing assumptions imply for the \( f_i \)?
   (b) Let \( w \) and \( r \) denote the equilibrium prices of capital, and \( P_i \) the price of output \( i \). Equilibrium in this model requires for \( i = 1, 2 \)
   \[
   Y_i = F_i(K_i, L_i) \\
   P_i \frac{\partial F_i}{\partial K_i} = r \\
   P_i \frac{\partial F_i}{\partial L_i} = w \\
   K_1 + K_2 = K \\
   L_1 + L_2 = L \\
   P_1 Y_1 = rK \\
   P_2 Y_2 = wL
   \]
   Interpret these equations, including the demand conditions (4).
   (c) Rewrite these conditions in terms of the aggregate capital/Labor ratio \( k = K/L \), the output/labor ratios \( y_i \), the labor shares \( l_i = L_i/L \) and the wage/rent ratio \( \omega = w/r \).
   (d) Compute \( dk_i/d\omega \) and use this to prove that the capital/labor ratio in each sector is uniquely determined by the wage/rent ratio.
(e) Find an equation that implicitly defines the equilibrium wage-rent ration \( \omega \) in terms of the capital-labor ratio.