1. Are the following matrices productive? Prove your answers.

\[ A_1 = \begin{bmatrix} 0.6 & 0.2 & 0.1 \\ 0.3 & 0.2 & 0.4 \\ 0.2 & 0.4 & 0.3 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0.6 & 0.5 \\ 0.1 & 0.5 \end{bmatrix} \]

2. Prove that \( A \) is productive if and only if \((I - A)^{-1}\) exists and is non-negative.

3. Prove that if \( A \) is productive, then there is at least one row sum of \( A \) which is less than 1.

4. Let

\[ A = \begin{bmatrix} 0.1 & 0.4 & 0.3 \\ 0.2 & 0.7 & 0.0 \\ 0.1 & 0.1 & 0.5 \end{bmatrix} \]

Suppose too that the vector of labor requirements is \((1, 1, 1)\). Compute competitive equilibrium prices for this economy.

5. Associated with any \( n \times n \) non-negative matrix \( A \) is a graph \( G \) where there is a directed edge from \( i \) to \( j \) iff \( a_{ij} > 0 \). A matrix is said to be irreducible iff the graph is strongly connected; that is iff there is a path from any \( i \) to any \( j \).

   (a) Show that a square matrix \( A \) is irreducible iff for each pair \( i \) and \( j \) there is an \( m \) such that the \( i, j \)th element of \( A^m \) is positive. [Hint: What does it mean for an element of \( A^2 \) to be positive, in terms of the graph?]

   (b) Use the preceding fact to show that the assumption \( a_0 \gg 0 \) can be replaced with the assumptions \( a_0 > 0 \) and \( A \) is irreducible in the characterization of equilibrium in the simple Leontief model.

6. In this question we develop the Ricardian trade model. There are two countries, England and Spain. (Portugal is traditional here, but then we have too many \( p \)'s floating around.) Each country can make both wine (\( v \)) and mutton (\( m \)). England requires \( a_{ve} \) units of English labor to produce a unit of wine, and \( a_{me} \) units of English labor to produce a unit of mutton. Spain requires \( a_{ve} \) units of Spanish labor to produce a unit of wine, and \( a_{me} \) units of Spanish labor to produce a unit of mutton. Neither good is used as an intermediate product in the production of any good; itself or the other. England and Spain have \( l_\epsilon \) and \( l_s \) units of labor, respectively, and because of BREXIT, no laborer may cross the Channel to work in the other country.

   (a) Describe the production of wine and mutton as a general Leontief model. How many primary factors are there? What are \( A \) and \( B \)?

   (b) Describe the production possibility set as a convex polyhedron, that is, as a solution set to a system of linear inequalities. Show that it is closed and convex.
(c) The production possibility set is convex. The convex support function is one way of characterizing a closed convex set $C$. It is defined as

$$h(p) = \max\{p \cdot x : x \in C\}.$$ 

The convex support function is one way of giving the dual description of $C$, the set of all half spaces containing it. (The concave support function does this with min’s.) Write down a linear programming problem that gives the convex support function. Hint: It begins, $h(\alpha, \beta) = \max \alpha v + \beta m$ over some variables including, obviously, $v$ and $m$, subject to some constraints.

(d) What is the dual of your program?

(e) Using complementary slackness, interpret solutions to the dual as competitive prices. There is no loss of generality in assuming $\alpha, \beta > 0$.

(f) Under what conditions can all activities be used?

(g) Suppose that there is an optimal solution in which $x_{me} > 0$ and $x_{vs} > 0$. What can you infer about the ratios $a_{me}/a_{ve}$ and $a_{ms}/a_{vs}$?

7. There are two countries, and a continuum of goods indexed by $z \in [0, 1]$. There is one input, labor. Production of each good is CRS, and $a(z)$ and $b(z)$ are the unit labor requirements in countries $A$ and $B$, respectively. Without loss of generality, goods are ordered so that $\alpha(z) = a(z)/b(z)$ is decreasing. Assume it is strictly decreasing. Let $p(z)$ denote the free-trade world price of good $z$. Let $w_a$ and $w_b$ denote the wage rate in countries $A$ and $B$, respectively, and let $L_A$ and $L_B$ denote their respective labor endowments.

(a) Describe the implications of profit maximization that must hold for all goods $z$ in countries $A$ and $B$.

(b) What additional relations does profit maximization imply for goods produced in $A$ and in $B$?

(c) Proposition: In any equilibrium, there exists $z^* \in [0, 1]$ such that $A$ produces all goods $z < z^*$ and $B$ produces all goods $z > z^*$. Prove it.

(d) Suppose that $\alpha$ is continuous. Show that profit maximization requires $\alpha(z^*) = w_B/w_A$.

Suppose that consumers have identical Cobb-Douglas preferences, and let $s(z)$ denote the share of expenditures on good $z$. So $s(z) = p(z)c_i(z)/w_iL_i$ for $i = A, B$.

(e) Why must $\int_0^1 s(z)dz = 1$?

Let $\theta(z^*) = \int_{z^*}^1 s(z)dz$ denote the fraction of income spent by both countries on good produced in country $A$.

(f) What are the equilibrium budget constraints for both countries on expenditures at home and abroad?

(g) Use these relations to derive a relationship between $z^*$ and the equilibrium ratio $w_A/w_B$. Denote this relationship as $w_B/w_A = \beta(z^*)$. 

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(h) What is an increase of $L_A$ on the equilibrium $z^*$?