Chapter 6: Random Signals, Correlations, and Noise

In this lecture you will learn:

• Random Signals
• Shot Noise
• Correlated Noise: Bunching and Antibunching
• Partition Noise
• Langevin Equation
• Noise Spectral Densities and Weiner-Kinchine Theorem
• Brownian and Diffusion Noise
Fourier Transforms of Signals

\[ x(\omega) = \int_{-\infty}^{\infty} e^{+i\omega t} x(t) \, dt \]

\[ x(t) = \int_{-\infty}^{+\infty} e^{-i\omega t} x(\omega) \frac{d\omega}{2\pi} \]
Random Signals

A random signal \( x(t) \) can be any signal from a set of signals \( \{x_1(t), x_2(t), x_3(t), \ldots \} \)

The set \( \{x_1(t), x_2(t), x_3(t), \ldots \} \) is the sample space

The probability that \( x(t) \) will equal \( x_n(t) \) is: \( P_{x(t)}[x_n(t)] \)

Mean:

\[
m_x(t) = \langle x(t) \rangle = \sum_n x_n(t) P_{x(t)}[x_n(t)]
\]

Auto-correlation:

\[
R_{xx}(t_1, t_2) = \langle x(t_1) x(t_2) \rangle = \sum_n x_n(t_1) x_n(t_2) P_{x(t)}[x_n(t)]
\]

Cross-correlation:

\[
R_{xy}(t_1, t_2) = \langle x(t_1) y(t_2) \rangle = \sum_{n,m} x_n(t_1) y_m(t_2) P_{x(t)y(t)}[x_n(t_1), y_n(t_2)]
\]
Stationary Signals and Ergodic Signals

**Stationary Signals:** Are those whose characteristics do not depend upon the time origin

Stationary signals have the following properties:

a) The mean values are independent of time, i.e.:

\[ m_x(t) = \langle x(t) \rangle = m_x \]

b) The auto- and cross-correlation functions are functions of the time difference only, i.e.,

\[ R_{xx}(t_1, t_2) = \langle x(t_1)x(t_2) \rangle = R_{xx}(t_1 - t_2) \quad R_{xy}(t_1, t_2) = \langle x(t_1)y(t_2) \rangle = R_{xy}(t_1 - t_2) \]

**Ensemble Averages and Time Averages:**

Averages of signals can be done in two ways,

a) Ensemble Averages:

\[ \langle x(t) \rangle = \sum_n x_n(t) \ P_x(t)[x_n(t)] \]

b) Time Averages:

\[ \overline{x(t)} = \lim_{T \to \infty} \frac{1}{T/2} \int_{-T/2}^{T/2} x(t) \ dt \]
Stationary Signals and Ergodic Signals

**Ergodic Signals:** When all ensemble averages equal the corresponding time averages the signal is called ergodic. Ergodicity implies stationarity but it is not the other way around.

\[
\langle x(t) \rangle = \bar{x}(t) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) dt = m_x
\]

\[
R_{xy}(t_1 - t_2) = \langle x(t_1)y(t_2) \rangle = \bar{x}(t_1)y(t_2) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t + t_1 - t_2)y(t) dt
\]

Ergodicity implies that each signal in the sample set is representative of the whole set.
Power Spectral Densities of Signals

Signal Energy:

The total “energy” of a signal $x(t)$ can be infinite:

$$\int_{-\infty}^{\infty} x^2(t) dt = \int_{-\infty}^{\infty} |x(\omega)|^2 \frac{d\omega}{2\pi}$$

So it is better to work with signal power and not with signal energies

Signal Power:

Given $x(t)$, define a truncated signal as:

$$x_T(t) = \begin{cases} x(t) & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0 & \text{otherwise} \end{cases}$$

The power in the signal $x(t)$ is:

$$= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t) dt = \frac{1}{T} \int_{-\infty}^{\infty} x_T^2(t) dt$$

$$= \frac{1}{T} \int_{-\infty}^{\infty} |x_T(\omega)|^2 \frac{d\omega}{2\pi}$$
Power Spectral Densities of Signals

The ensemble averaged signal power is:

\[ \frac{1}{T} \left\langle \int_{-\infty}^{\infty} \left| x_T(\omega) \right|^2 \frac{d\omega}{2\pi} \right\rangle = \int_{-\infty}^{\infty} \frac{1}{T} \left\langle \left| x_T(\omega) \right|^2 \right\rangle \frac{d\omega}{2\pi}. \]

The **power spectral density** of the signal \( x(t) \) is defined as:

\[ S_{xx}(\omega) = \lim_{T \to \infty} \frac{1}{T} \left\langle \left| x_T(\omega) \right|^2 \right\rangle \]

**Weiner-Kinchine Theorem:**

The power spectral density of a stationary signal is the Fourier transform of the auto-correlation function:

\[ S_{xx}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} R_{xx}(t) dt \]

**A Useful Relation:** Consider a stationary random signal \( x(t) \):

\[ x(\omega) = \int_{-\infty}^{\infty} dt \, e^{i\omega t} \, x(t) \]

\[ \implies \left\langle x^*(\omega') x(\omega) \right\rangle = 2\pi \delta(\omega - \omega') S_{xx}(\omega) \]

\[ \implies S_{xx}(\omega) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \left\langle x^*(\omega') x(\omega) \right\rangle \]
Shot Noise

Suppose we are looking at a process that consists of a set of discrete events happening randomly in time.

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[Particle stream]  
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Suppose the \( j \)-th event happens at time \( t_j \).

The rate \( r(t) \) of the events (i.e. the number of events happening per unit time) is:

\[
r(t) = \sum_j \delta(t - t_j)
\]

The times \( t_j \) constitute the random part.

We will assume the following:

i) The times \( t_j \) are completely independent of each other.

ii) The probability that there is an event in a very short time interval \( dt \) is given by \( \alpha dt \) where \( \alpha \) is called the average rate of the process.
Shot Noise

Ensemble Average Rate:

$$\langle r(t) \rangle = \left\langle \sum_j \delta(t - t_j) \right\rangle = \int_{-\infty}^{\infty} \alpha \, dt' \, \delta(t - t') = \alpha$$

Auto-Correlation:

$$R_{rr}(\tau) = \left\langle r(t + \tau) r(t) \right\rangle$$

$$= \left\langle \sum_j \delta(t + \tau - t_j) \delta(t - t_m) \right\rangle$$

$$= \int_{-\infty}^{\infty} \alpha \, dt' \, \delta(t + \tau - t') \, \delta(t - t') + \left\langle \sum_j \sum_m \delta(t + \tau - t_j) \delta(t - t_m) \right\rangle_{j \neq m}$$

$$\alpha \, \delta(\tau) + \left\langle \sum_j \sum_m \delta(t + \tau - t_j) \delta(t - t_m) \right\rangle_{j \neq m} = \alpha^2$$
Shot Noise

Auto-Correlation:

\[ R_{rr}(\tau) = \alpha \, \delta(\tau) + \alpha^2 \]

Spectral Density:

\[ S_{rr}(\omega) = \alpha + \alpha^2 2\pi \delta(\omega) \]

Noise: The noise in the signal \( r(t) \) is:

\[ n(t) = r(t) - \langle r(t) \rangle = r(t) - \alpha \]

Noise Auto-Correlation and Spectral Density:

\[ R_{nn}(\tau) = R_{rr}(\tau) - \alpha^2 = \alpha \, \delta(\tau) \]

\[ \Rightarrow S_{nn}(\omega) = S_{rr}(\omega) - \alpha^2 2\pi \delta(\omega) \]

\[ S_{nn}(\omega) = \alpha \]

i) The noise spectral density is white (independent of frequency) or flat

ii) The noise spectral density is equal to the average rate

The noise is not present because the events are discreet
The noise is present because the events happen randomly in time
Poisson Statistics and Shot Noise

Particle stream with shot noise

Let $P(n,T)$ be the probability of having $n$ events in time $t$:

$$P(n,T + \Delta T) = P(n-1,T) \alpha \Delta T + P(n,T)(1 - \alpha \Delta T)$$

$$\Rightarrow \frac{P(n,T + \Delta T) - P(n,T)}{\Delta T} = P(n-1,T)\alpha - P(n,T)\alpha$$

$$\Rightarrow \frac{d}{dT} P(n,T) + \alpha P(n,T) = \alpha P(n-1,T)$$

Solution given the boundary condition $P(n,T = 0) = \delta_{n0}$ is:

$$P(n,T) = \frac{(\alpha T)^n}{n!} e^{-\alpha T}$$

The average events in time $T$ is:

$$\langle N_T \rangle = \sum_{n=0}^{\infty} n P(n,T) = \sum_{n=1}^{\infty} n \frac{(\alpha T)^n}{n!} e^{-\alpha T}$$

$$= \sum_{n=1}^{\infty} \frac{(\alpha T)^{n-1}}{(n-1)!} e^{-\alpha T} (\alpha T) = \alpha T$$

As expected
Poisson Statistics and Shot Noise

The standard deviation is:

\[ \langle \Delta N_T^2 \rangle = \langle (N_T - \langle N_T \rangle)^2 \rangle = \langle N_T^2 \rangle - \langle N_T \rangle^2 \]

\[ \langle N_T^2 \rangle = \sum_{n=0}^{\infty} n^2 \frac{(\alpha T)^n}{n!} e^{-\alpha T} = \sum_{n=1}^{\infty} n \frac{(\alpha T)^{n-1}}{(n-1)!} e^{-\alpha T} (\alpha T) \]

\[ = \sum_{n=1}^{\infty} (n - 1 + 1) \frac{(\alpha T)^{n-1}}{(n-1)!} e^{-\alpha T} (\alpha T) = (\alpha T)^2 + (\alpha T) \]

\[ \Rightarrow \langle \Delta N_T^2 \rangle = \langle N_T^2 \rangle - \langle N_T \rangle^2 = (\alpha T) = \langle N_T \rangle \]

Standard deviation is equal to the mean
Bunching and Antibunching

Particle stream

Detector

Shot Noise: For $t'' \neq t'$:

$$P(t'', t') = P(t'' | t')P(t') = (\alpha dt'')(\alpha dt')$$

Bunching Example: For $t'' \neq t'$:

$$P(t'', t') = P(t'' | t')P(t') = (\alpha dt'')\left(1 + e^{-\gamma|t'' - t'|}\right)(\alpha dt')$$

$$\Rightarrow S_{nn}(\omega) = \alpha\left(1 + \frac{2\alpha\gamma}{\omega^2 + \gamma^2}\right)$$

Antibunching Example: For $t'' \neq t'$ and $\alpha \leq \frac{\gamma}{2}$:

$$P(t'', t') = P(t'' | t')P(t') = (\alpha dt'')\left(1 - e^{-\gamma|t'' - t'|}\right)(\alpha dt')$$

$$\Rightarrow S_{nn}(\omega) = \alpha\left(1 - \frac{2\alpha\gamma}{\omega^2 + \gamma^2}\right)$$
Bunching and Antibunching

Shot noise signal
(All frequencies are present in the noise)

Bunched noise signal
(Lower frequencies are dominant in the noise)

Antibunched noise signal
(Higher frequencies are dominant in the noise)
Average Rates:

\[
\langle i(t) \rangle = \alpha \\
\langle f(t) \rangle = \eta \langle i(t) \rangle = \eta \alpha \\
\langle r(t) \rangle = (1 - \eta) \langle i(t) \rangle = (1 - \eta) \alpha.
\]

Noise in the Streams:

\[
\begin{align*}
n^i(t) &= i(t) - \langle i(t) \rangle = i(t) - \alpha \\
n^f(t) &= f(t) - \eta \alpha \\
n^r(t) &= r(t) - (1 - \eta) \alpha
\end{align*}
\]
i(t) = \sum_j \delta(t - t_j)

f(t) = \sum_j f_j \delta(t - t_j)

r(t) = \sum_j r_j \delta(t - t_j)

\Rightarrow \langle f(t) \rangle = \langle \sum_j f_j \delta(t - t_j) \rangle = \eta \langle \sum_j \delta(t - t_j) \rangle = \eta \langle i(t) \rangle = \eta \alpha

\Rightarrow \langle r(t) \rangle = \langle \sum_j r_j \delta(t - t_j) \rangle = (1 - \eta) \langle \sum_j \delta(t - t_j) \rangle = (1 - \eta) \langle i(t) \rangle = (1 - \eta) \alpha

\Rightarrow f_j \Rightarrow \{0, 1\}

P_{f_j}(1) = \eta

P_{f_j}(0) = 1 - \eta

r_j \Rightarrow \{0, 1\}

P_{r_j}(1) = 1 - \eta

P_{r_j}(0) = \eta
Auto-Correlation:

\[ R_{ff}(\tau) = \left\langle f(t + \tau)f(t) \right\rangle \]

\[ = \left\langle \sum_{j} \delta(t + \tau - t_j)\delta(t - t_m)f_j f_m \right\rangle \]

\[ R_{ff}(\tau) = \left\langle \sum_{j} f_j^2 \delta(t + \tau - t_j)\delta(t - t_j) \right\rangle + \left\langle \sum_{j,m} f_j f_m \delta(t + \tau - t_j)\delta(t - t_m) \right\rangle_{j \neq m} \]

Note that:

\[ \left\langle f_j \right\rangle = \eta \]
\[ \left\langle f_j^2 \right\rangle = \eta \]
\[ \left\langle f_j f_m \right\rangle_{j \neq m} = \left\langle f_j \right\rangle \left\langle f_m \right\rangle = \eta^2 \]
Auto-Correlation:

\[ R_{ff}(\tau) = \eta \left( \sum_j \delta(t+\tau-t_j)\delta(t-t_j) \right) + \eta^2 \left( \sum_j \sum_m \delta(t+\tau-t_j)\delta(t-t_m) \right) \]

\[ = \eta(1-\eta) \left( \sum_j \delta(t+\tau-t_j)\delta(t-t_j) \right) \]

\[ + \eta^2 \left( \sum_j \delta(t+\tau-t_j)\delta(t-t_j) \right) + \left( \sum_j \sum_m \delta(t+\tau-t_j)\delta(t-t_m) \right) \]

\[ \Rightarrow R_{ff}(\tau) = \eta(1-\eta) \left( \sum_j \delta(t+\tau-t_j)\delta(t-t_j) \right) + \eta^2 R_{ii}(\tau) \]
Partition Noise

Particle stream

Detector

\[ i(t) \quad \eta \quad f(t) \]

\[ r(t) \quad \text{splitter} \]

**Auto-Correlation:**

\[ R_{ff}(\tau) = \alpha \eta (1-\eta) \delta(\tau) + \eta^2 R_{ii}(\tau) \]

\[ \Rightarrow S_{ff}(\omega) = \alpha \eta (1-\eta) + \eta^2 S_{ii}(\omega) \]

**Noise:**

\[ n_f^f(t) = f(t) - \langle f(t) \rangle = f(t) - \eta \alpha \]

**Noise Auto-Correlation:**

\[ R_{n_f n_f}(\tau) = R_{ff}(\tau) - \eta^2 \alpha^2 = \alpha \eta (1-\eta) \delta(\tau) + \eta^2 \left[ R_{ii}(\tau) - \alpha^2 \right] \]

\[ = \alpha \eta (1-\eta) \delta(\tau) + \eta^2 R_{n_i n_i}(\tau) \]

**Noise Spectral Density:**

\[ S_{n_f n_f}(\omega) = \alpha \eta (1-\eta) + \eta^2 S_{n_i n_i}(\omega) \]
Noise Spectral Density:

\[ S_{n_f n_f} (\omega) = \alpha \eta (1 - \eta) + \eta^2 S_{n_i n_i} (\omega) \]

Case i: \( \eta << 1 \)

\[ S_{n_f n_f} (\omega) \approx \alpha \eta \]

Output stream has shot noise irrespective of the noise in the input stream

Case ii: \( S_{n_i n_i} (\omega) = \alpha \) (input stream has shot noise)

\[ S_{n_f n_f} (\omega) = \alpha \eta \]

Output stream has shot noise as well

Case iii: \( \eta = 1 \)

\[ S_{n_f n_f} (\omega) = S_{n_i n_i} (\omega) \]

As expected
Langevin Equations: Introduction

Velocity of a Particle in Brownian Motion

Assume 1D problem
Suppose we take velocity to be a random signal and write:

\[
\frac{dv(t)}{dt} = -\gamma v(t)
\]

\[
\Rightarrow v(t) = v(t = 0) e^{-\gamma t}
\]

\[
\Rightarrow v^2(t) = v^2(t = 0) e^{-2\gamma t}
\]

But in steady state (in 1D) statistical physics tells us:

\[
\left\langle \frac{1}{2} m v^2(t) \right\rangle = \frac{1}{2} k_B T \Rightarrow \left\langle v^2(t) \right\rangle = \frac{k_B T}{m}
\]
Langevin Equations: Introduction

We need a better model for the kicks the particle velocity receives in collisions:

\[ m \frac{dv(t)}{dt} = -\gamma m v(t) + F(t) \]

We assume:

\[ \langle F(t) \rangle = 0 \]
\[ \langle F(t_1)F(t_2) \rangle = A \delta(t_1 - t_2) \quad \rightarrow \quad \text{We don’t know } A \text{ (yet)} \]

Solution:

\[ v(t) = v(t = 0)e^{-\gamma t} + \frac{1}{m} \int_0^t F(t_1) e^{-\gamma(t-t_1)}dt_1 \]

\[ \Rightarrow v^2(t) = v^2(t = 0)e^{-2\gamma t} + \frac{1}{m^2} \int_0^t dt_1 \int_0^t dt_2 F(t_1)F(t_2) e^{-\gamma(2t-t_1-t_2)} \]

\[ + 2v(t = 0) \frac{e^{-\gamma t}}{m} \int_0^t dt_1 F(t_1) e^{-\gamma(t-t_1)} \]
Langevin Equations: Introduction

\[
\langle v^2(t) \rangle = \langle v^2(t = 0) \rangle e^{-2\gamma t} + 2 \frac{e^{-\gamma t}}{m} \int_0^t \langle v(t = 0)F(t_1) \rangle e^{-\gamma (t-t_1)} dt_1 \\
+ \frac{1}{m^2} \int_0^t dt_1 \int_0^t dt_2 \langle F(t_1)F(t_2) \rangle e^{-\gamma (2t-t_1-t_2)}
\]

In steady state, when \( t \) can be assumed to be large, the only term that survives is the last term:

\[
\lim_{t \to \infty} \langle v^2(t) \rangle = \lim_{t \to \infty} \frac{1}{m^2} \int_0^t dt_1 \int_0^t dt_2 A \delta(t_1 - t_2) e^{-\gamma (2t-t_1-t_2)}
\]

\[
= \lim_{t \to \infty} \frac{A}{m^2} \int_0^t dt_1 e^{-2\gamma (t-t_1)}
\]

\[
= \lim_{t \to \infty} \frac{A}{m^2} \frac{(1 - e^{-2\gamma t})}{2r} = \frac{A}{2\gamma m^2}.
\]

Enforce the condition from statistical physics:

\[
\langle v^2(t) \rangle = \frac{K_B T}{m} \quad \Rightarrow \quad \frac{A}{2\gamma m^2} = \frac{K_B T}{m} \quad \Rightarrow \quad A = 2\gamma m K_B T
\]
Langevin Equations: Introduction

Langevin Equations for the Velocity of a Particle in Brownian Motion

\[
\frac{dv(t)}{dt} = -\gamma v(t) + \frac{F(t)}{m}
\]

\[
\langle F(t) \rangle = 0
\]

\[
\langle F(t_1)F(t_2) \rangle = 2\gamma mK_B T \delta(t_1 - t_2)
\]

Spectral Density:

\[
v(t) = v(t = 0)e^{-\gamma t} + \frac{1}{m} \int_0^t F(t_1) e^{-\gamma(t-t_1)}
\]

\[
\langle v(t')v(t'') \rangle = \langle v^2(0)e^{-\gamma(t'+t'')} \rangle + \frac{1}{m^2} \int_0^{t'} \int_0^{t''} \langle F(t_1)F(t_2) \rangle e^{-\gamma(t'+t'')} e^{\gamma(t_1+t_2)}
\]

\[
+ \left\langle \frac{v(0)}{m} e^{-\gamma t''} \int_0^{t''} F(t_2) e^{-\gamma(t''-t_2)} dt_2 \right\rangle + \left\langle \frac{v(0)}{m} e^{-\gamma t'} \int_0^{t'} \int_0^{t''} F(t_1) e^{-\gamma(t'-t_1)} dt_1 dt_2 \right\rangle
\]

\[
\Rightarrow \langle v(t')v(t'') \rangle = \lim_{t' \to \infty} \lim_{t'' \to \infty} \frac{2\gamma K_B T}{m} \int_0^{t'} \int_0^{t''} \delta(t_1 - t_2) e^{-\gamma(t'+t'')} e^{\gamma(t_1+t_2)}
\]
Langevin Equations: Introduction

\[ R_{vv}(t-t') = \langle v(t')v(t'') \rangle = \frac{K_B T}{m} e^{-\gamma|t'-t''|} \]

The correlation function points to a stationary process in steady state

\[ S_{vv}(\omega) = \frac{K_B T}{m} \int_{-\infty}^{\infty} d\tau \ e^{i\omega \tau} e^{-\gamma|\tau|} \]

\[ S_{vv}(\omega) = \frac{2\gamma K_B T}{m} \frac{1}{(\omega^2 + \gamma^2)} \]
Langevin Equations: Introduction

Spectral Density: Frequency Domain Technique

\[
\frac{dv(t)}{dt} = -\gamma \ v(t) + \frac{F(t)}{m} \quad \begin{cases} 
\langle F(t) \rangle = 0 \\
\langle F(t_1)F(t_2) \rangle = 2\gamma \ m \ K_B T \ \delta(t_1 - t_2)
\end{cases}
\]

Note that:

\[
F(\omega) = \int_{-\infty}^{\infty} dt \ e^{i\omega t} \ F(t)
\]

\[
\Rightarrow \langle F(\omega) \rangle = \int_{-\infty}^{\infty} dt \ e^{i\omega t} \langle F(t) \rangle = 0
\]

\[
\langle F^*(\omega_1)F(\omega_2) \rangle = \int_{-\infty}^{\infty} dt_1 e^{-i\omega_1 t_1} \int_{-\infty}^{\infty} dt_2 e^{-i\omega_2 t_2} \langle F(t_1)F(t_2) \rangle
\]

\[
= 2\gamma \ m \ K_B T \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 e^{-i\omega_1 t_1} e^{+i\omega_2 t_2} \ \delta(t_1 - t_2)
\]

\[
= 2\gamma \ m \ K_B T \int_{-\infty}^{\infty} dt_1 e^{-i(\omega_1 - \omega_2) t_1}
\]

\[
\langle F^*(\omega_1)F(\omega_2) \rangle = 2\gamma \ m \ K_B T \left[2\pi \ \delta(\omega_1 - \omega_2) \right]
\]
Langevin Equations: Introduction

Start from:

\[
\frac{dv(t)}{dt} = -\gamma v(t) + \frac{F(t)}{m}
\]

Fourier transform it:

\[
-i\omega v(\omega) = -\gamma v(\omega) + F(\omega)
\]

\[
v(\omega) = \frac{F(\omega) / m}{-i[\omega + i\gamma]}
\]

Compute the spectral density:

\[
S_{vv}(\omega) = \int_{-\infty}^{\infty} \left\langle v^*(\omega')v(\omega) \right\rangle \frac{d\omega'}{2\pi}
\]

\[
= \frac{1}{m^2} \int_{-\infty}^{\infty} \frac{\left\langle F^*(\omega')F(\omega) \right\rangle}{[\omega' - i\gamma][\omega + i\gamma]} \frac{d\omega'}{2\pi}
\]

\[
= \int_{-\infty}^{\infty} \frac{m}{[\omega' + i\gamma][\omega - i\gamma]} 2\pi \delta(\omega' - \omega) \frac{d\omega'}{2\pi}
\]

\[
S_{vv}(\omega) = \frac{m}{\omega^2 + \gamma^2}
\]
Stationary Processes, Correlation functions, and the Regression Theorem

The famous regression theorem

If \( \frac{d \langle \tilde{x}(t) \rangle}{dt} = \overline{A} \langle \tilde{x}(t) \rangle \) holds for all signals in the sample set

\[ \text{Markov Process (no memory!)} \]

Then:

\[ \frac{d \langle \tilde{x}(t+\tau) \otimes \tilde{x}(t) \rangle}{d\tau} = \overline{A} \langle \tilde{x}(t+\tau) \otimes \tilde{x}(t) \rangle \]

Regression of correlation functions obey the same dynamical equations as the mean values - also known as the Onsager’s Regression Hypothesis

\[ \langle \tilde{x}(t+\tau) \otimes \tilde{x}(t) \rangle = \langle \tilde{x}(t-\tau) \otimes \tilde{x}(t) \rangle = \langle \tilde{x}(t) \otimes \tilde{x}(t+\tau) \rangle \]

Microscopic reversibility

Stationarity
Stationary Processes, Correlation functions, and the Regression Theorem

Consider the Langevin equations of several coupled random variables (in vector notation):

\[
\frac{d\bar{x}(t)}{dt} = \bar{A} \bar{x}(t) + \bar{F}(t)
\]

\[
\Rightarrow \frac{d\langle \bar{x}(t) \rangle}{dt} = \bar{A} \langle \bar{x}(t) \rangle
\]

The solution subject to an initial condition \( \langle \bar{x}(t_o) \rangle = \bar{x}_o \) would be:

\[
\langle \bar{x}(t) \rangle = e^{\bar{A}(t-t_o)} \langle \bar{x}(t_o) \rangle = e^{\bar{A}(t-t_o)} \bar{x}_o
\]

One can write the “conditional” average as:

\[
\langle \bar{x}(t) \rangle = \int d^d\bar{x} \bar{x} P_{\bar{x}}(\bar{x}, t | \bar{x}_o, t_o)
\]

If the random variable is stationary:

\[
\langle \bar{x}(t) \rangle = \int d^d\bar{x} \bar{x} P_{\bar{x}}(\bar{x}, t-t_o | \bar{x}_o, 0)
\]
Stationary Processes, Correlation functions, and the Regression Theorem

Now consider the correlation function matrix:

\[
\overline{R}(t - t') = \langle \tilde{x}(t) \otimes \tilde{x}(t') \rangle = \int d^d \tilde{x}_1 \int d^d \tilde{x}_2 \ \tilde{x}_1 \otimes \tilde{x}_2 \ P_{\tilde{x}}(\tilde{x}_1, t; \tilde{x}_2, t')
\]

\[
= \int d^d \tilde{x}_1 \int d^d \tilde{x}_2 \ \tilde{x}_1 \otimes \tilde{x}_2 \ P_{\tilde{x}}(\tilde{x}_1, t | \tilde{x}_2, t') P_{\tilde{x}}(\tilde{x}_2, t')
\]

\[
= \int d^d \tilde{x}_1 \int d^d \tilde{x}_2 \ \tilde{x}_1 \otimes \tilde{x}_2 \ P_{\tilde{x}}(\tilde{x}_1, t - t' | \tilde{x}_2, 0) P_{\tilde{x}}(\tilde{x}_2, t')
\]

Stationarity implies that the “unconditional” probability distribution \( P_{\tilde{x}}(\tilde{x}_2, t') \) cannot be a function of time, and if there is a steady state then:

\[ P_{\tilde{x}}(\tilde{x}_2, t') = P_{\tilde{x}}(\tilde{x}_2)^{\text{steady state}} \]

This implies:

\[
\overline{R}(t - t') = \langle \tilde{x}(t) \otimes \tilde{x}(t') \rangle = \int d^d \tilde{x}_1 \int d^d \tilde{x}_2 \ \tilde{x}_1 \otimes \tilde{x}_2 \ P_{\tilde{x}}(\tilde{x}_1, t - t' | \tilde{x}_2, 0) P_{\tilde{x}}(\tilde{x}_2)^{\text{steady state}}
\]
Stationary Processes, Correlation functions, and the Regression Theorem

We get:

\[
\bar{R}(t-t') = \langle \tilde{x}(t) \otimes \tilde{x}(t') \rangle = \int d^d \tilde{x}_1 \int d^d \tilde{x}_2 \; \tilde{x}_1 \otimes \tilde{x}_2 \; P_{\tilde{x}}(\tilde{x}_1, t-t' | \tilde{x}_2, 0) \; P_{\tilde{x}}(\tilde{x}_2)_{\text{steady state}}
\]

\[
\Rightarrow \frac{d}{dt} \bar{R}(t-t') = \frac{d}{dt} \langle \tilde{x}(t) \otimes \tilde{x}(t') \rangle = \bar{A} \langle \tilde{x}(t) \otimes \tilde{x}(t') \rangle
\]

Which needs to be solved subject to the initial condition:

\[
\bar{R}(0) = \langle \tilde{x}(t') \otimes \tilde{x}(t') \rangle = \int d^d \tilde{x}_1 \int d^d \tilde{x}_2 \; \tilde{x}_1 \otimes \tilde{x}_2 \; P_{\tilde{x}}(\tilde{x}_1, 0 | \tilde{x}_2, 0) \; P_{\tilde{x}}(\tilde{x}_2)_{\text{steady state}}
\]

\[
= \int d^d \tilde{x}_2 \; \tilde{x}_2 \otimes \tilde{x}_2 \; P_{\tilde{x}}(\tilde{x}_2)_{\text{steady state}} = \langle \tilde{x}(t') \otimes \tilde{x}(t') \rangle_{\text{steady state}} = \langle \tilde{x} \otimes \tilde{x} \rangle_{\text{steady state}}
\]

And the solution is:

\[
R_{xx}(t-t') = \langle \tilde{x}(t) \otimes \tilde{x}(t') \rangle = e^{\bar{A}(t-t')} \langle \tilde{x}(t') \otimes \tilde{x}(t') \rangle_{\text{steady state}}
\]

This is the famous regression theorem!
Stationary Processes, Correlation functions, and the Regression Theorem: An Example

Consider the Brownian motion equation:

\[
\frac{dv(t)}{dt} = -\gamma v(t) + \frac{F(t)}{m}
\]

We know that in steady state:

\[
\langle v(t + \tau)v(t) \rangle_{\tau \to 0} = \langle v(t)v(t) \rangle = \langle v^2 \rangle_{\text{steady state}} = \frac{K_BT}{m}
\]

Using the regression theorem:

\[
\frac{d}{d\tau} \langle v(t + \tau)v(t) \rangle = -\gamma \langle v(t + \tau)v(t) \rangle
\]

Solution for \( \tau > 0 \) is:

\[
\langle v(t + \tau)v(t) \rangle = e^{-\gamma \tau} \langle v(t)v(t) \rangle_{\text{steady state}} = \frac{K_BT}{m} e^{-\gamma \tau}
\]

If then system has time reversal symmetry then:

\[
\langle v(t + \tau)v(t) \rangle = \langle v(t - \tau)v(t) \rangle = \langle v(t)v(t + \tau) \rangle = e^{-\gamma \tau} \langle v(t)v(t) \rangle_{\text{steady state}} = \frac{K_BT}{m} e^{-\gamma |\tau|}
\]

Initial condition:

\[
R_{vv}(\tau = 0) = \langle v^2 \rangle_{\text{steady state}} = \frac{K_BT}{m}
\]
Particle Diffusion

The diffusion of a particle (in 1D) making random jumps is also described by a Langevin equation

\[
\frac{dx(t)}{dt} = W(t)
\]

\[\langle W(t) \rangle = 0\]

\[\langle W(t_1)W(t_2) \rangle = 2D \delta(t_1 - t_2)\]

The probability of the particle being at a particular location is given by the diffusion equation

\[
\frac{\partial P(x, t)}{\partial t} = D \frac{\partial^2 P(x, t)}{\partial x^2}
\]

If the initial condition is: \( P(x, t_o) = \delta(x - x(t_o)) \)

Solution is:

\[
P(x, t | x(t_o), t_o) = \frac{1}{\sqrt{2\pi 2D(t - t_o)}} \exp \left[ -\frac{(x - x(t_o))^2}{2[2D(t - t_o)]} \right]
\]
Particle Diffusion

\[ \frac{dx(t)}{dt} = W(t) \]

Solution is:

\[ x(t) = x(0) + \int_{0}^{t} W(t) dt \]

\[ \Rightarrow \langle x(t) \rangle = x(0) \]

Variance:

\[ \langle x^2(t) \rangle = x^2(0) + 2x(0)\int_{0}^{t} \langle W(t) dt \rangle + \left( \int_{0}^{t} dt_{1} \int_{0}^{t} dt_{2} W(t_{1})W(t_{2}) \right) \]

\[ \langle x^2(t) \rangle = x^2(0) + \int_{0}^{t} dt_{1} \int_{0}^{t} dt_{2} 2D \delta(t_{1} - t_{2}) \]

\[ = x^2(0) + 2D \int_{0}^{t} dt_{1} \]

\[ = x^2(0) + 2Dt \]

\[ \langle x^2(t) \rangle - x^2(0) = 2Dt \]

\[ \Rightarrow \langle [x(t) - x(0)]^2 \rangle = 2Dt \]
Particle Diffusion

Auto-Correlation:

\[
\langle x(t_1)x(t_2) \rangle = x^2(0) + \int_0^{t_1} dt_1 \int_0^{t_2} dt_2 \langle w(t_1)w(t_2) \rangle
\]

\[
= x^2(0) + \int_0^{t_1} dt_1 \int_0^{t_2} dt_2 2D \delta(t_1 - t_2)
\]

\[
\langle x(t_1)x(t_2) \rangle = x^2(0) + 2D \min(t_1, t_2) \quad \Rightarrow \quad \text{smaller of } t_1 \text{ and } t_2
\]

\[
\Rightarrow \langle [x(t_1) - x(t_2)]^2 \rangle = 2D |t_1 - t_2|
\]

Not a stationary process!!

Diffusion Equation Result:

\[
P(x, t \mid x(t_o), t_o) = \frac{1}{\sqrt{2\pi} 2D(t - t_o)} \exp \left[ - \frac{(x - x(t_o))^2}{2[2D(t - t_o)]} \right]
\]

\[
\Rightarrow \langle [x(t) - x(t_o)]^2 \rangle = 2D (t - t_o) \quad \{\text{for } t \geq t_o\}
\]
Langevin Equations Vs Fokker-Planck Equations

For every Langevin equation of the form:

\[
\frac{da(t)}{dt} = -A(a(t), t) + F(t) \quad \text{and} \quad \frac{dv(t)}{dt} = -\gamma v(t) + \frac{F(t)}{m}
\]

\[ \langle F(t) \rangle = 0 \]
\[ \langle F(t)F(t') \rangle = B(a(t), t) \delta(t - t') \]

there is a Fokker-Planck equation (a generalized diffusion equation) of the form:

\[
\frac{\partial P(a, t)}{\partial t} = -\frac{\partial}{\partial a} \left( A(a, t)P(a, t) \right) + \frac{1}{2} \frac{\partial^2}{\partial a^2} \left( B(a, t)P(a, t) \right)
\]

which gives the full probability distribution of the random signal at any time

In most cases, one is never interested in the full probability distribution but only in the correlations which are measured experimentally.
Particle Diffusion

Spectral Density \( (\omega) \):

\[
-i\omega \ x(\omega) = W(\omega)
\]

\[
x(\omega) = \frac{W(\omega)}{-i\omega}
\]

\[
S_{xx}(\omega) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \langle x(\omega')x(\omega) \rangle
\]

\[
= \int \frac{d\omega'}{2\pi} \frac{W(\omega')W(\omega)}{\omega' \omega}
\]

\[
= \int \frac{d\omega'}{2\pi} 2D \frac{2\pi}{\omega' \omega} \delta(\omega' - \omega)
\]

\[
S_{xx}(\omega) = \frac{2D}{\omega^2} \quad \text{Signature of diffusion!}
\]
Consider a narrow band signal:

\[ y(t) = \int \frac{d\omega}{2\pi} y(\omega) e^{-i\omega t} \]

\[ y(t) = \int_{\omega_o-\Delta/2}^{\omega_o+\Delta/2} \frac{d\omega}{2\pi} y(\omega) e^{-i\omega t} + \int_{-\omega_o-\Delta/2}^{-\omega_o+\Delta/2} \frac{d\omega}{2\pi} y(\omega) e^{-i\omega t} \]

Factor out the fast time dependence:

\[ y(t) = e^{-i\omega_o t} \int_{\omega_o-\Delta/2}^{\omega_o+\Delta/2} \frac{d\omega}{2\pi} y(\omega) e^{-i(\omega-\omega_o)t} + e^{i\omega_o t} \int_{-\omega_o-\Delta/2}^{-\omega_o+\Delta/2} \frac{d\omega}{2\pi} y(\omega) e^{-i(\omega+\omega_o)t} \]

\[ y(t) = e^{-i\omega_o t} \frac{x(t)}{2} + e^{i\omega_o t} \frac{x^*(t)}{2} \]
Phasor Representation and Signal Quadratures

\[ y(t) = e^{-i\omega_o t} \frac{x(t)}{2} + e^{i\omega_o t} \frac{x^*(t)}{2} \]

\[ \Rightarrow y(t) = \text{Re} \left\{ x(t) e^{-i\omega_o t} \right\} \]

Let:

\[ x(t) = x_1(t) + i \ x_2(t) \]

Then:

\[ y(t) = \text{Re} \left\{ [x_1(t) + i \ x_2(t)] e^{-i\omega_o t} \right\} \]

\[ y(t) = x_1(t) \cos \omega_o t + x_2(t) \sin \omega_o t \]

Two quadratures of \( y(t) \)
Phasor Representation and Signal Quadratures

Generalized Quadratures:

\[ x(t) = x_\theta(t) e^{i\theta} + x_{\theta + \pi/2}(t)e^{i(\theta + \pi/2)} \]
\[ = [x_\theta(t) + i x_{\theta + \pi/2}(t)] e^{i\theta} \]

Example 1:

\[ x(t) = x_o e^{i\phi} \]
\[ x(t) = x_o \cos \phi + i x_o \sin \phi \]

\[ y(t) = \text{Re} \left\{ x(t)e^{-i\omega_0 t} \right\} \]
\[ = \text{Re} \left\{ x_o e^{i\phi} e^{-i\omega_0 t} \right\} \]
\[ y(t) = x_o \cos (\omega_0 t - \phi) \]
Phasor Representation and Signal Quadratures

Example 2:

\[ x(t) = x_o + \eta_1(t) \]
\[ \eta_1(t) \ll x_o \]
\[ y(t) = x(t) \cos \omega_o t \]
\[ = (x_o + \eta_1(t)) \cos \omega_o t \]

Example 3:

\[ x(t) = x_o + i\eta_2(t) \]
\[ \eta_2(t) \ll x_o \]
\[ x(t) = [x_o + i\eta_2(t)] = x_o \left[ 1 + i \frac{\eta_2(t)}{x_o} \right] \]
\[ \approx x_o e^{i \frac{\eta_2(t)}{x_o}} \]
\[ \Rightarrow y(t) \approx x_o \cos \left[ \omega_o t - \frac{\eta_2(t)}{x_o} \right] \]
Phasor Representation and Signal Quadratures

Example 4: \( \eta_1(t), \eta_2(t) \) are zero-mean random signals
\[ \eta_1(t), \eta_2(t) \ll x_o \]

\[ x(t) = x_o + \eta_1(t) + i\eta_2(t) \]

Example 5:
\[ x(t) = x_o e^{i\phi} + \eta(t) \quad |\eta(t)| \ll x_o \]

\[ x(t) = x_o e^{i\phi} + [\eta_\phi(t) + i\eta_{\phi+\pi/2}] e^{i\phi} \]

\[ x(t) \approx [x_o + \eta_\phi(t)] \left[ 1 + i \frac{\eta_{\phi+\pi/2}(t)}{x_o} \right] e^{i\phi} \]

\[ \approx [x_o + \eta_\phi(t)] e^{i \phi + \eta_{\phi+\pi/2}(t) / x_o} \]

\[ y(t) \approx [x_o + \eta_\phi(t)] \cos \left[ \omega_o t - \phi - \frac{\eta_{\phi+\pi/2}(t)}{x_o} \right] \]
Phasor Representation and Signal Quadratures