In this lecture you will learn:

- Classical Rate Equations
- Phase Transition in a Laser
- Quantum Description of a Laser
- Phase Noise in a Laser and Laser Linewidth
- Photon Number Fluctuations and Photon Flux Noise
- Quantum State of a Laser: Coherent State vs a Statistical Mixture
- Phase Coherence and Interference in Quantum Optics: Lessons from Bose-Einstein Condensates
The Rate Equation Description

$N$ two-level systems in a cavity

$N$ Two-Level systems inside the laser cavity

$$N_2 = \sum_{j=1}^{N} N_{2,j}$$

$$N_1 = \sum_{j=1}^{N} N_{1,j}$$

$$N_d = \sum_{j=1}^{N} N_{d,j}$$

$$N_2 + N_1 = \sum_{j=1}^{N} N_{2,j} + N_{1,j} = N$$
The Rate Equation Description

Level 3

Level 2

Level 1

Pump: $T_p$

Non-radiative relaxation: $T_1$

Stimulated emission

Spontaneous emission

$N$ two-level systems in a cavity

Cavity

Waveguide

$z = 0$

\[
\frac{dN_2(t)}{dt} = \frac{N_1(t)}{T_p} - \frac{N_2(t)}{T_1} - 2g_d \left( N_2(t) - N_1(t) \right) n(t) - 2g_g N_2(t)
\]

\[
\frac{dN_1(t)}{dt} = -\frac{N_1(t)}{T_p} + \frac{N_2(t)}{T_1} + 2g_d \left( N_2(t) - N_1(t) \right) n(t) + 2g_d N_2(t)
\]

\[
\frac{dn(t)}{dt} = \left( 2g_d \left( N_2(t) - N_1(t) \right) - \frac{1}{\tau_p} \right) n(t) + 2g_d N_2(t)
\]

\[
\frac{dN_d(t)}{dt} = N \left( \frac{1}{T_p} - \frac{1}{T_1} \right) - N_d(t) \left( \frac{1}{T_p} + \frac{1}{T_1} \right) - 4g_d N_d(t) n(t)
\]
Rate Equation Description: Steady State

Steady state solution of the rate equations:

\[ N \left( \frac{1}{T_p} - \frac{1}{T_1} \right) - N_d \left( \frac{1}{T_p} + \frac{1}{T_1} \right) - 4g_d N_d n = 0 \]  \hspace{1cm} (1)

\[ N_d = \frac{N \left( \frac{1}{T_p} - \frac{1}{T_1} \right)}{\left( \frac{1}{T_p} + \frac{1}{T_1} \right) + 4g_d n} \]

\[ \left( 2g_d N_d - \frac{1}{\tau_p} \right) n(t) + g_d \left[ N + N_d \right] = 0 \]  \hspace{1cm} (2)

Lasing Condition:

\[ 2g_d N_{dth} = \frac{1}{\tau_p} \]

Cavity Gain = Cavity Loss

\[ 2g_d N_d n + 2g_d N_2 = n \]
Laser Below Threshold: ASE

\[ N_d = \frac{N \left( \frac{1}{T_p} - \frac{1}{T_1} \right)}{\left( \frac{1}{T_p} + \frac{1}{T_1} \right)} + 4g_d n \]

Lasing Condition:

\[ 2g_d N_{dth} = \frac{1}{\tau_p} \]

Cavity Gain = Cavity Loss

\[ 2g_d N_d n + 2g_d N_2 = \frac{n}{\tau_p} \]

\[ N_{dth} = \frac{N \left( \frac{1}{T_{pth}} - \frac{1}{T_1} \right)}{\left( \frac{1}{T_{pth}} + \frac{1}{T_1} \right)} \]

\[ n = \frac{g_d [N + N_d]}{\frac{1}{\tau_p} - 2g_d N_d} \]
Laser at Threshold

\[ N_d = \frac{N \left( \frac{1}{T_p} - \frac{1}{T_1} \right)}{\left( \frac{1}{T_p} + \frac{1}{T_1} \right)} + 4g_d n \]

\[ n = \frac{g_d [N + N_d]}{1 - 2g_d N_d} \]

Lasing Condition:

\[ 2g_d N_{dth} = \frac{1}{\tau_p} \]

Cavity Gain = Cavity Loss

\[ 2g_d N_d n + 2g_d N_2 = \frac{n}{\tau_p} \]
Laser Above Threshold: Gain Clamping

\[ N \left( \frac{1}{T_p} - \frac{1}{T_1} \right) - N_d \left( \frac{1}{T_p} + \frac{1}{T_1} \right) - 4g_d N_d n = 0 \]

At threshold:

\[ N \left( \frac{1}{T_{pth}} - \frac{1}{T_1} \right) - N_{dth} \left( \frac{1}{T_{pth}} + \frac{1}{T_1} \right) = 0 \]

Subtract the two:

\[ N \left( \frac{1}{T_p} - \frac{1}{T_{pth}} \right) - 4g_d N_{dth} n = 0 \]

\[ \Rightarrow N \left( \frac{1}{T_p} - \frac{1}{T_{pth}} \right) - \frac{2}{\tau_p} n = 0 \]

\[ \Rightarrow n = \frac{N\tau_p}{2} \left( \frac{1}{T_p} - \frac{1}{T_{pth}} \right) \]

Output power:

\[ P_{out} = \hbar \omega_o \frac{n}{\tau_p} = \hbar \omega_o \frac{N}{2} \left( \frac{1}{T_p} - \frac{1}{T_{pth}} \right) \]
The Complete Set of Quantum Equations

\[ \hat{N}_2(t) + \hat{N}_1(t) = N \]

**Population Equations:**

\[
\frac{d\hat{N}_2(t)}{dt} = \frac{\hat{N}_1(t)}{T_p} - \frac{\hat{N}_2(t)}{T_1} - \frac{i}{\hbar} \left[ k\hat{J}_+(t)\hat{a}(t) - k^*\hat{a}^+(t)\hat{J}_-(t) \right] - \hat{D}_N(t) + \hat{D}_{\text{pump}}(t)
\]

\[
\frac{d\hat{N}_1(t)}{dt} = -\frac{\hat{N}_1(t)}{T_p} + \frac{\hat{N}_2(t)}{T_1} + \frac{i}{\hbar} \left[ k\hat{J}_+(t)\hat{a}(t) - k^*\hat{a}^+(t)\hat{J}_-(t) \right] + \hat{D}_N(t) - \hat{D}_{\text{pump}}(t)
\]

**Polarization Equations:**

\[
\frac{d\hat{J}_+(t)}{dt} = \left( i \frac{\Delta \varepsilon}{\hbar} - \frac{1}{T_2} \right) \hat{J}_+(t) - \frac{i}{\hbar} k^* \hat{a}^+(t) \left[ \hat{N}_2(t) - \hat{N}_1(t) \right] + \hat{D}_+(t)
\]

\[
\frac{d\hat{J}_-(t)}{dt} = \left( -i \frac{\Delta \varepsilon}{\hbar} - \frac{1}{T_2} \right) \hat{J}_-(t) + \frac{i}{\hbar} k \left[ \hat{N}_2(t) - \hat{N}_1(t) \right] \hat{a}(t) + \hat{D}_-(t)
\]
The Complete Set of Quantum Equations

Field Equations:

\[
\frac{d\hat{a}(t)}{dt} = \left(-i\omega_0 - \frac{1}{2\tau_p}\right)\hat{a}(t) - \frac{i}{\hbar} k^* \hat{J}_{-}(t) + \sqrt{\frac{1}{\tau_p}} \hat{S}_{in}(t)
\]

\[
\frac{d\hat{a}^+(t)}{dt} = \left(i\omega_0 - \frac{1}{2\tau_p}\right)\hat{a}^+(t) + \frac{i}{\hbar} k \hat{J}_{+}(t) + \sqrt{\frac{1}{\tau_p}} \hat{S}^+_{in}(t)
\]

\[
\hat{S}_{in}(t) = \sqrt{v_g} b_L (z = 0, t) e^{-i\omega_0 t}
\]

\[
\hat{S}^+_{in}(t) = \sqrt{v_g} b^+_L (z = 0, t) e^{i\omega_0 t}
\]

\[
\hat{S}_{out}(t) = \sqrt{v_g} b_R (z = 0, t) e^{-i\omega_0 t} = \sqrt{\frac{1}{\tau_p}} \hat{a}(t) - \sqrt{v_g} b_L (z = 0, t) e^{-i\omega_0 t}
\]

\[
\hat{S}^+_{out}(t) = \sqrt{v_g} b^+_R (z = 0, t) e^{i\omega_0 t} = \sqrt{\frac{1}{\tau_p}} \hat{a}^+(t) - \sqrt{v_g} b^+_L (z = 0, t) e^{i\omega_0 t}
\]
The Complete Set of Quantum Equations

Assume fast decoherence

Solve the polarization equation assuming steady state:

\[
\frac{d\hat{J}_-(t)}{dt} = \left(-i \frac{\Delta \varepsilon}{\hbar} - \frac{1}{T_2}\right) \hat{J}_-(t) + \frac{i}{\hbar} k \left[ \hat{N}_2(t) - \hat{N}_1(t) \right] \hat{a}(t) + \hat{G}_-(t)
\]

\[
\Rightarrow \hat{J}_-(t) = -k \left[ \frac{\hat{N}_2(t) - \hat{N}_1(t)}{(\hbar \omega_o - \Delta \varepsilon) + i\hbar/2T_2} \right] \hat{a}(t) + \frac{i\hbar \hat{D}_-(t)}{(\hbar \omega_o - \Delta \varepsilon) + i\hbar/2T_2} + \hat{F}_{sp}(t)
\]

Plug it into the field equation:

\[
\frac{d\hat{a}(t)}{dt} = \left(-i \omega_o - \frac{1}{2\tau_p}\right) \hat{a}(t) - \frac{i}{\hbar} k^* \hat{J}_-(t) + \sqrt{\frac{1}{\tau_p}} \hat{S}_{in}(t)
\]

\[
\Rightarrow \frac{d\hat{a}(t)}{dt} = \left(-i \omega_o - \frac{1}{2\tau_p}\right) \hat{a}(t) + g_d \left[ \hat{N}_2(t) - \hat{N}_1(t) \right] \hat{a}(t) + \hat{F}_{sp}(t) + \sqrt{\frac{1}{\tau_p}} \hat{S}_{in}(t)
\]
The Complete Set of Quantum Equations

Population and Photon Density Equations in the limit of Fast Decoherence:

\[
\frac{d\hat{N}_2(t)}{dt} = \frac{\hat{N}_1(t)}{T_p} - \frac{\hat{N}_2(t)}{T_1} - 2g_d \left[ \hat{N}_2(t) - \hat{N}_1(t) \right] \hat{a}^+ (t) \hat{a} (t) - \hat{D}_N(t) + \hat{D}_{pump}(t)
\]

\[- \left\{ \hat{a}^+ (t) \hat{F}_{sp}(t) + \hat{F}_{sp}^+(t) \hat{a}(t) \right\} \]

\[
\frac{d\hat{N}_1(t)}{dt} = -\frac{\hat{N}_1(t)}{T_p} + \frac{\hat{N}_2(t)}{T_1} + 2g_d \left[ \hat{N}_2(t) - \hat{N}_1(t) \right] \hat{a}^+ (t) \hat{a} (t) + \hat{D}_N(t) - \hat{D}_{pump}(t)
\]

\[+ \left\{ \hat{a}^+ (t) \hat{F}_{sp}(t) + \hat{F}_{sp}^+(t) \hat{a}(t) \right\} \]

\[
\frac{d\hat{n}(t)}{dt} = -\frac{\hat{n}(t)}{\tau_p} + 2g_d \left[ \hat{N}_2(t) - \hat{N}_1(t) \right] \hat{n}(t) + \left\{ \hat{a}^+ (t) \hat{F}_{sp}(t) + \hat{F}_{sp}^+(t) \hat{a}(t) \right\}
\]

\[+ \sqrt{\frac{1}{\tau_p}} \left\{ \hat{a}^+ (t) \hat{S}_{in}(t) + \hat{S}_{in}^+(t) \hat{a}(t) \right\} \]
The Semi-Classical Field Equation

Assume fast decoherence

Start from:

\[ \frac{d\hat{a}(t)}{dt} = \left( -i\omega_0 - \frac{1}{2\tau_p} \right)\hat{a}(t) + g_d \left[ \hat{N}_2(t) - \hat{N}_1(t) \right] \hat{a}(t) + \hat{F}_{sp}(t) + \sqrt{\frac{1}{\tau_p}} \hat{S}_{in}(t) \]

Assume:

\[ \langle \hat{a}(t) \rangle = \beta(t)e^{-i\omega_0 t} \quad \Rightarrow \quad \langle \hat{n}(t) \rangle = \langle \hat{a}^+(t)\hat{a}(t) \rangle = |\beta(t)|^2 \]

Complex

Take the average of the above operator equation to get:

\[ \Rightarrow \frac{d\beta(t)}{dt} = \left[ g_d N_d(t) - \frac{1}{2\tau_p} \right] \beta(t) \quad \{ \text{where } N_d(t) = \langle \hat{N}_2(t) - \hat{N}_1(t) \rangle \} \]
The Semi-Classical Field Equation

We have the field equation:

\[ \implies \frac{d\beta(t)}{dt} = \left[ g_d N_d(t) - \frac{1}{2\tau_p} \right] \beta(t) \]

\[ \text{where } N_d(t) = \langle \tilde{N}_2(t) - \tilde{N}_1(t) \rangle \]

The population equation becomes upon averaging:

\[ \frac{dN_d(t)}{dt} = N \left( \frac{1}{T_p} - \frac{1}{T_1} \right) - N_d(t) \left( \frac{1}{T_p} + \frac{1}{T_1} \right) - 4g_d N_d(t) |\beta(t)|^2 \]

If one is looking at long time scales then safe to set the LHS=0 in the above equation:

\[ N_d(t) \approx N \left( \frac{1}{T_p} - \frac{1}{T_1} \right) \left( \frac{1}{T_p} + \frac{1}{T_1} \right) + 4g_d |\beta(t)|^2 \]

Valid over long time scales only

The field equation then becomes:

\[ \frac{d\beta(t)}{dt} = \left[ g_d N \left( \frac{1}{T_p} - \frac{1}{T_1} \right) \left( \frac{1}{T_p} + \frac{1}{T_1} \right) + 4g_d |\beta(t)|^2 \right] - \frac{1}{2\tau_p} \beta(t) \]
The Semi-Classical Field Equation

Write the field average value as:
\[ \beta(t) = x_1(t) + i x_2(t), \]

Define:
\[ \bar{r}(t) = x_1(t) \hat{x} + x_2(t) \hat{y} \quad \Rightarrow \quad r^2(t) = \bar{r}(t) \cdot \bar{r}(t) = x_1^2(t) + x_2^2(t) = |\beta(t)|^2 \]

The field equation becomes:
\[
\frac{d\beta(t)}{dt} = g_d N \frac{\left( \left( \frac{1}{T_p} - \frac{1}{T_1} \right) \right)}{\left( \frac{1}{T_p} + \frac{1}{T_1} \right) + 4g_d|\beta(t)|^2} - \frac{1}{2\tau_p} \beta(t) \quad \Rightarrow \quad \frac{d\bar{r}(t)}{dt} = -\vec{\nabla} V(\bar{r}(t))
\]

The steady state field value corresponds to the minimum of the potential:
\[
V(\bar{r}) = \frac{r^2}{4\tau_p} - \frac{1}{2} \sqrt{\frac{g_d}{\frac{1}{T_p} + \frac{1}{T_1}} N \left( \frac{1}{T_p} - \frac{1}{T_1} \right)} \tan^{-1} \left[ 2r \sqrt{\frac{g_d}{\frac{1}{T_p} + \frac{1}{T_1}}} \right]
\]
The Semi-Classical Field Potential

1/\(T_p < 1/\tau_{pth}\)

1/\(T_p > 1/\tau_{pth}\)

\(|\beta|^2 = r_{min}^2 \approx \frac{N\tau_p}{2} \left( \frac{1}{T_p} - \frac{1}{\tau_{pth}} \right) = \text{steady state photon number} = n\)
Laser Spectrum and Linewidth Below Threshold

Spectrum is related to the first order coherence function:

\[ S(\omega) = \int_{-\infty}^{\infty} d\tau \, g_1(t, t + \tau) \, e^{i\omega \tau} \]
\[ = \int_{-\infty}^{\infty} d\tau \frac{\langle \hat{a}^+(t)\hat{a}(t + \tau) \rangle}{\sqrt{\langle \hat{a}^+(t)\hat{a}(t) \rangle \langle \hat{a}^+(t + \tau)\hat{a}(t + \tau) \rangle}} e^{i\omega \tau} \]

The field equation below threshold is:

\[
\frac{d\hat{a}(t)}{dt} = \left( -i\omega_o + \left( g_d N_d - \frac{1}{2\tau_p} \right) \right) \hat{a}(t) + \hat{F}_{sp}(t) + \sqrt{\frac{1}{\tau_p}} \hat{S}_{in}(t)
\]

Solution:

\[ S(\omega) = \frac{\Delta \omega}{(\omega - \omega_o)^2 + \left( \frac{\Delta \omega}{2} \right)^2} \]

\[ \Delta \omega = \frac{1}{\tau_p} - 2g_d N_d \]

Note:

\[ \left( \frac{1}{\tau_p} - 2g_d N_d \right)n = 2g_d N_2 = \text{spontaneous emission rate} \]
The Laser State Above Threshold

Candidates:

1) There is no steady state. Phase is always diffusing. The quantum state is a coherent state with a diffusing phase:

\[ \hat{\rho}(t) = |\beta| e^{i\phi(t)} \langle |\beta| e^{i\phi(t)} | \rightarrow \langle \hat{n} \rangle = |\beta|^2 = r_{min}^2 \]

2) Phase is always diffusing. The quantum state is a statistical mixture of coherent states and all phases are possible:

\[ \hat{\rho} = \frac{2\pi}{0 \, 2\pi} \left| \beta \right| e^{i\phi} \langle |\beta| e^{i\phi} | \right. \\
= \sum_{n=0}^{\infty} \exp \left( -|\beta|^2 \right) \frac{\left( |\beta|^2 \right)^n}{n!} \left| n \right\rangle \langle n \left| \rightarrow \langle \hat{n} \rangle = |\beta|^2 = r_{min}^2 \]

3) The entanglement with the atom degrees of freedom makes any superposition of number states very unlikely in steady state. The reduced density matrix for photons corresponds to a quantum state which is just a statistical mixture of number states:

\[ \hat{\rho} = \sum_{n=0}^{\infty} P(n) \left| n \right\rangle \langle n \right. \\
\rightarrow \langle \hat{n} \rangle = |\beta|^2 = r_{min}^2 \]

Same as:

\[ \hat{\rho} = \int_0^\infty d|\beta| \frac{2\pi}{0 \, 2\pi} \left| \beta \right| e^{i\phi} \Pi(|\beta|) \left| \beta \right| e^{i\phi} \langle |\beta| e^{i\phi} | \right. \\
= \int_0^\infty d|\beta| \left[ P(n) = \int_0^\infty d|\beta| \, \Pi(|\beta|) \exp \left( -|\beta|^2 \right) \frac{\left( |\beta|^2 \right)^n}{n!} \right]
The Phase of the Field above Threshold and Laser Linewidth

In steady state above threshold, the phase is NOT well-defined so:

\[ \langle \hat{a}(t) \rangle = 0 \]

We assume that:

\[ \hat{a}(t) = e^{-i \omega_0 t \sqrt{n + 1}} e^{i \hat{\phi}(t)} = e^{-i \omega_0 t} e^{i \hat{\phi}(t)} \sqrt{n} = e^{-i \omega_0 t} e^{i \hat{\phi}(t)} \sqrt{|\beta|^2 + \Delta n(t)} \]

\[ \approx e^{-i \omega_0 t} e^{i \hat{\phi}(t)} \left[ \beta + \frac{\Delta n(t)}{2|\beta|} \right] \]

Not Hermitian

Now we know that the phase operator is not Hermitian and that:

\[ \left[ e^{i \hat{\phi}(t)}, e^{-i \hat{\phi}^+(t)} \right] = |0 \rangle \langle 0 | \]

But if our laser state has very little weight on the vacuum state then the phase operator can be assumed to be Hermitian to a very good approximation, and:

\[ \left[ e^{i \hat{\phi}(t)} \right]^+ = e^{-i \hat{\phi}(t)} \]
The Phase of the Field above Threshold and Laser Linewidth

We start again from the equation:

$$\frac{d \hat{a}(t)}{dt} = \left( -i\omega_o - \frac{1}{2\tau_p} \right) \hat{a}(t) + d \left( \hat{N}_2(t) - \hat{N}_1(t) \right) \hat{a}(t) + e^{-i\omega_o t} \hat{F}_{sp}(t) + \sqrt{\frac{1}{\tau_p}} \hat{S}_{in}(t) e^{-i\omega_o t}$$

We use:

$$\hat{a}(t) \approx e^{-i\omega_o t} e^{i\phi(t)} \left[ \beta \left| + \frac{\Delta \hat{n}(t)}{2|\beta|} \right. \right]$$

in:

$$\frac{d \hat{a}(t)}{dt} = \left( -i\omega_o - \frac{1}{2\tau_p} \right) \hat{a}(t) + g_d \hat{N}_{dth} \hat{a}(t) + \hat{F}_{sp}(t) + \sqrt{\frac{1}{\tau_p}} \hat{S}_{in}(t)$$

$$\frac{d \hat{a}^+(t)}{dt} = \left( i\omega_o - \frac{1}{2\tau_p} \right) \hat{a}^+(t) + g_d \hat{a}^+(t) \hat{N}_{dth} + \hat{F}_{sp}^+(t) + \sqrt{\frac{1}{\tau_p}} \hat{S}_{in}^+(t)$$

$$\langle \hat{N}_2(t) - \hat{N}_1(t) \rangle = N_d = N_{dth}$$

$$|\beta|^2 = n = \text{avg photon number}$$
The Phase of the Field above Threshold and Laser Linewidth

To get for the phase operator:

\[
\frac{d \hat{\phi}(t)}{dt} = \frac{1}{|\beta|} \left[ e^{-i\hat{\phi}(t)} \hat{F}_{sp}(t) - e^{i\hat{\phi}(t)} \hat{F}_{sp}^+(t) \right] + \frac{1}{|\beta|} \sqrt{\frac{1}{\tau_p}} \left[ \hat{S}_{in}(t) e^{-i\hat{\phi}(t)} - e^{i\hat{\phi}(t)} \hat{S}_{in}^+(t) \right]
\]

The phase operator obeys a diffusion equation again!

Solution:

\[
\left\langle \left[ \hat{\phi}(t) - \hat{\phi}(t') \right]^2 \right\rangle = \frac{1}{2} \left( \frac{1}{\tau_p} - 2g_d N_d \right) |t - t'| = \frac{2g_d N_d}{2n} |t - t'|
\]

\[
= \frac{2g_d N_d n_{sp}}{2n} |t - t'| = \frac{1}{2\tau_p} \left( \frac{n_{sp}}{n} \right) |t - t'|
\]

Has contributions from both spontaneous emission and vacuum fluctuations in equal parts!
The Phase of the Field above Threshold and Laser Linewidth

We need to evaluate:

\[
S(\omega) = \int_{-\infty}^{\infty} d\tau \; g_1(t, t+\tau) e^{i\omega\tau} = \int_{-\infty}^{\infty} d\tau \; \frac{\langle \hat{a}^+(t)\hat{a}(t+\tau) \rangle}{\sqrt{\langle \hat{a}^+(t)\hat{a}(t) \rangle \langle \hat{a}^+(t+\tau)\hat{a}(t+\tau) \rangle}} e^{i\omega\tau}
\]

We have:

\[
\hat{a}(t) \approx |\beta| e^{-i\omega_0 t} e^{i\hat{\phi}(t)}
\]

Denominator:

\[
\Rightarrow \sqrt{\langle \hat{a}^+(t)\hat{a}(t) \rangle \langle \hat{a}^+(t+\tau)\hat{a}(t+\tau) \rangle} \approx |\beta|^2
\]

Numerator:

\[
\langle \hat{a}^+(t)\hat{a}(t+\tau) \rangle \approx |\beta|^2 e^{-i\omega_0 \tau} \langle e^{-i\hat{\phi}(t)} e^{i\hat{\phi}(t+\tau)} \rangle
\]

\[
= |\beta|^2 e^{-i\omega_0 \tau} \left[ e^{\frac{-|\tau|^2}{2}} \right]
\]

\[
= |\beta|^2 e^{-i\omega_0 \tau} \left[ e^{\frac{-|\tau|^2}{4\tau_p \left( \frac{n_{sp}}{n} \right)}} \right]
\]
Schawlow-Townes Expression for Laser Linewidth

\[ S(\omega) = \int_{-\infty}^{\infty} d\tau \ g_1(t, t+\tau) e^{i\omega \tau} \]

\[ = \int_{-\infty}^{\infty} d\tau \ e^{i\omega \tau} \left( \frac{n_{sp}}{n} \right) e^{-\frac{\tau}{4\tau_p}} \]

\[ = \frac{\Delta \omega}{(\omega - \omega_0)^2 + \left( \frac{\Delta \omega}{2} \right)^2} \]

FWHM Linewidth:

\[ \Delta \omega = \frac{1}{2\tau_p} \left( \frac{n_{sp}}{n} \right) \]

Note that the spontaneous emission rate above threshold is:

\[ \frac{1}{\tau_p} - 2g_d N_{dth} \]

\[ = \frac{2g_d N_2}{n} = \frac{2g_d N_{dth} n_{sp}}{n} = \frac{1}{\tau_p} \left( \frac{n_{sp}}{n} \right) \]
Suppose the average operator value is:

\[ \langle \hat{a}(t) \rangle = \sqrt{n} \ e^{-i\omega_0 t + i\phi(t)} \]

At time “t” a spontaneously emitted photon with phase \( \theta \) (wrt \( \phi(t) \)) was added to the cavity. Right afterwards, the field operator’s average value is,

\[ \langle \hat{a}(t) \rangle = \sqrt{n} \ e^{-i\omega_0 t + i\phi(t)} + \sqrt{1} \ e^{-i\omega_0 t + i\phi(t) + i\theta} \]

\[ = \sqrt{\left( \sqrt{n} \cos \theta \right)^2 + \sin^2 \theta} \ e^{-i\omega_0 t + i\phi(t) + i\Delta \phi} \]

\[ \Delta \phi = \frac{\sin \theta}{\sqrt{n}} \quad \langle \cos \theta \rangle = \langle \sin \theta \rangle = 0 \]

\[ \langle \cos^2 \theta \rangle = \langle \sin^2 \theta \rangle = \frac{1}{2} \]

After some time, the amplitude increase has relaxed back to its steady state value, so:

\[ \langle \hat{a}(t) \rangle = \sqrt{n} \ e^{-i\omega_0 t + i\phi(t) + i\Delta \phi} \]
After some time, the amplitude increase has relaxed back to its steady state value, so:

$$\langle \hat{a}(t) \rangle = \sqrt{n} \ e^{-i\omega t + i\phi(t) + i\Delta\phi}$$

=> Every spontaneous emission event kicks the phase by an amount equal to:

$$\Delta\phi = \frac{\sin \theta}{\sqrt{n}}$$

Total phase change after time "τ"

$$\phi(t + \tau) - \phi(t) = \sum_k \Delta\phi_k = \frac{1}{\sqrt{n}} \sum_k \sin \theta_k$$

$$\Rightarrow \langle \phi(t + \tau) - \phi(t) \rangle = \sum_k \langle \Delta\phi_k \rangle = \frac{1}{\sqrt{n}} \sum_k \langle \sin \theta_k \rangle = 0$$

$$\Rightarrow \langle \left[ \phi(t + \tau) - \phi(t) \right]^2 \rangle = \frac{1}{n} \sum_k \langle \sin^2 \theta_k \rangle = \frac{1}{2n} \left( 2g_d N_{dth} n_{sp} \right) |\tau| = \frac{1}{2\tau_p} \left( \frac{n_{sp}}{n} \right) |\tau|$$
Laser Linewidth above Threshold: A Semi-Classical Interpretation

Therefore, the field correlation is:

\[
\langle \hat{a}^+(t) \hat{a}(t+\tau) \rangle \approx n e^{-i\omega_0 \tau} \langle e^{-i\phi(t)} e^{i\phi(t+\tau)} \rangle = ne^{-i\omega_0 \tau} e^{-\frac{\tau}{4\tau_p}} \left( \frac{n_{sp}}{n} \right) \]

And the laser spectrum can then be found from the first order coherence function:

\[
S(\omega) = \int_{-\infty}^{\infty} d\tau \ g_1(t,t+\tau) e^{i\omega \tau} = \int_{-\infty}^{\infty} d\tau \ e^{-\frac{\tau}{4\tau_p}} \left( \frac{n_{sp}}{n} \right) e^{-i\omega_0 \tau} e^{i\omega \tau} \]

\[
= \frac{\Delta \omega}{(\omega - \omega_o)^2 + \left( \frac{\Delta \omega}{2} \right)^2} \]

**FWHM Linewidth:**

\[
\Delta \omega = \frac{1}{2\tau_p} \left( \frac{n_{sp}}{n} \right) \]

**Question:** whatever happened to vacuum fluctuations?
Laser Photon Number Noise Above Threshold

Consider a laser operating above threshold:

\[
\frac{d \hat{N}_2(t)}{dt} = \frac{\hat{N}_1(t)}{T_p} - \frac{\hat{N}_2(t)}{T_1} - 2g_d \left[ \hat{N}_2(t) - \hat{N}_1(t) \right] \hat{a}^+(t) \hat{a}(t) - \hat{D}_N(t) + \hat{D}_{\text{pump}}(t)
\]

\[
\frac{d \hat{N}_1(t)}{dt} = -\frac{\hat{N}_1(t)}{T_p} + \frac{\hat{N}_2(t)}{T_1} + 2g_d \left[ \hat{N}_2(t) - \hat{N}_1(t) \right] \hat{a}^+(t) \hat{a}(t) + \hat{D}_N(t) - \hat{D}_{\text{pump}}(t)
\]

\[
\frac{d \hat{n}(t)}{dt} = -\frac{\hat{n}(t)}{\tau_p} + 2g_d \left[ \hat{N}_2(t) - \hat{N}_1(t) \right] \hat{n}(t) + \left\{ \hat{a}^+(t) \hat{F}_{sp}(t) + \hat{F}_{sp}^+(t) \hat{a}(t) \right\}
\]

\[
+ \sqrt{\frac{1}{\tau_p}} \left\{ \hat{a}^+(t) \hat{S}_{in}(t) + \hat{S}_{in}^+(t) \hat{a}(t) \right\}
\]

\[
\hat{F}_v(t)
\]
Laser Photon Number Noise Above Threshold

Assume:

\[
\begin{align*}
\hat{N}_2(t) &= N_2 + \Delta \hat{N}_2(t) \\
\hat{N}_1(t) &= N_1 + \Delta \hat{N}_1(t) \\
\hat{n}(t) &= n + \Delta \hat{n}(t)
\end{align*}
\]

\[
\Rightarrow \quad \hat{N}_d(t) = N_d + \Delta \hat{N}_d(t)
\]

After linearizing the population and photon number equations we get:

\[
\frac{d}{dt} \begin{bmatrix}
\Delta \hat{N}_d(t) \\
\Delta \hat{n}(t)
\end{bmatrix} =
\begin{bmatrix}
-\frac{1}{\tau_r} & -\frac{2}{\tau_p} \\
2g_d n & 0
\end{bmatrix}
\begin{bmatrix}
\Delta \hat{N}_d(t) \\
\Delta \hat{n}(t)
\end{bmatrix} +
\begin{bmatrix}
-2 \hat{D}_N(t) + 2 \hat{D}_{pump}(t) - 2 \hat{F}_{st}(t) \\
\hat{F}_{st}(t) + \hat{F}_{V(t)}
\end{bmatrix}
\]

\[
\omega_r^2 = \frac{4g_d n}{\tau_p}
\]

\[
\frac{1}{\tau_r} = \frac{1}{T_p} + \frac{1}{T_1} + 4g_d n
\]
\[ \langle \hat{F}_{st}(t)\hat{F}_{st}(t') \rangle = \left[ 2g_dN_2(n+1) + 2g_dN_1n \right] \delta(t-t') \]

\[ \langle \hat{F}_v(t)\hat{F}_v(t') \rangle = \frac{n}{\tau_p} \delta(t-t') \]

\[ \langle \hat{D}_{\text{pump}}(t)\hat{D}_{\text{pump}}(t') \rangle = \frac{N_1}{T_p} \delta(t-t') \]

\[ \langle \hat{D}_N(t)\hat{D}_N(t') \rangle = \frac{N_2}{T_1} \delta(t-t') \]
Classical Stability Analysis and Laser Relaxation Oscillations

\[ \hat{N}_2(t) = N_2 + \Delta \hat{N}_2(t) \]
\[ \hat{N}_1(t) = N_1 + \Delta \hat{N}_1(t) \]
\[ \hat{n}(t) = n + \Delta \hat{n}(t) \]

\[ \frac{d}{dt} \begin{bmatrix} \Delta \hat{N}_d(t) \\ \Delta \hat{n}(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{\tau_r} & -\frac{2}{\tau_p} \\ 2g_d n & 0 \end{bmatrix} \begin{bmatrix} \Delta \hat{N}_d(t) \\ \Delta \hat{n}(t) \end{bmatrix} + \begin{bmatrix} -2\hat{D}_N(t) + 2\hat{D}_{pump}(t) - 2\hat{F}_{st}(t) \\ \hat{F}_{st}(t) + \hat{F}_v(t) \end{bmatrix} \]

Upon averaging:

\[ \frac{d}{dt} \begin{bmatrix} \langle \Delta \hat{N}_d(t) \rangle \\ \langle \Delta \hat{n}(t) \rangle \end{bmatrix} = \begin{bmatrix} -\frac{1}{\tau_r} & -\frac{2}{\tau_p} \\ 2g_d n & 0 \end{bmatrix} \begin{bmatrix} \langle \Delta \hat{N}_d(t) \rangle \\ \langle \Delta \hat{n}(t) \rangle \end{bmatrix} \]

Damped second order linear system

\[ \omega_r^2 = \frac{4g_d n}{\tau_p} \]
Laser Photon Number Noise Above Threshold

Assume:
\[ \hat{N}_2(t) = N_2 + \Delta \hat{N}_2(t) \]
\[ \hat{N}_1(t) = N_1 + \Delta \hat{N}_1(t) \]
\[ \hat{n}(t) = n + \Delta \hat{n}(t) \]

After linearizing the population and photon number equations we get:

\[
\frac{d}{dt} \begin{bmatrix} \Delta \hat{N}_d(t) \\ \Delta \hat{n}(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{\tau_r} & -\frac{2}{\tau_p} \\ 2g_d n & 0 \end{bmatrix} \begin{bmatrix} \Delta \hat{N}_d(t) \\ \Delta \hat{n}(t) \end{bmatrix} + \begin{bmatrix} -2\hat{D}_N(t) + 2\hat{D}_{pump}(t) - 2\hat{F}_{st}(t) \\ \hat{F}_{st}(t) + \hat{F}_v(t) \end{bmatrix}
\]

Solve in frequency domain to get:

\[
\begin{bmatrix} \Delta \hat{N}_d(\omega) \\ \Delta \hat{n}(\omega) \end{bmatrix} = \frac{H(\omega)}{\omega_r^2} \begin{bmatrix} -i\omega & -\frac{2}{\tau_p} \\ 2g_d n & -i\omega + \frac{1}{\tau_r} \end{bmatrix} \begin{bmatrix} -2\hat{D}_N(\omega) + 2\hat{D}_{pump}(\omega) - 2\hat{F}_{st}(\omega) \\ \hat{F}_{st}(\omega) + \hat{F}_v(\omega) \end{bmatrix}
\]

\[
\omega_r^2 = \frac{4g_d n}{\tau_p}
\]
\[
\frac{1}{\tau_r} = \frac{1}{T_p} + \frac{1}{T_1} + 4g_d n
\]
Laser Photon Number Noise Above Threshold

\[ H(\omega) = \frac{\omega_r^2}{(\omega_r^2 - \omega^2) - i \omega/\tau_r} \]

Laser modulation response function
Laser Photon Number Noise Above Threshold

\[
S_{\Delta n \Delta n}(\omega) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \langle \Delta \hat{n}^* (\omega') \Delta \hat{n}(\omega) \rangle
\]

\[
S_{\Delta n \Delta n}(\omega) = \frac{H(\omega)^2}{\omega_r^4} \left\{ (2g_d n)^2 \left[ \frac{4 N_2}{T_1} + \frac{4 N_1}{T_p} \right] + \left( \omega^2 + \left( \frac{1}{\tau_r} - 4g_d n \right)^2 \right] \right. 
\]

\[
+ \left( \omega^2 + \left( \frac{1}{\tau_r} \right)^2 \right) \left[ \frac{n}{\tau_p} \right] \left\} \right.
\]

\[
\langle \Delta \hat{n}^2(t) \rangle = \tau_r \left( \frac{n_{sp}}{T_1} + \frac{n_{sp} - 1}{T_p} \right)
\]

\[
+ \tau_r \left[ \frac{1}{2} + \left( \frac{1}{T_p} + \frac{1}{T_1} \right)^2 \frac{\tau_p}{8g_d n} \right] \frac{2n_{sp} - 1}{\tau_p}
\]

\[
+ \tau_r \left[ \frac{1}{2} + \left( \frac{1}{\tau_r} \right)^2 \frac{\tau_p}{8g_d n} \right] \frac{1}{\tau_p}
\]

---

Recombination noise and pump noise

Stimulated emission and absorption noise

Vacuum noise (photon loss noise)
Laser Photon Number Noise Above Threshold

Assume operation much above threshold:

\[ \frac{1}{T_p} \gg \frac{1}{T_{\text{th}}}, \frac{1}{T_1} \]

Then in this limit one obtains:

\[
\frac{\langle \Delta \hat{n}^2(t) \rangle}{n} \approx \frac{1}{2} \left[ \frac{n_{sp} - 1}{n_{sp}} \right] + \frac{1}{4n_{sp}} + \frac{1}{2} \left[ \frac{2n_{sp}}{2n_{sp} - 1} \right]
\]

\[
\sim 1 \quad \text{(order of unity when } n_{sp} \gg 1 \text{ and pump noise is present)}
\]

Laser noise inside the cavity is dominated by the pump noise and the vacuum noise much above threshold.

But pump noise is not fundamental – electrically pumped semiconductor lasers have little or no pump noise.
Photon Flux Noise in the Laser Output Above Threshold

Noise outside the cavity is not the same as noise inside the cavity

\[ \hat{S}_{out}(t) = \sqrt{v_g} \hat{b}_R(z = 0, t) e^{-i\omega_0 t} \]
\[ = \sqrt{\frac{1}{\tau_p}} \hat{a}(t) - \sqrt{v_g} \hat{b}_L(z = 0, t) e^{-i\omega_0 t} = \sqrt{\frac{1}{\tau_p}} \hat{a}(t) - \hat{S}_{in}(t) \]

Photon Flux Operator:

\[ \hat{F}_R(z = 0, t) = \hat{S}_{out}^+(t) \hat{S}_{out}(t) \]
\[ = \frac{\hat{n}(t)}{\tau_p} - \sqrt{\frac{1}{\tau_p}} \left\{ \hat{a}^+(t) \hat{S}_{in}(t) + \hat{S}_{in}^+(t) \hat{a}(t) \right\} + \hat{S}_{in}^+(t) \hat{S}_{in}(t) \]

Average Flux:

\[ \langle \hat{F}_R(z = 0, t) \rangle = \frac{\langle \hat{n}(t) \rangle}{\tau_p} \]

Flux Noise Operator:

\[ \Delta \hat{F}_R(z = 0, t) = \frac{\Delta \hat{n}(t)}{\tau_p} - \hat{F}_V(t) \]
\[ \Delta \hat{F}_R(z = 0, \omega) = \frac{\Delta \hat{n}(\omega)}{\tau_p} - \hat{F}_V(\omega) \]
Photon Flux Noise in the Laser Output Above Threshold

We had:

\[ \Delta \hat{n}(\omega) = \frac{H(\omega)}{\omega_r^2} \left[ 2g_d n \left( -2\hat{D}_N(\omega) + 2\hat{D}_{\text{pump}}(\omega) \right) - \left( 4g_d n + i\omega - \frac{1}{\tau_r} \right) \hat{F}_{st}(\omega) - \left( i\omega - \frac{1}{\tau_r} \right) \hat{F}_v(\omega) \right] \]

So we get:

\[ \Delta \hat{F}_R(z = 0, \omega) = \frac{\Delta \hat{n}(\omega)}{\tau_p} - \hat{F}_v(\omega) \]

\[ = \frac{H(\omega)}{\omega_r^2 \tau_p} \left[ 2g_d n \left( -2\hat{D}_N(\omega) + 2\hat{D}_{\text{pump}}(\omega) \right) - \left( 4g_d n + i\omega - \frac{1}{\tau_r} \right) \hat{F}_{st}(\omega) \right] - \hat{F}_v(\omega) \left[ 1 + \frac{H(\omega)}{\omega_r^2 \tau_p} \left( i\omega - \frac{1}{\tau_r} \right) \right] \]

High Frequency Limit:

Assume: \( \omega \gg \omega_r \rightarrow H(\omega) \approx 0 \)

\[ \Rightarrow \Delta \hat{F}_R(z = 0, \omega) = -\hat{F}_v(\omega) \]

\[ \Rightarrow S_{\Delta F_R \Delta F_R}(\omega \gg \omega_r) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \left\langle \Delta \hat{F}_R^*(z = 0, \omega') \Delta \hat{F}_R(z = 0, \omega) \right\rangle \]

\[ = \frac{n}{\tau_p} = \left\langle \hat{F}_R(z = 0, t) \right\rangle \quad \text{Shot noise!} \quad (\text{why?}) \]
Photon Flux Noise in the Laser Output Above Threshold

Low Frequency Limit:

Assume: $\omega \ll \omega_r \quad \Rightarrow \quad H(\omega) \approx 1$

And: $1/T_p \gg 1/T_{\text{pth}}, 1/T_1$

$$\frac{S_{\Delta F_R \Delta F_R}}{n/\tau_p} \approx \frac{2(n_{sp} - 1)}{2n_{sp} - 1} + \frac{1}{2n_{sp} - 1} + \frac{1}{(2n_{sp} - 1)^2} \approx 1$$

(order of unity when $n_{sp} \gg 1$ and pump noise is present)

Laser noise outside the cavity is dominated by the pump noise much above threshold!

But pump noise is not fundamental – electrically pumped semiconductor lasers have little or no pump noise

Low freq. photon flux noise in semiconductor lasers can be $\sim 10-13$ dB below shot noise!
 Photon Flux Noise in the Laser Output Above Threshold

Photon Flux Noise Spectral Density of a Optically Pumped Lasers

![Graph showing photon flux noise spectral density for optically pumped lasers.](image)

Increasing pumping

Photon flux has noise less than shot noise

Photon anti-bunching!!

Photon Flux Noise Spectral Density of Electrically Pumped Semiconductor Lasers

![Graph showing photon flux noise spectral density for electrically pumped lasers.](image)

Increasing pumping
The Laser State Above Threshold

Candidates:

1) There is no steady state. Phase is always diffusing. The quantum state is a coherent state with a diffusing phase:

\[ \hat{\rho}(t) = |\beta| e^{i\phi(t)} \langle |\beta| e^{i\phi(t)} | \rightarrow \langle \hat{n} \rangle = |\beta|^2 = r_{\min}^2 \]

2) Phase is always diffusing. The quantum state is a statistical mixture of coherent states and all phases are possible:

\[ \hat{\rho} = \int_0^{2\pi} \frac{d\phi}{2\pi} |\beta| e^{i\phi} \langle |\beta| e^{i\phi} | \rightarrow \sum_{n=0}^{\infty} \exp\left(-|\beta|^2\right) \frac{(|\beta|^2)^n}{n!} \langle n \rangle \langle n \rangle \rightarrow \langle \hat{n} \rangle = |\beta|^2 = r_{\min}^2 \]

3) The entanglement with the atom degrees of freedom makes any superposition of number states very unlikely in steady state. The reduced density matrix for photons corresponds to a quantum state which is just a statistical mixture of number states:

\[ \hat{\rho} = \sum_{n=0}^{\infty} P(n) |n\rangle \langle n| \rightarrow \langle \hat{n} \rangle = |\beta|^2 = r_{\min}^2 \]

Same as: \[ \hat{\rho} = \int_0^\infty d|\beta| \int_0^{2\pi} \frac{d\phi}{2\pi} \Pi(|\beta|) |\beta| e^{i\phi} \langle |\beta| e^{i\phi} | \]

\[ P(n) = \int_0^\infty d|\beta| \Pi(|\beta|) \exp\left(-|\beta|^2\right) \frac{(|\beta|^2)^n}{n!} \]
Lessons from Bose-Einstein Condensates

What is the quantum state of the BEC?

Option 1:

\[ |\psi\rangle = e^{\alpha \hat{a}^+} |0\rangle \]

\[ \langle \psi | \hat{a} | \psi \rangle = \alpha = \sqrt{N} e^{i \phi} \]

\[ \langle \psi | \hat{a}^+ a | \psi \rangle = |\alpha|^2 = N \]

Option 2:

\[ |N\rangle = \left( \frac{\hat{a}^+}{\sqrt{N}} \right)^N |0\rangle \]

\[ \text{Completely uncertain phase!} \]
Lessons from Bose-Einstein Condensates

Observation of Interference Between Two Bose Condensates

* See all authors and affiliations

Science 31 Jan 1997:
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\[ |\psi\rangle = e^{i\phi_1 \hat{a}_{k_1}^+} + e^{i\phi_2 \hat{a}_{k_2}^+} |0\rangle \]

\[ \hat{n}(\vec{r}) = \left( \sum_k e^{-i\vec{k}.\vec{r}} \hat{a}_k^+ \right) \left( \sum_k e^{i\vec{k}.\vec{r}} \hat{a}_k \right) \]

\[ \langle \psi | \hat{n}(\vec{r}) |\psi\rangle = \left| \sqrt{N_1} e^{i\phi_1} \frac{e^{i\vec{k}_1.\vec{r}}}{\sqrt{V}} + \sqrt{N_2} e^{i\phi_2} \frac{e^{i\vec{k}_2.\vec{r}}}{\sqrt{V}} \right|^2 \]

\[ = \frac{N_1}{V} + \frac{N_2}{V} + 2 \frac{\sqrt{N_1N_2}}{V} \cos \left[ (\vec{k}_1 - \vec{k}_2).\vec{r} + (\phi_1 - \phi_2) \right] \]
Lessons from Bose-Einstein Condensates

Observation of Interference Between Two Bose Condensates

* See all authors and affiliations

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\[ |\psi\rangle = \frac{\left(\hat{a}_{k_1}^+\right)^{N_1}}{\sqrt{N_1}} \frac{\left(\hat{a}_{k_2}^+\right)^{N_2}}{\sqrt{N_2}} |0\rangle \]

\[ \hat{n}(\vec{r}) = \left( \sum_{k} \frac{e^{-i\vec{k} \cdot \vec{r}}}{\sqrt{V}} \hat{a}_{k}^+ \right) \left( \sum_{k} \frac{e^{i\vec{k} \cdot \vec{r}}}{\sqrt{V}} \hat{a}_{k} \right) \]

\[ \langle \psi | \hat{n}(\vec{r}) |\psi\rangle = \frac{N_1}{V} + \frac{N_2}{V} \]

Coherent state description wins !!! .......?
Lessons from Bose-Einstein Condensates

Observation of Interference Between Two Bose Condensates

See all authors and affiliations

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\[
|\psi\rangle = \frac{(\hat{a}_{\vec{k}_1}^+)^{N_1}}{\sqrt{N_1}} \frac{(\hat{a}_{\vec{k}_2}^+)^{N_2}}{\sqrt{N_2}} |0\rangle
\]

\[
\hat{n}(\vec{r}) = \left( \sum_k \frac{e^{-i\vec{k}.\vec{r}}}{\sqrt{V}} \hat{a}_{\vec{k}}^+ \right) \left( \sum_k \frac{e^{i\vec{k}.\vec{r}}}{\sqrt{V}} \hat{a}_{\vec{k}} \right)
\]

\[
\langle \psi | \hat{n}(\vec{r}) \hat{n}(\vec{r}') |\psi\rangle \approx \left( \frac{N_1 + N_2}{V} \right)^2 
+ \frac{N_1^2 + N_2^2}{V^2} 
+ 2 \frac{N_1 N_2}{V^2} \cos \left[ (\vec{k}_1 - \vec{k}_2) \cdot (\vec{r} - \vec{r}') \right]
\]

Fringes!!!
How come???
But not very deep!
Suppose we know: \[ |\psi(t = 0)\rangle \]

Suppose at time “\(t_1\)” a photon is detected at the output port 4, then immediately afterwards the quantum state of the two cavities must be:

\[ |\psi(t_1)\rangle \propto \frac{(-\hat{a}_1 + \hat{a}_2)}{\sqrt{2}} |\psi(t = 0)\rangle \]

Suppose later at time “\(t_2\)” a second photon is detected at the output port 3, then immediately afterwards the quantum state of the two cavities must be:

\[ |\psi(t_2)\rangle \propto \frac{\hat{a}_1 + \hat{a}_2}{\sqrt{2}} \frac{(-\hat{a}_1 + \hat{a}_2)}{\sqrt{2}} |\psi(t = 0)\rangle \]
Interference in Optical Cavities: Coherent States

\[
\left| \alpha e^{i\phi_1} \right>_1 \quad \hat{b}_3 \quad \text{Cavity 1} \quad \hat{b}_1 \quad \hat{b}_2 \quad \hat{b}_4 \quad \text{Cavity 2} \\
\begin{bmatrix}
\hat{b}_3(z_o, t) \\
\hat{b}_4(z_o, t)
\end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix}
1 & 1 \\
-1 & 1
\end{bmatrix} 
\begin{bmatrix}
\hat{b}_1(z_o, t) \\
\hat{b}_2(z_o, t)
\end{bmatrix}
\]

Suppose we know:

\[
|\psi(t = 0)\rangle = |\alpha e^{i\phi_1}\rangle_1 \otimes |\alpha e^{i\phi_2}\rangle_2
\]

Suppose at time “\(t_1\)” a photon is detected

The probability that the photon is detected in port 4 is \(\sin^2\left(\frac{\phi_1 - \phi_2}{2}\right)\) and the state immediately afterwards is:

\[
|\psi(t_1)\rangle \propto (-\hat{a}_1 + \hat{a}_2)|\psi(t = 0)\rangle = -2i|\alpha|e^{i\left(\frac{\phi_1 + \phi_2}{2}\right)}\sin\left(\frac{\phi_1 - \phi_2}{2}\right)|\psi(t = 0)\rangle
\]

The probability that the photon is detected in port 3 is \(\cos^2\left(\frac{\phi_1 - \phi_2}{2}\right)\) and the state immediately afterwards is:

\[
|\psi(t_1)\rangle \propto (\hat{a}_1 + \hat{a}_2)|\psi(t = 0)\rangle = 2|\alpha|e^{i\left(\frac{\phi_1 + \phi_2}{2}\right)}\cos\left(\frac{\phi_1 - \phi_2}{2}\right)|\psi(t = 0)\rangle
\]
Interference in Optical Cavities: Coherent States

After some time “ $T$ ”, when a total of $N$ photons have been detected with $N_4$ in port 4 and $N-N_4=N_3$ in port 3, the state is:

$$\psi(t = T) \propto (\hat{a}_1 \pm \hat{a}_2)^{N_3} (-\hat{a}_1 \pm \hat{a}_2)^{N_4} \psi(t = 0)$$

$$\propto (\hat{a}_1 \pm \hat{a}_2)^{N-N_4} (-\hat{a}_1 \pm \hat{a}_2)^{N_4} \psi(t = 0)$$

$$= 2^N e^{iN\frac{\phi_1 + \phi_2}{2}} (i)^{N_4} \left( \cos\left(\frac{\phi_1 - \phi_2}{2}\right) \right)^{N-N_4} \left( \sin\left(\frac{\phi_1 - \phi_2}{2}\right) \right)^{N_4} \psi(t = 0)$$
Suppose in time “T”, N photons have been detected

The probability that $N_4$ photons have been detected in port 4, and $N-N_4=N_3$ photons have been detected in port 3, is:

$$
\frac{N!}{(N-N_4)!} \frac{N_3!}{N_4!} \left[ \sin^2 \left( \frac{\phi_1 - \phi_2}{2} \right) \right]^{N_4} \left[ \cos^2 \left( \frac{\phi_1 - \phi_2}{2} \right) \right]^{N-N_4} 
$$

$$
\approx \frac{\lambda^{N_4}}{N_4!} e^{-\lambda}
$$

Given the measured values of $N$ and $N_4$, the phase difference can be determined very optimally (up to a negative sign) using ML detection
Interference in Optical Cavities: Coherent States

Suppose the phase difference was found to be: \( \phi_1 - \phi_2 = \frac{\pi}{3} \) (or \( -\frac{\pi}{3} \))

One can then vary \( \Delta \phi_1 \) to get a complete interference pattern
Interference in Optical Cavities: Phase Difference States

One can define states of cavity 1 and cavity 2 with a well define phase difference $\phi$:

$$|\phi\rangle_m = \frac{e^{-im(\phi_1 + \phi_2)/2}}{\sqrt{2^m m!}} \left( \hat{a}_1^+ e^{i\phi_1} + \hat{a}_2^+ e^{i\phi_2} \right)^m |0\rangle_1 \otimes |0\rangle_2$$

$$= \sum_{p=0}^{m} e^{-i(m-2p)\phi} \frac{m!}{\sqrt{2^m p!(m-p)!}} |p\rangle_1 \otimes |m-p\rangle_2$$

$$\phi = \frac{\phi_1 - \phi_2}{2}$$

Of course, since the phase difference is well specified, the number difference is uncertain.

Furthermore, any state belonging to the system with total number of particles $m$ can be expanded in terms of this over-complete set of phase difference states:

$$|\psi\rangle_m = \frac{\pi/2}{\pi} \int_{-\pi/2}^{\pi/2} d\phi \ c(\phi) |\phi\rangle_m$$

$$m \langle \phi | \phi' \rangle_m = \cos^m (\phi - \phi') \approx \sqrt{\frac{2\pi}{m}} \delta(\phi - \phi')$$
Interference in Optical Cavities: Phase Difference States

\[ |\phi\rangle_m = \frac{e^{-im(\phi_1 + \phi_2)}}{\sqrt{2^m m!}} \left( \hat{a}_1^+ e^{i\phi_1} + \hat{a}_2^+ e^{-i\phi_2} \right)^m |0\rangle_1 \otimes |0\rangle_2 \]

\[ = \frac{1}{\sqrt{2^m m!}} \left( \hat{a}_1^+ e^{i\phi} + \hat{a}_2^+ e^{-i\phi} \right)^m |0\rangle_1 \otimes |0\rangle_2 \]

\[ = \sum_{p=0}^{m} e^{-i(m-2p)\phi} \frac{m!}{\sqrt{2^m p! (m-p)!}} |p\rangle_1 \otimes |m-p\rangle_2 \]

\[ \phi = \frac{\phi_1 - \phi_2}{2} \]

After some time “ \( T \)”, when a total of \( N \) photons have been detected with \( N_4 \) in port 4 and \( N-N_4=N_3 \) in port 3, the state is:

\[ |\psi(t=T)\rangle \propto (\hat{a}_1 + \hat{a}_2)^{N_3} (-\hat{a}_1 + \hat{a}_2)^{N_4} |\psi(t=0)\rangle \]

\[ \propto (\hat{a}_1 + \hat{a}_2)^{N-N_4} (-\hat{a}_1 + \hat{a}_2)^{N_4} |\phi\rangle_m \]

\[ = \sqrt{\frac{m! 2^N}{(m-N)!}} (i)^{N_4} \left( \cos\left(\frac{\phi_1 - \phi_2}{2}\right) \right)^{N-N_4} \left( \sin\left(\frac{\phi_1 - \phi_2}{2}\right) \right)^{N_4} |\phi\rangle_{m-N} \]
Interference in Optical Cavities: Phase Difference States

\[ |\psi(t = 0)\rangle = |\phi\rangle_m \]

\[ |\psi(t = T)\rangle \propto \sqrt{\frac{m!2^N}{(m-N)!}} (i)^{N_4} \left( \cos\left(\frac{\phi_1 - \phi_2}{2}\right) \right)^{N - N_4} \left( \sin\left(\frac{\phi_1 - \phi_2}{2}\right) \right)^{N_4} |\phi\rangle_{m-N} \]

The probability that \(N_4\) photons have been detected in port 4, and \(N - N_4 = N_3\) photons have been detected in port 3, is:

\[ \frac{N!}{(N-N_4)! \ N_4!} \sin^2\left(\frac{\phi_1 - \phi_2}{2}\right)^{N_4} \cos^2\left(\frac{\phi_1 - \phi_2}{2}\right)^{N - N_4} \approx \frac{\lambda^{N_4}}{N_4!} e^{-\lambda} \]

\[ \lambda = N \sin^2\left(\frac{\phi_1 - \phi_2}{2}\right) \]

Given the measured values of \(N\) and \(N_4\), the phase difference can be determined very optimally (up to a negative sign) using ML detection.

Peaks when \(N_4\) equals \(\lambda\)
Interference in Optical Cavities: Phase Difference States

\[
|\psi(t = 0)\rangle = |\phi\rangle_m
\]

Suppose the phase difference was found to be: \(\phi_1 - \phi_2 = \frac{\pi}{3}\) (or \(-\frac{\pi}{3}\))

One can then vary \(\Delta \phi_1\) to get a complete interference pattern

\[
|\psi(t = 0)\rangle = |\phi\rangle_m
\]
Suppose:

\[ \left[ \hat{b}_3(z_o, t) \right] = \frac{1}{\sqrt{2}} \left[ \begin{array}{cc} 1 & 1 \\ -1 & 1 \end{array} \right] \left[ \hat{b}_1(z_o, t) \right] \]

\[ \left[ \hat{b}_4(z_o, t) \right] = \frac{1}{\sqrt{2}} \left[ \begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array} \right] \left[ \hat{b}_2(z_o, t) \right] \]

What happens now??!!

\[ |\psi(t = 0)\rangle = \frac{m}{2} \langle 1 \otimes \frac{m}{2} \rangle \]

Will we see the interference pattern??

\[ P(N_3) \]

\[ \Delta \phi_1 \text{ (units of } \pi) \]
Suppose: \[
\begin{bmatrix}
\hat{b}_3(z_o, t) \\
\hat{b}_4(z_o, t)
\end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix}
1 & 1 \\
-1 & 1
\end{bmatrix} \begin{bmatrix}
\hat{b}_1(z_o, t) \\
\hat{b}_2(z_o, t)
\end{bmatrix}
\]

Suppose at time “t₁” a photon is detected

The probability that the photon is detected in port 4 is \(\frac{1}{2}\) and the state immediately afterwards is:

\[|\psi(t = 0)\rangle = \frac{m}{2} |1\rangle \otimes \frac{m}{2} |2\rangle\]

The probability that the photon is detected in port 3 is \(\frac{1}{2}\) and the state immediately afterwards is:

\[|\psi(t_1)\rangle \propto \frac{(-\hat{a}_1 + \hat{a}_2)}{\sqrt{2}} |\psi(t = 0)\rangle \propto \frac{1}{\sqrt{2}} \left[ \sqrt{\frac{m}{2}} \frac{m}{2} - 1 \right] \frac{m}{2} + \sqrt{\frac{m}{2}} \frac{m}{2} \frac{m}{2} - 1 \right]

Entangled states!
Interference in Optical Cavities: Number States

\[ |m/2\rangle \]

\[ \sqrt{1/2} (4) \]

\[ \sqrt{1/2} (3) \]

\[ \begin{pmatrix} -\sqrt{m/2 \ m/2-1} |m/2\rangle + \sqrt{m/2 \ m/2-1} |m/2-1\rangle \end{pmatrix} \]

\[ \begin{pmatrix} \sqrt{m/2 \ m/2-1} |m/2\rangle + \sqrt{m/2 \ m/2-1} |m/2-1\rangle \end{pmatrix} \]

\[ \sqrt{3/4} (4) \]

\[ \sqrt{1/4} (3) \]

\[ \begin{pmatrix} \sqrt{m/2 \ m/2-1} |m/2\rangle + \sqrt{m/2 \ m/2-1} |m/2-2\rangle + 2\sqrt{m/2 \ m/2-1} |m/2-1\rangle \end{pmatrix} \]

Cavity 1

\[ \hat{b}_1 \]

\[ \hat{b}_2 \]

\[ \hat{b}_3 \]

\[ \hat{b}_4 \]

Cavity 2
Interference in Optical Cavities: Number States

Suppose:

\[ |\psi(t = 0)\rangle = \left| \frac{m}{2} \right\rangle_1 \otimes \left| \frac{m}{2} \right\rangle_2 \]

For this initial state, when expanded as:

\[ |\psi(t = 0)\rangle = \frac{\pi}{2} \int_{-\pi/2}^{\pi/2} \frac{d\phi}{\pi} c(\phi) |\phi\rangle_m \]

|c(\phi)|^2 is independent of \( \phi \) (i.e. the initial state has a uniform phase difference distribution)

Suppose after some time \( "T" \), when a total of \( N \) photons have been detected, with \( N_4 \) in port 4 and \( N-N_4=N_3 \) in port 3, the state is:

\[ |\psi(t = T)\rangle \propto \left( \hat{a}_1 + \hat{a}_2 \right)^{N_3} \left( -\hat{a}_1 + \hat{a}_2 \right)^{N_4} |\psi(t = 0)\rangle \]

\[ \propto \int_{-\pi/2}^{\pi/2} \frac{d\phi}{\pi} \left( \hat{a}_1 + \hat{a}_2 \right)^{N-N_4} \left( -\hat{a}_1 + \hat{a}_2 \right)^{N_4} |\phi\rangle_m \]

\[ \propto \int_{-\pi/2}^{\pi/2} \frac{d\phi}{\pi} (\cos \phi)^{N-N_4} (\sin \phi)^{N_4} |\phi\rangle_{m-N} \]

\[ \phi = \frac{\phi_1 - \phi_2}{2} \]
Interference in Optical Cavities: Number States

Now if \( N \) is large, then as a function of \( \phi \) the integrand peaks when \( \phi \) equals \( \phi_0 \) where:

\[
\sin^2 \phi_0 = \frac{N_4}{N} \\
\cos^2 \phi_0 = \frac{N_3}{N}
\]

So we finally get:

\[
|\psi(t = T)\rangle \propto \int_{-\pi/2}^{\pi/2} \frac{d\phi}{\pi} (\cos \phi)^{N-N_4} (\sin \phi)^{N_4} |\phi\rangle_{m-N}
\]

After \( N=50 \) photon detections
Interference in Optical Cavities: Number States

As the photon detection continues, and more and more photons are detected, and the uncertainty in the photon number difference between the cavities increases, the joint quantum state of the two cavities becomes more and more a state of precise phase difference (up to a negative sign)!!!

\[ |\psi(t = T)\rangle \propto \int_0^\pi d\phi \left[ e^{-N(\phi-\phi_0)^2} + (-1)^N e^{-N(\phi+\phi_0)^2} \right] |\phi\rangle_{m-N} \]

\[ \sin^2 \phi_0 = \frac{N_4}{N} \quad \cos^2 \phi_0 = \frac{N_3}{N} \]

Suppose the phase difference was found to be:

\[ \phi_0 = \frac{\phi_{10} - \phi_{20}}{2} = \frac{\pi}{6} \quad \text{or} \quad -\frac{\pi}{6} \]

One can then vary \( \Delta \phi_1 \) to get a complete interference pattern as before!
Number State Splitting and Phase Coherence

\[ |\psi(t = 0)\rangle \]

Suppose:

\[ |\psi(t = 0)\rangle = \frac{1}{\sqrt{m!}} \left( \int_{-\infty}^{z_0} d z' A(z') \hat{b}_1^+(z',0) \right)^m |0\rangle = |m\rangle_1 \]

Then at later time when the state has passed through the beam splitter:

\[ |\psi(t)\rangle = \frac{1}{\sqrt{m!}} \left\{ \int_{-\infty}^{z_0} d z' A(z') e^{-i\omega_0 t} \left[ t |e^{i\phi} \hat{b}_3^+(z' + v_g t, 0) + r |e^{i\phi} \hat{b}_4^+(z' + v_g t, 0) \right] \right\}^m |0\rangle_3 \otimes |0\rangle_4 \]

Does the transmitted packet have any sort of phase coherence??
Number State Splitting and Phase Coherence

\[ |\psi(t = 0)\rangle \]

\[
\begin{align*}
\hat{b}_1 & \rightarrow \hat{b}_2 \\
\hat{b}_3 & \rightarrow z_o \\
\hat{b}_4 & \rightarrow z_o
\end{align*}
\]

\[
\begin{bmatrix}
\hat{b}_3(z_o, t) \\
\hat{b}_4(z_o, t)
\end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix}
t e^{i\phi_t} & -r e^{-i\phi_r} \\
re^{i\phi_r} & te^{-i\phi_t}
\end{bmatrix} \begin{bmatrix}
\hat{b}_1(z_o, t) \\
\hat{b}_2(z_o, t)
\end{bmatrix}
\]

We can always expand the state at later time in terms of the phase-difference states:

\[
|\psi(t)\rangle = \int_{-\pi/2}^{\pi/2} \frac{d\phi}{\pi} c(\phi) |\phi\rangle_m
\]

The largest weight will be for the following phase-difference state:

\[
|\phi = \frac{\phi_t - \phi_r}{2}\rangle_m = \frac{1}{\sqrt{2^m m!}} \left\{ \int_{-\infty}^{z_o} dz' A(z') e^{-i\omega_0 t} \left[ e^{i\phi} \hat{b}_3^+ (z' + v_g t, 0) + e^{-i\phi} \hat{b}_4^+ (z' + v_g t, 0) \right] \right\}^m |0\rangle_3 \otimes |0\rangle_4
\]

And the width of the distribution \(c(\phi)\) will become more and more sharply peaked around the above state as the number of photons “\(m\)” increases!

If a quantum state of photons is split, the split parts enjoy phase coherence among each other!!
Laser Coherence: A Convenient Fiction?

Optical coherence: A convenient fiction

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We conjecture that optical coherences, i.e., quantum-mechanical coherences between states separated by Bohr frequencies in the optical regime, do not exist in optics experiments. We claim the exact vanishing of optical field amplitudes and atomic dipole expectation values, and we discuss the seemingly contradictory success of assigning finite values to such quantities in theoretical calculations. We show that our conjecture is not at variance with the observed interference between different light sources. The connection to the concept of spontaneous symmetry breaking and the identification of entangled states as pointer basis states is discussed.

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Main conclusion of the paper:

Laser state cannot be, and is not, a coherent state!
The Laser State Above Threshold: Final Conclusion

Candidates:

1) There is no steady state. Phase is always diffusing. The quantum state is a coherent state with a diffusing phase:

$$\hat{\rho}(t) = |\beta| e^{i\phi(t)} \langle |\beta| e^{i\phi(t)} | \langle n | \langle n | \rightarrow \langle \hat{n} \rangle = |\beta|^2 = r_{\text{min}}^2$$

2) Phase is always diffusing. The quantum state is a statistical mixture of coherent states and all phases are possible:

$$\hat{\rho} = \frac{2\pi}{2\pi} \int_{0}^{2\pi} |\beta| e^{i\phi} \langle |\beta| e^{i\phi} | \langle n | \langle n | \rightarrow \langle \hat{n} \rangle = |\beta|^2 = r_{\text{min}}^2$$

3) The entanglement with the atom degrees of freedom makes any superposition of number states very unlikely in steady state. The reduced density matrix for photons corresponds to a quantum state which is just a statistical mixture of number states:

$$\hat{\rho} = \sum_{n=0}^{\infty} P(n) |n\rangle \langle n| \rightarrow \langle \hat{n} \rangle = |\beta|^2 = r_{\text{min}}^2$$

Most likely candidate!

Same as: $$\hat{\rho} = \int_{0}^{\infty} d|\beta| \int_{0}^{2\pi} \frac{d\phi}{2\pi} \Pi(|\beta|) |\beta| e^{i\phi} \langle |\beta| e^{i\phi} |$$

$$P(n) = \int_{0}^{\infty} d|\beta| \Pi(|\beta|) \exp(-|\beta|^2) \frac{(|\beta|^2)^n}{n!}$$