Chapter 1: Review of Quantum Mechanics

In this lecture you will learn (…..all that you might have forgotten):

• Postulates of quantum mechanics
• Commutation relations
• Schrodinger and Heisenberg pictures
• Time development
• Density operators and density matrices
• Decoherence in quantum mechanics

To brush up your quantum physics, you can also see my ECE 4060 slides at:
https://courses.cit.cornell.edu/ece4060/
Postulates of Quantum Mechanics: 1-3

There are several postulates of quantum mechanics. These are postulates since they cannot be derived from some other deeper theory. They are known only from experiments.

First three postulates are:

1) The state of physical system at time ‘t’ is described by a vector (or a ket), denoted by $|\psi(t)\rangle$, which belongs to a Hilbert space.

2) Every measurable quantity $A$ (like position or momentum of a particle) is described by an operator $\hat{A}$ that acts in the Hilbert space.

3) The only possible outcome of a measurement of the quantity $A$ is one of the eigenvalues of the operator.
Hilbert Spaces

Hilbert space is just a fancy name for a vector space with the following properties:

Vectors $|v\rangle$, $|u\rangle$, $|w\rangle$…… belong to a Hilbert space if and only if:

1) For some operation, denoted by ‘+’, $|v\rangle + |u\rangle \in H$ if $|v\rangle \in H$ and $|u\rangle \in H$

2) $|v\rangle + |u\rangle = |u\rangle + |v\rangle$ and $|v\rangle + (|u\rangle + |w\rangle) = (|v\rangle + |u\rangle) + |w\rangle$

3) $|v\rangle + 0 = |v\rangle$

4) $|v\rangle + (-|v\rangle) = 0$

5) $\alpha |v\rangle \in H$ if $|v\rangle \in H$

6) $\alpha(|v\rangle + |u\rangle) = \alpha |v\rangle + \alpha |u\rangle$ and $(\alpha + \beta) |v\rangle = \alpha |v\rangle + \beta |v\rangle$

7) Inner product: for $|v\rangle \in H$ and $|u\rangle \in H$, the inner product has the properties:

$$\langle u|v \rangle = \langle v|u \rangle^*$$

$$\langle v|v \rangle \geq 0$$

8) An operator $\hat{O}$ in the Hilbert space has the property that if $|v\rangle \in H$ then $\hat{O}|v\rangle \in H$. 
Properties of Hilbert Spaces

Example of a Hilbert Space: Column vectors

\[|v\rangle = \begin{bmatrix} a \\ b \end{bmatrix} \quad |u\rangle = \begin{bmatrix} c \\ d \end{bmatrix} \quad \langle u|v\rangle = [c^*d^*] \begin{bmatrix} a \\ b \end{bmatrix} \]

Eigenvectors and Eigenvalues:

\[\hat{O}|v\rangle = \lambda |v\rangle \rightarrow \text{Eigenvector} \]
\[|v\rangle \langle u| = [a^*b^*] [c^*d^*] = \begin{bmatrix} a^*c^* \\ b^*d^* \end{bmatrix} \]

Adjoint Operators: The adjoint operator \(\hat{O}^+\) of \(\hat{O}\) is defined by:

\[\langle u|\hat{O}|v\rangle = \langle v|\hat{O}^+|u\rangle^* \]

Let: \(|w\rangle = \hat{O}|v\rangle\) then,

\[\langle u|w\rangle = \langle w|u\rangle^* \rightarrow \text{This follows from the definition of inner product} \]

Therefore: \(\langle w| = \langle v|\hat{O}^+ \)

Hermitian Operators: An operator \(\hat{O}\) is Hermitian or self-adjoint if, \(\hat{O} = \hat{O}^+ \)

Hermitian operators have real eigenvalues:

\[\hat{O}|v\rangle = \lambda |v\rangle \]

\[\Rightarrow \langle v|\hat{O}|v\rangle = \lambda \langle v|v\rangle \Rightarrow \langle v|\hat{O}^+|v\rangle^* = \lambda^* \langle v|v\rangle \]

\[\Rightarrow \lambda^* \langle v|v\rangle = \lambda \langle v|v\rangle \Rightarrow \lambda^* = \lambda \]
Properties of Hilbert Spaces

**Basis Vectors:** A set of vectors that “span” a Hilbert space is called a basis.

If \( |v_1\rangle, |v_2\rangle, |v_3\rangle, \ldots, |v_n\rangle \) “span” a Hilbert space, then any vector \( |w\rangle \) in the Hilbert space can be written as:

\[
|w\rangle = \sum_{k=1}^{n} a_k |v_k\rangle
\]

The minimum number of vectors needed to span a Hilbert space is called the **dimensionality** of the Hilbert space.

**Orthogonal and Orthonormal Basis:** A basis set is orthogonal if

\[
\langle v_j | v_k \rangle = 0 \quad \text{if} \quad j \neq k
\]

A basis set is orthonormal if,

\[
\langle v_j | v_k \rangle = \delta_{jk}
\]

For an orthonormal basis set, if \( |w\rangle = \sum_{k=1}^{n} a_k |v_k\rangle \) then the coefficients of expansion are:

\[
a_j = \langle v_j | w \rangle
\]
Properties of Hilbert Spaces

Complete Basis: An orthonormal basis is considered complete if:

\[ \sum_{k=1}^{n} |v_k \rangle \langle v_k | = \hat{1} \rightarrow \text{Completeness relation} \]

\[ |w\rangle = \hat{1} |w\rangle = \left( \sum_{k=1}^{n} |v_k \rangle \langle v_k | \right) |w\rangle \]

\[ = \sum_{k=1}^{n} |v_k \rangle \langle v_k | w\rangle \]

\[ = \sum_{k=1}^{n} \langle v_k | w\rangle v_k \]

Eigenvectors of Hermitian Operators:

i) All eigenvectors of a Hermitian operator form a complete set

ii) Eigenvectors can be chosen to be orthonormal

\[ \hat{O} |v\rangle = \alpha |v\rangle \quad \hat{O} |u\rangle = \beta |u\rangle \]

\[ \Rightarrow \langle u | \hat{O} |v\rangle = \alpha \langle u | v\rangle \quad \Rightarrow \langle v | \hat{O}^+ |u\rangle^* = \alpha \langle u | v\rangle \]

\[ \Rightarrow \langle v | \hat{O}^+ |u\rangle^* = \beta \langle u | v\rangle \quad \Rightarrow (\beta - \alpha) \langle u | v\rangle = 0 \]

\[ \Rightarrow \langle u | v\rangle = 0 \]
Postulate 4: Measurement and Probabilities in Quantum Mechanics

Suppose the quantum state of a system is known: \( |\psi\rangle \)

**Question:** if a physical observable \( A \) is measured, what is the result?

We know from postulate 3 that the result has to be one of the eigenvalues of the operator \( \hat{A} \)

\[
\hat{A} |v_k\rangle = \lambda_k |v_k\rangle
\]

4) We cannot know for certain the result of a measurement before it is made, but the **probability** of getting \( \lambda_k \) as a result is given by:

\[
|\langle v_k | \psi \rangle|^2
\]

**Discussion:** we can write:

\[
|\psi\rangle = \sum_{k=1}^{n} a_k |v_k\rangle
\]

The state is a linear superposition of the eigenvectors of \( A \)

The probability of getting \( \lambda_k \) as result is then:

\[
|a_k|^2 = |\langle v_k | \psi \rangle|^2
\]

The probabilities must all add up to unity:

\[
1 = \langle \psi | \psi \rangle = \langle \psi | \hat{1} | \psi \rangle = \langle \psi | \sum_{k=1}^{n} v_k \langle v_k | \psi \rangle
\]

\[
= \sum_{k=1}^{n} |\langle v_k | \psi \rangle|^2 = \sum_{k=1}^{n} |a_k|^2
\]
Postulate 5: Collapse of the Quantum State upon Measurement

Suppose the quantum state of a system is: $|\psi\rangle$

Suppose a physical observable $A$ is measured

$$\hat{A}|v_k\rangle = \lambda_k |v_k\rangle$$

$$|\psi\rangle = \sum_{k=1}^{n} a_k |v_k\rangle$$

And the result is: $\lambda_j$

**Question:** What is the state of the system just after the measurement?

i) The quantum state represents knowledge all that is knowable about the system

ii) The quantum state post measurement must reflect the result of the measurement

iii) If second measurement of $A$ is made immediately after the first measurement, the result $\lambda_j$ must be obtained with probability 1

**Answer:** The state immediately after the first measurement must be: $|v_j\rangle$

$$|\psi\rangle = \sum_{k=1}^{n} a_k |v_k\rangle \xrightarrow{\text{measurement}} \frac{\lambda_j}{|\lambda_j|^2} |v_j\rangle$$

5) This sudden change in the quantum state upon measurement is called the “collapse” of the quantum state
Postulate 5: Collapse of the Quantum State upon Measurement

Suppose the quantum state of a system is: \( |\psi\rangle \)

Suppose a physical observable \( A \) is measured

\[
\hat{A}|v_k\rangle = \lambda_k |v_k\rangle
\]

\[|\psi\rangle = \sum_{k=1}^{n} a_k |v_k\rangle\]

But now \( A \) has two degenerate eigenvalues:

\[\lambda_1 = \lambda_2 = \lambda\]

Suppose the result of the measurement is: \( \lambda \)

**Question:** What is the state of the system just after the measurement?

i) The measurement cannot distinguish between \( |v_1\rangle \) and \( |v_2\rangle \)

ii) The state right after the measurement must be in the eigensubspace corresponding to the eigenvalue \( \lambda \):

\[
\left( \frac{a_1|v_1\rangle + a_2|v_2\rangle}{\sqrt{|a_1|^2 + |a_2|^2}} \right)
\]

Measurement projects the quantum state into the eigensubspace corresponding to the measured eigenvalue
Some Common Observables in Quantum Mechanics

The Position Operator in 1D:
\[ \hat{x} | x \rangle = x | x \rangle \]

Orthogonality: \[ \langle x' | x \rangle = \delta(x - x') \]

Completeness: \[ \int_{-\infty}^{\infty} dx | x \rangle \langle x | = \hat{1} \]

The Momentum Operator in 1D:
\[ \hat{p} | p \rangle = p | p \rangle \]

Orthogonality: \[ \langle p' | p \rangle = \delta(p' - p) \]

Completeness: \[ \int_{-\infty}^{\infty} dp | p \rangle \langle p | = \hat{1} \]

Position Wavefunction:
\[ | \psi \rangle = \hat{1} | \psi \rangle = \left( \int_{-\infty}^{\infty} dx | x \rangle \langle x | \right) | \psi \rangle = \int_{-\infty}^{\infty} dx \langle x | \psi \rangle | x \rangle = \int_{-\infty}^{\infty} dx \psi(x) | x \rangle \]
\[ \langle x | \psi \rangle = \psi(x) \]

Momentum Wavefunction:
\[ | \psi \rangle = \int_{-\infty}^{\infty} dp \psi(p) | p \rangle \]
\[ \langle p | \psi \rangle = \psi(p) \]
Momentum and Position Commutation Relation

Suppose my quantum state is a position eigenstate:

\[ |\psi\rangle = |x\rangle \]

And I want to expand it in momentum basis:

\[ |\psi\rangle = \hat{1}|\psi\rangle = \left( \int_{-\infty}^{\infty} dp \langle p |\psi\rangle |p\rangle \right) |\psi\rangle = \int_{-\infty}^{\infty} dp \langle p |\psi\rangle |p\rangle = \int_{-\infty}^{\infty} dp \langle p |x\rangle |p\rangle \]

What is this?

We need to know something more about the relationship between position and momentum in order to find \( \langle p |x\rangle \)

What does \( \langle p |x\rangle \) mean? What if it equals 1? What if it equals 0?

There is an intimate connection between position and momentum measurements and \( \langle p |x\rangle \)

This connection is most elegantly expressed by the operator commutation relation

\[ [\hat{x}, \hat{p}] = i\hbar \quad \rightarrow \quad \text{Fundamental property} \]

\[ \Rightarrow \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar \]
Momentum and Position Commutation Relation

Commutation Relation:
\[
[\hat{x}, \hat{p}] = i\hbar \quad \rightarrow \quad \text{Fundamental property}
\]

Can this give \( \langle p \mid x \rangle \)?

\[
[\hat{x}, \hat{p}] = i\hbar
\]

\( \Rightarrow \langle p \mid [\hat{x}, \hat{p}] \mid x \rangle = i\hbar \langle p \mid x \rangle \)

\( \Rightarrow \langle p \mid \hat{x} \hat{p} \mid x \rangle - xp\langle p \mid x \rangle = i\hbar \langle p \mid x \rangle \)

\( \Rightarrow (i\hbar + xp)\langle p \mid x \rangle = \langle p \mid \hat{x} \hat{p} \mid x \rangle \)

\( \Rightarrow (i\hbar + xp)\langle p \mid x \rangle = \langle p \mid \hat{x} \hat{p} \mid x \rangle = \langle p \mid \hat{x} \left[ \int dp' \langle p' \mid \hat{p} \rangle \langle p' \mid \hat{p} \rangle \right] \hat{p} \mid x \rangle \)

\( \Rightarrow (i\hbar + xp)\langle p \mid x \rangle = \int dp' \langle p \mid \hat{x} \rangle \langle p' \rangle \langle p' \rangle \langle p' \rangle \langle p' \rangle \hat{p} \mid x \rangle \)

\( \Rightarrow (i\hbar + xp)\langle p \mid x \rangle = \int dp' \int dx' p' x' \langle p \mid x' \rangle \langle x' \rangle \langle x' \rangle \langle x' \rangle \hat{p} \mid x \rangle \)

\( \Rightarrow (i\hbar + xp)\langle p \mid x \rangle = \int dp' \int dx' p' x' \langle p \mid x' \rangle \langle x' \rangle \langle x' \rangle \langle x' \rangle \hat{p} \mid x \rangle \)

This is an integral equation for \( \langle x \mid p \rangle \) and the solution is:

\[
\langle p \mid x \rangle = e^{-i\frac{px}{\hbar}} \quad \Rightarrow \quad \langle x \mid p \rangle = e^{i\frac{px}{\hbar}}
\]

\[
\sqrt{2\pi \hbar}
\]
Momentum and Position Commutation Relation

It follows that:

\[ |\psi\rangle = \int dx \psi(x) |x\rangle \quad \Rightarrow \quad \psi(x) = \langle x | \psi \rangle = \int dp \langle x | p \rangle \langle p | \psi \rangle = \int dp \frac{e^{i px}}{\sqrt{2\pi \hbar}} \psi(p) \]

\[ |\psi\rangle = \int dp \psi(p) |p\rangle \quad \Rightarrow \quad \psi(p) = \int dx \psi(x) \frac{e^{-i px}}{\sqrt{2\pi \hbar}} \]

There is a Fourier transform relation between the momentum and position wavefunctions!

1) \( \langle x | \hat{x} | \psi \rangle \):

\[ \langle x | \hat{x} | \psi \rangle = (\langle \psi | \hat{x}^+ | x \rangle)^* = (\langle \psi | \hat{x} | x \rangle)^* = (x \langle \psi | x \rangle)^* \]

\[ = x \langle x | \psi \rangle = x \psi(x) \]

2) \( \langle x | \hat{p} | \psi \rangle \):

\[ \langle x | \hat{p} | \psi \rangle = \langle x | \hat{p} | \psi \rangle = \langle x | \hat{p} \int dp \langle p | \psi \rangle = \int dp \langle x | p \rangle \langle p | \psi \rangle \]

\[ = \int dp \int dp \frac{e^{i px}}{\sqrt{2\pi \hbar}} \psi(p) = \frac{\hbar}{i \partial x} \left[ \int dp \frac{e^{i px}}{\sqrt{2\pi \hbar}} \psi(p) \right] = \frac{\hbar}{i} \frac{\partial \psi(x)}{\partial x} \]
Momentum and Position Commutation Relation

3) $\langle p \mid \hat{x} \mid \psi \rangle$

\[
\langle p \mid \hat{x} \mid \psi \rangle = \langle p \mid \hat{x} \hat{1} \mid \psi \rangle = \langle p \mid \hat{x} \int dx \mid x \mid \psi \rangle = \int dx \langle p \mid x \rangle x \psi(x)
\]

\[
= \int dx \frac{i \ h}{\sqrt{2 \pi \ h}} x \psi(x) = -\frac{\hbar}{i} \frac{\partial}{\partial p} \psi(p)
\]

4) $\langle p \mid \hat{p} \mid \psi \rangle$

\[
\langle p \mid \hat{p} \mid \psi \rangle = (\langle \psi \mid \hat{p} \mid p \rangle)^* = (\langle \psi \mid \hat{p} \mid p \rangle)^* = (p \langle \psi \mid p \rangle)^*
\]

\[
= p \langle p \mid \psi \rangle = p \psi(p)
\]
Mean (or Expectation) Values and Standard Deviations of Operators

Mean Value:
\[ \langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle \]

Suppose:
\[ \hat{A} | \psi_k \rangle = \lambda_k | \psi_k \rangle \quad | \psi \rangle = \sum_k a_k | \psi_k \rangle \]

Then:
\[ \langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle = \sum_k |a_k|^2 \lambda_k \]

Standard Deviation:

Define:
\[ \Delta \hat{A} = \hat{A} - \langle \hat{A} \rangle \]

Then:
\[ \Rightarrow \Delta \hat{A}^2 = \hat{A}^2 + \langle \hat{A} \rangle^2 - 2\hat{A} \langle \hat{A} \rangle \]

Then:
\[ \langle \Delta \hat{A}^2 \rangle = \langle \psi | \hat{A}^2 + \langle \hat{A} \rangle^2 - 2\hat{A} \langle \hat{A} \rangle | \psi \rangle = \langle \psi | \hat{A}^2 | \psi \rangle - \langle \psi | \hat{A} | \psi \rangle^2 \]
\[ \langle \Delta \hat{A}^2 \rangle = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2 \]
The Fourier Transform Relation Between Momentum and Position Wavefunctions

\[ \psi(x) = \int_{-\infty}^{\infty} dp \frac{e^{i px}}{\sqrt{2\pi\hbar}} \psi(p) \]

\[ \psi(p) = \int_{-\infty}^{\infty} dx \psi(x) \frac{e^{-i px}}{\sqrt{2\pi\hbar}} \]

\[ \Rightarrow \langle \Delta \hat{x}^2 \rangle \langle \Delta \hat{p}^2 \rangle \geq \frac{\hbar^2}{4} \]

Can we generalize this observation to all observables and operators?
Heisenberg Uncertainty Relations

If: \[ [\hat{A}, \hat{B}] = iC \]

Then for any quantum state: \[ \langle \Delta \hat{A}^2 \rangle \langle \Delta \hat{B}^2 \rangle \geq \frac{C^2}{4} \]

Example:

We know that: \[ [\hat{x}, \hat{p}] = i\hbar \]

Therefore for any quantum state: \[ \langle \Delta \hat{x}^2 \rangle \langle \Delta \hat{p}^2 \rangle \geq \frac{\hbar^2}{4} \]

This is not a surprise since the same Fourier transform relation,

\[ \psi(x) = \int_{-\infty}^{\infty} dp \frac{e^{ipx}}{\sqrt{2\pi \hbar}} \psi(p) \]

already told us that:

\[ \langle \Delta \hat{x}^2 \rangle \langle \Delta \hat{p}^2 \rangle \geq \frac{\hbar^2}{4} \]
Heisenberg Uncertainty Relations: Proof

If: \[ \begin{bmatrix} \hat{A} \\ \hat{B} \end{bmatrix} = i C \quad \{ \text{C is a real number, and } \hat{A} \text{ and } \hat{B} \text{ are Hermitian operators} \} \]

then for any quantum state: \[ \langle \Delta \hat{A}^2 \rangle \langle \Delta \hat{B}^2 \rangle \geq \frac{C^2}{4} \]

For some real number \( \lambda \), consider the state: \[ |\phi\rangle = (\Delta \hat{A} + i \lambda \Delta \hat{B})|\psi\rangle \]

Now we know that \( \langle \phi | \phi \rangle \geq 0 \), therefore:

\[
\begin{align*}
\Rightarrow & \quad \langle \psi | (\Delta \hat{A}^+ - i \lambda \Delta \hat{B}^+) (\Delta \hat{A} + i \lambda \Delta \hat{B}) |\psi\rangle \geq 0 \\
\Rightarrow & \quad \langle \psi | \Delta \hat{A}^2 + \lambda^2 \Delta \hat{B}^2 + i \lambda \left[ \Delta \hat{A}, \Delta \hat{B} \right] |\psi\rangle \geq 0 \\
\Rightarrow & \quad \langle \psi | \Delta \hat{A}^2 + \lambda^2 \Delta \hat{B}^2 - \lambda C |\psi\rangle \geq 0 \\
\Rightarrow & \quad \langle \Delta \hat{A}^2 \rangle + \lambda^2 \langle \Delta \hat{B}^2 \rangle - \lambda C \geq 0
\end{align*}
\]

The above must hold for all values of \( \lambda \) and this can only happen if,

\[ C^2 - 4 \langle \Delta \hat{A}^2 \rangle \langle \Delta \hat{B}^2 \rangle \leq 0 \]

\[ \Rightarrow \langle \Delta \hat{A}^2 \rangle \langle \Delta \hat{B}^2 \rangle \geq \frac{C^2}{4} \]
Heisenberg Uncertainty Relations and Measurements

\[ [\hat{A}, \hat{B}] = iC \quad \Rightarrow \quad \langle \Delta \hat{A}^2 \rangle \langle \Delta \hat{B}^2 \rangle \geq \frac{C^2}{4} \]

Consider an experiment in which one measures the observable \( A \) and the state post measurement is: \( |\psi\rangle \)

\[ \langle \phi | \Delta \hat{B}^2 | \phi \rangle \geq \frac{C^2}{4 \langle \phi | \Delta \hat{A}^2 | \phi \rangle} \]

The greater the accuracy/uncertainty in determining \( A \), the bigger the uncertainty in \( B \) just after the measurement

Measurements and Heisenberg Uncertainty Relation:

Turns out that the above inequality is a bit deceiving. Under the assumption that the state observables all commute with the observables of the measurement apparatus, which is a reasonable assumption, the correct relation is:

\[ \langle \psi | \Delta \hat{B}^2 | \psi \rangle \geq \frac{C^2}{\langle \psi | \Delta \hat{A}^2 | \psi \rangle} \]

\( \text{LHS is 4 times larger!!} \)

Heisenberg Uncertainty Relations and Measurements

\[
[\hat{A}, \hat{B}] = i\hbar \quad \Rightarrow \quad \langle \Delta \hat{A}^2 \rangle \langle \Delta \hat{B}^2 \rangle \geq \frac{\hbar^2}{4}
\]

Consider an experiment in which one measures the observable \( A \) and the state post measurement is: \( |\psi\rangle \)

\[
\text{Measurement of } A \quad \Rightarrow \quad |\psi\rangle
\]

\[
\Rightarrow \langle \psi | \Delta \hat{B}^2 | \psi \rangle \geq \frac{\hbar^2}{\langle \psi | \Delta \hat{A}^2 | \psi \rangle}
\]

The greater the accuracy/uncertainty in determining \( A \), the bigger the uncertainty in \( B \) just after the measurement

Example: Position and Momentum (Heisenberg Microscope)

Suppose one tries to measure the position of an electron with certainty \( \Delta x \)

One needs a photon of wavelength \( \lambda \) where, \( \lambda \sim \Delta x \)

\[
\Rightarrow \text{Need a photon of momentum: } p \sim \frac{\hbar}{\lambda} = \frac{\hbar}{\Delta x}
\]

Resulting uncertainty in the particle momentum after scattering:

\[
\Delta p \sim p \sim \frac{\hbar}{\Delta x}
\]

\[
\Rightarrow \Delta x \Delta p \sim \hbar
\]
Heisenberg Uncertainty Relations and Measurements

\[ [\hat{x}, \hat{p}] = i\hbar \]

\[ \langle p | x \rangle = e^{-\frac{i p x}{\hbar}} = \langle x | p \rangle^* \]

\[ \psi(p) = \int_{-\infty}^{\infty} dx \psi(x) e^{-\frac{i p x}{\hbar}} \]

\[ \langle \Delta \hat{x}^2 \rangle \langle \Delta \hat{p}^2 \rangle \geq \frac{\hbar^2}{4} \]
The Hamiltonian Operator

The operator corresponding to energy (a physical observable) is the Hamiltonian operator

For a particle of mass \( m \) in a potential \( V(x) \), the Hamiltonian operator is:

\[
\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})
\]
Postulate 6: Time Development in Quantum Mechanics

The time evolution of a quantum state $|\psi(t)\rangle$ is given by the equation:

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle$$

If the Hamiltonian is time independent then the formal solution subject to the boundary condition that at time $t=0$ the state is $|\psi(t = 0)\rangle$ is:

$$|\psi(t)\rangle = e^{\frac{-i\hat{H}t}{\hbar}} |\psi(t = 0)\rangle = \left(1 + \left(-\frac{i}{\hbar} \hat{H}t\right) + \frac{1}{2!}\left(-\frac{i}{\hbar} \hat{H}t\right)^2 + \ldots\right) |\psi(t = 0)\rangle$$

Stationary states: Eigenstates of the Hamiltonian are called stationary states

Suppose:

$$\hat{H} |e_k\rangle = \varepsilon_k |e_k\rangle$$

Then:

$$|\psi(t)\rangle = e^{\frac{-i\hat{H}t}{\hbar}} \sum_k c_k |e_k\rangle = \sum_k c_k e^{\frac{-i\varepsilon_k t}{\hbar}} |e_k\rangle$$

The probability of being in a particular energy eigenstate does not change with time:

$$\langle e_j |\psi(t)\rangle^2 = \langle e_j |\psi(t = 0)\rangle^2 = |c_j|^2$$
Matrix Representation of Operators

Suppose we have a complete basis set:

\[ \sum_k \omega_k \langle \omega_k | \hat{A} \rangle = \hat{1} \]

Then we can write any operator as:

\[ \hat{A} = \hat{1} \hat{A} \hat{1} \]

\[ = \sum_k \omega_k \langle \omega_k | \hat{A} \sum_j \omega_j \rangle \omega_j \]

\[ = \sum_{kj} \langle \omega_k | \hat{A} | \omega_j \rangle \omega_k \rangle \omega_j \]

\[ = \sum_{kj} A_{kj} \langle \omega_k | \hat{A} | \omega_j \rangle \]

\[ A_{kj} = \langle \omega_k | \hat{A} | \omega_j \rangle \]

Matrix representation:

\[
\begin{bmatrix}
1 \\
0 \\
0 \\
\vdots
\end{bmatrix}
\begin{bmatrix}
0 \\
1 \\
0 \\
\vdots
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
1 \\
\vdots
\end{bmatrix}
\ldots
\]

\[
\hat{A} =
\begin{bmatrix}
A_{11} & A_{12} & \ldots \\
A_{21} & A_{22} & \ldots \\
\vdots & \vdots & \ddots
\end{bmatrix}
\]

Matrix representation is a mapping between the Hilbert space of physical states and the Hilbert space of column vectors.
Matrix Representation of Operators: The Hamiltonian Operator

Suppose:

$$\hat{H}|e_k\rangle = \varepsilon_k|e_k\rangle$$

Then:

$$\hat{H} = \hat{1} \hat{H} \hat{1}$$

$$= \left(\sum_k |e_k\rangle \langle e_k|\right) \hat{H} \left(\sum_j |e_j\rangle \langle e_j|\right)$$

$$= \sum_j \sum_k |e_k\rangle \langle e_k| e_j \langle e_j| \varepsilon_j \langle e_j|$$

$$= \sum_j \sum_k |e_k\rangle \delta_{kj} \varepsilon_j \langle e_j|$$

$$\hat{H} = \sum_k \varepsilon_k |e_k\rangle \langle e_k|$$

Every operator is diagonal in its own eigenbasis

$$\hat{H} = \begin{bmatrix} \varepsilon_1 & 0 & \cdots \\ 0 & \varepsilon_2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$
The Hamiltonian Operator with Added Perturbation

Suppose we had a Hamiltonian of a particle in a potential:

\[ \hat{H}_o = \frac{\hat{p}^2}{2m} + V(\hat{x}) \]

And we had found all its eigenstates and eigenvalues:

\[ \hat{H}_o |e_k\rangle = \varepsilon_k |e_k\rangle \]

Now suppose an additional potential is added to the Hamiltonian:

\[ \hat{H} = \hat{H}_o + U(\hat{x}) = \frac{\hat{p}^2}{2m} + V(\hat{x}) + U(\hat{x}) \]

Let's write the new full Hamiltonian in matrix representation using the eigenbasis of the original Hamiltonian:

\[
\hat{H} = \sum_k \varepsilon_k |e_k\rangle \langle e_k| + \sum_k \sum_j U_{kj} |e_k\rangle \langle e_j| \quad \{ U_{kj} = \langle e_k |U(\hat{x})| e_j \rangle \}
\]
The Hamiltonian for two completely isolated potential wells (when they are far away from each other) is approximately:

\[ \hat{H}_o = \varepsilon |e_1\rangle \langle e_1| + \varepsilon |e_2\rangle \langle e_2| \]

Only diagonal elements

The Hamiltonian for two potential wells coupled via tunneling is:

\[ \hat{H} = \varepsilon |e_1\rangle \langle e_1| + \varepsilon |e_2\rangle \langle e_2| - U |e_1\rangle \langle e_2| - U |e_2\rangle \langle e_1| \]

Off-diagonal elements!
**Dynamics of a Two-Level System**

Initial Condition: \( \psi(t = 0) = \left| e_1 \right> \)

**Question:** What is \( \psi(t) \)?

Write the Solution as: \( \psi(t) = c_1(t) \left| e_1 \right> + c_2(t) \left| e_2 \right> \)

And plug into:

\[
i\hbar \frac{\partial}{\partial t} \left[ c_1(t) \left| e_1 \right> + c_2(t) \left| e_2 \right> \right] = \hat{H} \left[ c_1(t) \left| e_1 \right> + c_2(t) \left| e_2 \right> \right]
\]

Multiply by bras \( \langle e_1 | \) and \( \langle e_2 | \) to get:

\[
i\hbar \frac{\partial}{\partial t} c_1(t) = \varepsilon c_1(t) - U c_2(t)
\]

\[
i\hbar \frac{\partial}{\partial t} c_2(t) = \varepsilon c_2(t) - U c_1(t)
\]

**Answer is:**

\[
c_1(t) = e^{-i\frac{\varepsilon t}{\hbar}} \cos \left( \frac{Ut}{\hbar} \right)
\]

\[
c_2(t) = i e^{-i\frac{\varepsilon t}{\hbar}} \sin \left( \frac{Ut}{\hbar} \right)
\]
Dynamics of a Two-Level System

Solution is:

\[ \lvert \psi(t) \rvert = e^{-\frac{i \epsilon t}{\hbar}} \cos \left( \frac{Ut}{\hbar} \right) \lvert e_1 \rangle + i e^{-\frac{i \epsilon t}{\hbar}} \sin \left( \frac{Ut}{\hbar} \right) \lvert e_2 \rangle \]

What are the probabilities of finding the particle in well 1 and well 2 at time \( t \)?

\[ \lvert \langle e_1 \lvert \psi(t) \rangle \rvert^2 = \lvert c_1(t) \rvert^2 = \cos^2 \left( \frac{Ut}{\hbar} \right) \]

\[ \lvert \langle e_2 \lvert \psi(t) \rangle \rvert^2 = \lvert c_2(t) \rvert^2 = \sin^2 \left( \frac{Ut}{\hbar} \right) \]

The particle oscillates between the two wells!
Dynamics of a Two-Level System

Solution Using the Exact Eigenstates of the Complete Hamiltonian:

$$\hat{H} = \hat{H}_o + U(\hat{x})$$

$$= \varepsilon |e_1\rangle \langle e_1| + \varepsilon |e_2\rangle \langle e_2| - U |e_1\rangle \langle e_2| - U |e_2\rangle \langle e_1|$$

Let: $$|e_1\rangle \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$ $$|e_2\rangle \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\hat{H} = \begin{bmatrix} \varepsilon & -U \\ -U & \varepsilon \end{bmatrix}$$

The eigenstates of the Hamiltonian are:

$$\varepsilon - U \quad \leftrightarrow \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\varepsilon + U \quad \leftrightarrow \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$|v_1\rangle = \frac{1}{\sqrt{2}} (|e_1\rangle + |e_2\rangle)$$

$$|v_2\rangle = \frac{1}{\sqrt{2}} (|e_1\rangle - |e_2\rangle)$$
Dynamics of a Two-Level System

Expand the initial state in the eigenstates:

\[ |\psi(t = 0)\rangle = |e_1\rangle = \frac{1}{\sqrt{2}} (|v_1\rangle + |v_2\rangle) \]

Find the state at any later time:

\[ |\psi(t)\rangle = e^{\frac{i}{\hbar} \hat{H} t} |\psi(t = 0)\rangle = \frac{1}{\sqrt{2}} \left[ e^{\frac{-i}{\hbar} (\epsilon - U) t} |v_1\rangle + e^{\frac{-i}{\hbar} (\epsilon + U) t} |v_2\rangle \right] \]

What are the probabilities of finding the particle in well 1 and well 2 at time \( t \)?

\[ |\langle e_1 | \psi(t) \rangle |^2 = \frac{1}{\sqrt{2}} (\langle v_1 | + \langle v_2 |) |\psi(t)\rangle |^2 = \cos^2 \left( \frac{Ut}{\hbar} \right) \]

\[ |\langle e_2 | \psi(t) \rangle |^2 = \sin^2 \left( \frac{Ut}{\hbar} \right) \]
Fermi’s Golden Rule

\[ |e_0\rangle \quad |e_k\rangle \quad D(E) = \sum_{k=1}^{\infty} \delta(E - \varepsilon_k) \]

Consider a single state coupled to many states – a continuum of states:

\[ \hat{H} = \varepsilon_0 |e_0\rangle \langle e_0| + \sum_{k=1}^{\infty} \varepsilon_k |e_k\rangle \langle e_k| - \sum_{k=1}^{\infty} \left[ U_k |e_0\rangle \langle e_k| + U^*_k |e_k\rangle \langle e_0| \right] \]

Suppose: \[ |\psi(t = 0)\rangle = |e_0\rangle \]

Then find: \[ \left| \langle e_0 | \psi(t) \rangle \right|^2 \]

Let: \[ |\psi(t)\rangle = b_0(t) e^{-\frac{i}{\hbar} \varepsilon_0 t} |e_0\rangle + \sum_{k=1}^{\infty} b_k(t) e^{-\frac{i}{\hbar} \varepsilon_k t} |e_k\rangle \]

And then use:

\[ i \hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle \]
Fermi’s Golden Rule

\[ D(E) = \sum_{k=1}^{\infty} \delta(E - \varepsilon_k) \]

One gets the following equations:

\[ \hbar \frac{\partial b_0(t)}{\partial t} = -\sum_{k=1}^{\infty} U_k e^{-\frac{i(\varepsilon_k - \varepsilon_0)t}{\hbar}} b_k(t) \]

\[ b_0(t = 0) = 1 \]
\[ b_k(t = 0) = 0 \quad \{k = 1, 2, 3, \ldots \} \]

\[ i\hbar \frac{\partial b_k(t)}{\partial t} = -U_k^* b_0(t) e^{\frac{i(\varepsilon_k - \varepsilon_0)t}{\hbar}} \]

Solve the second one:

\[ b_k(t) = \frac{i}{\hbar} U_k^* \int_0^t b_0(t') e^{\frac{i(\varepsilon_k - \varepsilon_0)t'}{\hbar}} \, dt' \]

And substitute in the first one to get:

\[ \frac{\partial b_0(t)}{\partial t} = -\frac{1}{\hbar^2} \sum_{k=1}^{\infty} \left| U_k \right|^2 \int_0^t dt' e^{-\frac{i(\varepsilon_k - \varepsilon_0)(t-t')}{\hbar}} b_0(t') \]

Note the correspondence between summation and integration for any quantity \( A \):

\[ \sum_{k=1}^{\infty} A_k \rightarrow \int_{-\infty}^{\infty} dE \ D(E) \ A(E) \]
Fermi's Golden Rule: Another Math Route

![Diagram of Fermi's Golden Rule]

\[ D(E) = \sum_{k=1}^{\infty} \delta(E - \varepsilon_k) \]

\[ \Rightarrow \frac{\partial b_0(t)}{\partial t} = -\frac{1}{\hbar^2} \sum_{k=1}^{\infty} |U_k|^2 \int_{0}^{t} dt' e^{-i\frac{(\varepsilon_k - \varepsilon_0)(t-t')}{\hbar}} b_0(t') \]

\[ = -\frac{1}{\hbar^2} \int_{0}^{t} dt' \left[ \int_{-\infty}^{+\infty} dE \ D(E) \ |U(E)|^2 \ e^{-i\frac{(E - \varepsilon_0)(t-t')}{\hbar}} \right] b_0(t') \]

Assume constant (energy independent) density of states and coupling

\[ \Gamma = \frac{2\pi}{\hbar} \sum_{k=1}^{\infty} |U_k|^2 \delta(\varepsilon_k - \varepsilon_0) \]

\[ \Gamma = \frac{2\pi}{\hbar} |U(\varepsilon_0)|^2 D(\varepsilon_0) \]
A Time-Dependent Two-Level System

Consider a two-level system consisting of a single potential well with two confined energy levels, as shown below

\[
\begin{align*}
|e_1\rangle & \quad \quad \quad \quad \quad |e_2\rangle \\
\varepsilon_1 & \quad \quad \quad \quad \quad \varepsilon_2
\end{align*}
\]

\[\hat{H} = \varepsilon_1 |e_1\rangle \langle e_1| + \varepsilon_2 |e_2\rangle \langle e_2|\]

In the presence of a time dependent sinusoidal electric field the Hamiltonian is,

\[
\begin{align*}
\phi(x,t) &= -x E_0 \cos(\omega t) \\
E_x(t) &= -\frac{\partial \phi(x,t)}{\partial x} = E_0 \cos(\omega t)
\end{align*}
\]

\[\hat{H}(t) = \varepsilon_1 |e_1\rangle \langle e_1| + \varepsilon_2 |e_2\rangle \langle e_2| + q \phi(\hat{x},t)\]

\[\hat{H}(t) = \varepsilon_1 |e_1\rangle \langle e_1| + \varepsilon_2 |e_2\rangle \langle e_2| - q \hat{x} E_0 \cos(\omega t)\]
A Time-Dependent Two-Level System

In the presence of a time dependent sinusoidal electric field the Hamiltonian is,

\[ \hat{H}(t) = \varepsilon_1 |e_1\rangle\langle e_1| + \varepsilon_2 |e_2\rangle\langle e_2| - q \hat{x} E_0 \cos(\omega t) \]

Resolution of the identity:

\[ \sum_j |e_j\rangle\langle e_j| = |e_1\rangle\langle e_1| + |e_2\rangle\langle e_2| = \hat{1} \]

\[ \hat{H}(t) = \hat{1} \hat{H}(t) \hat{1} \]

\[ \hat{H}(t) = \varepsilon_1 |e_1\rangle\langle e_1| + \varepsilon_2 |e_2\rangle\langle e_2| - \frac{\hbar \Omega_R}{2} \left[ 2 \cos(\omega t) |e_1\rangle\langle e_2| + 2 \cos(\omega t) |e_2\rangle\langle e_1| \right] \]

\[ \hbar \Omega_R = q E_0 \langle e_2 | \hat{x} | e_1 \rangle = q E_0 \langle e_1 | \hat{x} | e_2 \rangle \quad \text{Rabi energy} \]
A Time-Dependent Two-Level System: Rotating Wave Approximation

Rabi energy: \[ \hbar \Omega_R = q E_o \langle e_2 | \hat{x} | e_1 \rangle = q E_o \langle e_1 | \hat{x} | e_2 \rangle \]

\[ \hat{H}(t) = \varepsilon_1 |e_1\rangle \langle e_1| + \varepsilon_2 |e_2\rangle \langle e_2| - \frac{\hbar \Omega_R}{2} \left[ 2 \cos(\omega t) |e_1\rangle \langle e_2| + 2 \cos(\omega t) |e_2\rangle \langle e_1| \right] \]

Make the rotating wave approximation:

\[ \hat{H}(t) = \varepsilon_1 |e_1\rangle \langle e_1| + \varepsilon_2 |e_2\rangle \langle e_2| - \frac{\hbar \Omega_R}{2} \left[ \exp(i\omega t) |e_1\rangle \langle e_2| + \exp(-i\omega t) |e_2\rangle \langle e_1| \right] \]

Need to solve:

\[ i\hbar \frac{\partial |\psi(t)\rangle}{\partial t} = \hat{H}(t) |\psi(t)\rangle \]
Converting a Time-Dependent Two-Level System into a Time-Independent One

\[ \hat{H}(t) = \varepsilon_1 |e_1\rangle \langle e_1| + \varepsilon_2 |e_2\rangle \langle e_2| - \frac{\hbar \Omega_R}{2} \left[ \exp(i \omega t) |e_1\rangle \langle e_2| + \exp(-i \omega t) |e_2\rangle \langle e_1| \right] \]

Need to solve:

\[ i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H}(t) |\psi(t)\rangle \]

Define a new state as follows:

\[ |\phi(t)\rangle = \exp(-i \omega t |e_1\rangle \langle e_1|) |\psi(t)\rangle \]

The new state obeys the following equation (HW 1 will have the details):

\[ i\hbar \frac{\partial}{\partial t} |\phi(t)\rangle = \hat{H}_R |\phi(t)\rangle \]

Where \( \hat{H}_R \) is a time-independent Hamiltonian-like operator:

\[ \hat{H}_R = (\varepsilon_1 + \hbar \omega) |e_1\rangle \langle e_1| + \varepsilon_2 |e_2\rangle \langle e_2| - \frac{\hbar \Omega_R}{2} \left[ |e_1\rangle \langle e_2| + |e_2\rangle \langle e_1| \right] \]

The above looks like the Hamiltonian of a time-independent two-level system!
Suppose, given an initial state \( |\psi(t = 0)\rangle \), we need to find the mean value, \( \langle \psi(t) | \hat{A} | \psi(t) \rangle \)

at some later time

**Schroedinger Picture:**

1) Find \( |\psi(t)\rangle \) using:

\[
 i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle \quad \Rightarrow \quad |\psi(t)\rangle = e^{-\frac{i}{\hbar} \hat{H} t} |\psi(t = 0)\rangle
\]

2) Then calculate:

\[
 \langle \psi(t) | \hat{A} | \psi(t) \rangle
\]

**Heisenberg Picture:** Notice that,

\[
 \langle \psi(t) | \hat{A} | \psi(t) \rangle = \langle \psi(t = 0) | e^{\frac{i}{\hbar} \hat{H} t} \hat{A} e^{-\frac{i}{\hbar} \hat{H} t} | \psi(t = 0) \rangle = \langle \psi(t = 0) | \hat{A}(t) | \psi(t = 0) \rangle
\]

Define the time-dependent operator as:

\[
 \hat{A}(t) = e^{\frac{i}{\hbar} \hat{H} t} \hat{A} e^{-\frac{i}{\hbar} \hat{H} t}
\]
Schrodinger Picture and Heisenberg Pictures

\[ \hat{A}(t) = e^{\frac{i\hat{H}}{\hbar}t} \hat{A} e^{-\frac{i\hat{H}}{\hbar}t} \]

Differentiate the above w.r.t. to get:

\[ i\hbar \frac{d\hat{A}(t)}{dt} = [\hat{A}(t), \hat{H}] \quad \text{Heisenberg equation} \]

1) Find \( \hat{A}(t) \) using,

\[ i\hbar \frac{d\hat{A}(t)}{dt} = [\hat{A}(t), \hat{H}] \quad \text{Boundary condition:} \quad \hat{A}(t = 0) = \hat{A} \]

2) Then calculate:

\[ \langle \psi(t) \vert \hat{A} \vert \psi(t) \rangle = \langle \psi(t = 0) \vert \hat{A}(t) \vert \psi(t = 0) \rangle = \langle \hat{A}(t) \rangle \]

In the Schrodinger picture the states evolve in time but the operators do not evolve.

In the Heisenberg picture the operators evolve in time but the states do not evolve.
Heisenberg Picture

The Hamiltonian is always time-independent:

\[
\hat{H}(t) = e^{\frac{i\hat{H}}{\hbar}t} \hat{H} e^{-\frac{i\hat{H}}{\hbar}t} = \hat{H}
\]

The commutation relation do not change with time:

Suppose: \( [\hat{A}, \hat{B}] = \hat{C} \)

Then:

\[
[\hat{A}(t), \hat{B}(t)] = e^{\frac{i\hat{H}}{\hbar}t} (\hat{A} \hat{B} - \hat{B} \hat{A}) e^{-\frac{i\hat{H}}{\hbar}t} = e^{\frac{i\hat{H}}{\hbar}t} \hat{C} e^{-\frac{i\hat{H}}{\hbar}t} = \hat{C}(t)
\]

More precisely, the equal-time commutation relations maintain their form!
Dynamics of a Two Level System in the Heisenberg Picture

\[ \hat{H} = \varepsilon |e_1\rangle \langle e_1| + \varepsilon |e_2\rangle \langle e_2| - U |e_1\rangle \langle e_2| - U |e_2\rangle \langle e_1| \]

Define a few operators as follows:

\[ \hat{N}_1 = |e_1\rangle \langle e_1| \quad \hat{\sigma}_- = |e_1\rangle \langle e_2| \]
\[ \hat{N}_2 = |e_2\rangle \langle e_2| \quad \hat{\sigma}_+ = |e_2\rangle \langle e_1| \quad \{ \hat{\sigma}_- = \hat{\sigma}_+ \} \]

The Hamiltonian is then:

\[ \hat{H} = \varepsilon (\hat{N}_1 + \hat{N}_2) - U (\hat{\sigma}_- + \hat{\sigma}_+) \]

Given \( |\psi(t = 0)\rangle = |e_1\rangle \) we need to find \( |\langle e_1 | \psi(t) \rangle|^2 \)

\[ |\langle e_1 | \psi(t) \rangle|^2 = \langle \psi(t) | e_1 \rangle \langle e_1 | \psi(t) \rangle = \langle \psi(t) | \hat{N}_1 | \psi(t) \rangle = \langle \psi(t = 0) | \hat{N}_1(t) | \psi(t = 0) \rangle \]
\[ |\langle e_2 | \psi(t) \rangle|^2 = \langle \psi(t = 0) | \hat{N}_2(t) | \psi(t = 0) \rangle \]
Dynamics of a Two Level System in the Heisenberg Picture

You can verify the following commutation relations:

\[
\begin{align*}
\left[ \hat{N}_1, \hat{\sigma}_- \right] &= \hat{\sigma}_- & \left[ \hat{N}_2, \hat{\sigma}_- \right] &= -\hat{\sigma}_- \\
\left[ \hat{N}_1, \hat{\sigma}_+ \right] &= -\hat{\sigma}_+ & \left[ \hat{N}_2, \hat{\sigma}_+ \right] &= \hat{\sigma}_+ \\
\left[ \hat{\sigma}_+, \hat{\sigma}_- \right] &= \hat{N}_2 - \hat{N}_1
\end{align*}
\]

Example:

\[
\left[ \hat{N}_1, \hat{\sigma}_- \right] = \hat{N}_1 \hat{\sigma}_- - \hat{\sigma}_- \hat{N}_1 = |e_1\rangle\langle e_1| |e_1\rangle\langle e_2| - |e_1\rangle\langle e_2| |e_1\rangle\langle e_1| = |e_1\rangle\langle e_2| = \hat{\sigma}_-
\]

Find \( \hat{N}_1(t) \) using:

\[
\begin{align*}
i\hbar \frac{d\hat{N}_1(t)}{dt} &= \left[ \hat{N}_1(t), \hat{H} \right] = U \left[ \hat{\sigma}_+(t) - \hat{\sigma}_-(t) \right] = -i\hbar \frac{d\hat{N}_2(t)}{dt} \\
i\hbar \frac{d\hat{\sigma}_-(t)}{dt} &= \left[ \hat{\sigma}_-(t), \hat{H} \right] = U \left[ \hat{N}_2(t) - \hat{N}_1(t) \right] = -i\hbar \frac{d\hat{\sigma}_+(t)}{dt}
\end{align*}
\]

Notice that:

\[
\frac{d}{dt} \left[ \hat{N}_1(t) + \hat{N}_2(t) \right] = 0 \Rightarrow \hat{N}_1(t) + \hat{N}_2(t) = \hat{N}_1 + \hat{N}_2
\]
Define the population difference operator as:

\[ \hat{N}_d(t) = \hat{N}_2(t) - \hat{N}_1(t) \]

Equation for it is:

\[
\frac{d^2 \hat{N}_d(t)}{dt} = -\frac{4U^2}{\hbar^2} \hat{N}_d(t)
\]

Boundary conditions:

\[ \hat{N}_d(t = 0) = \hat{N}_d = \hat{N}_2 - \hat{N}_1 \]
\[ \frac{d\hat{N}_d(t)}{dt} \bigg|_{t=0} = \frac{2iU}{\hbar} (\hat{\sigma}_+ - \hat{\sigma}_-) \]

Solution is:

\[ \hat{N}_d(t) = \hat{N}_d \cos\left(\frac{2U}{\hbar} t\right) + i (\hat{\sigma}_+ - \hat{\sigma}_-) \sin\left(\frac{2U}{\hbar} t\right) \]

Finally:

\[
\hat{N}_1(t) = \frac{[\hat{N}_1(t) + \hat{N}_2(t)] - \hat{N}_d(t)}{2} = \frac{(\hat{N}_1 + \hat{N}_2 - \hat{N}_d(t)}{2}
\]

\[ \hat{N}_1(t) = \hat{N}_1 \cos^2\left(\frac{U}{\hbar} t\right) + \hat{N}_2 \sin^2\left(\frac{U}{\hbar} t\right) - \frac{i}{2} (\hat{\sigma}_+ - \hat{\sigma}_-) \sin\left(\frac{2U}{\hbar} t\right) \]
Dynamics of a Two Level System in the Heisenberg Picture

We had:

\[
\hat{N}_1(t) = \hat{N}_1 \cos^2 \left( \frac{Ut}{\hbar} \right) + \hat{N}_2 \sin^2 \left( \frac{Ut}{\hbar} \right) - \frac{i}{2} (\hat{\sigma}_+ - \hat{\sigma}_-) \sin \left( \frac{2Ut}{\hbar} \right)
\]

\[
\hat{N}_2(t) = \hat{N}_1 \sin^2 \left( \frac{Ut}{\hbar} \right) + \hat{N}_2 \cos^2 \left( \frac{Ut}{\hbar} \right) + \frac{i}{2} (\hat{\sigma}_+ - \hat{\sigma}_-) \sin \left( \frac{2Ut}{\hbar} \right)
\]

We needed to find:

\[
| \langle e_1 | \psi(t) \rangle |^2 = \langle \psi(t = 0) | \hat{N}_1(t) | \psi(t = 0) \rangle
\]

\[
= \langle e_1 | \hat{N}(t) | e_1 \rangle
\]

\[
= \cos^2 \left( \frac{Ut}{\hbar} \right)
\]

Same as obtained in the Schrodinger picture!

Similarly:

\[
| \langle e_2 | \psi(t) \rangle |^2 = \langle e_1 | \hat{N}_2(t) | e_1 \rangle = \sin^2 \left( \frac{Ut}{\hbar} \right)
\]
Heisenberg Uncertainty Relations and Measurements

\[ [\hat{A}, \hat{B}] = i\hbar C \quad \Rightarrow \quad \langle \Delta \hat{A}^2 \rangle \langle \Delta \hat{B}^2 \rangle \geq \frac{C^2}{4} \]

Consider an experiment in which one is able to measure the observable \( A \) with an accuracy \( \Delta A \)

\[ \langle \psi \mid \Delta \hat{A}^2 \mid \psi \rangle \sim \Delta A^2 \]

\[ \Rightarrow \langle \psi \mid \Delta \hat{B}^2 \mid \psi \rangle \geq \frac{C^2}{\Delta A^2} \]

The greater the accuracy/uncertainty in determining \( A \), the bigger the uncertainty in \( B \) just after the measurement

Example: Position and Momentum (Heisenberg Microscope)

Suppose one tries to measure the position of an electron with accuracy \( \Delta x \)
One needs a photon of wavelength \( \lambda \) where, \( \lambda \sim \Delta x \)

\[ \Rightarrow \text{Need a photon of momentum: } \quad p \sim \frac{\hbar}{\lambda} = \frac{\hbar}{\Delta x} \]

Resulting uncertainty in the particle momentum after scattering:

\[ \Delta p \sim p \sim \frac{\hbar}{\Delta x} \]

\[ \Rightarrow \Delta x \Delta p \sim \hbar \]
Measurements and Commutators in Quantum Mechanics

Commutation Relations and Common Eigenvectors: If two operators \( \hat{A} \) and \( \hat{B} \) commute, i.e.:
\[
\left[ \hat{A}, \hat{B} \right] = 0
\]
then they can have the same set of eigenvectors

Proof: Suppose:
\[
\hat{A} \left| v_k \right\rangle = \lambda_k \left| v_k \right\rangle \quad \text{and} \quad \left[ \hat{A}, \hat{B} \right] = 0
\]
\[
\Rightarrow \hat{A} \hat{B} \hat{B} \hat{A} = 0
\]
\[
\Rightarrow \left( \hat{A} \hat{B} \hat{B} \hat{A} \right) \left| v_k \right\rangle = 0
\]
\[
\Rightarrow \hat{A} \hat{B} \left| v_k \right\rangle - \lambda_k \hat{B} \left| v_k \right\rangle = 0
\]
\[
\Rightarrow \hat{A} \left( \hat{B} \left| v_k \right\rangle \right) = \lambda_k \left( \hat{B} \left| v_k \right\rangle \right)
\]
Therefore, \( \hat{B} \left| v_k \right\rangle \) is also an eigenvector of \( \hat{A} \) with the same eigenvalue as \( \left| v_k \right\rangle \)

1) If \( \hat{A} \) has all distinct eigenvalues, then \( \hat{B} \left| v_k \right\rangle \propto \left| v_k \right\rangle \) and therefore \( \left| v_k \right\rangle \) is also an eigenvector of \( \hat{B} \)

2) If has some degenerate eigenvalues, then \( \hat{B} \left| v_k \right\rangle \) at least lies in the same eigensubspace. In this case, the vectors in this eigensubspace can be chosen so that they are eigenvectors of both \( \hat{A} \) and \( \hat{B} \)
Measurements and Commutators in Quantum Mechanics

Commutation Relations and Simultaneous Measurements:

Consider two observables $A$ and $B$:

$$\hat{A} |v_k\rangle = \alpha_k |v_k\rangle \quad \hat{B} |u_k\rangle = \beta_k |u_k\rangle$$

$$[\hat{A}, \hat{B}] \neq 0$$

1) Suppose the observable $A$ is measured for a state $|\psi\rangle$.

$$\langle v_j | \psi \rangle^2$$

2) Suppose the observables $A$ and $B$ are measured for a state $|\psi\rangle$.

$$\langle v_j | \psi \rangle^2$$

Measurement of $B$ disturbs $A$ and measurement of $A$ disturbs $B$ when $A$ and $B$ do not commute. Accurate measurements of $A$ and $B$ cannot be done simultaneously.
Measurements and Commutators in Quantum Mechanics

Commutation Relations and Simultaneous Measurements:

Consider two observables $A$ and $B$:

$$
\hat{A} |\nu_k\rangle = \alpha_k |\nu_k\rangle \quad \hat{B} |u_k\rangle = \beta_k |u_k\rangle
$$

Since $A$ and $B$ commute, they have a common set of eigenvectors

$$
\hat{A} |\omega_k\rangle = \alpha_k |\omega_k\rangle \\
\hat{B} |\omega_k\rangle = \beta_k |\omega_k\rangle
$$

2) Suppose the observables $A$ and $B$ are measured for a state $|\psi\rangle$

Accurate measurements of $A$ and $B$ can be done simultaneously and will not affect each other
Superposition, Measurements, and Decoherence

Consider the following linear superposition state of a particle made from eigenkets of $A$:

$$\hat{A}\left|v_k\right\rangle = \lambda_k \left|v_k\right\rangle$$

$$\left|\psi\right\rangle = c_1 \left|v_1\right\rangle + c_2 \left|v_2\right\rangle$$

An observer makes a measurement to see if the particle was in state $\left|v_1\right\rangle$ or $\left|v_2\right\rangle$

Depending on the result the state collapses:

$$\left|\psi\right\rangle = c_1 \left|v_1\right\rangle + c_2 \left|v_2\right\rangle$$

Measurement of $A$

$$\left|c_1\right|^2 \lambda_1 \rightarrow \left|v_1\right\rangle$$

$$\left|c_2\right|^2 \lambda_2 \rightarrow \left|v_2\right\rangle$$

LESSON: If an observer “looks” at a quantum state, he destroys the linear superposition structure of the quantum state and collapses it
Superposition, Interaction, and Decoherence

Consider again the following linear superposition state of a particle made from eigenkets of $A$:

$$\hat{A}|v_k\rangle = \lambda_k |v_k\rangle$$

$$|\psi\rangle = c_1 |v_1\rangle + c_2 |v_2\rangle$$

**LESSON:** If environment degrees of freedom are changed by the interaction in a way that can let an observer determine the value of $A$ by looking at the environment, then this is equivalent to a direct measurement of $A$ and the quantum state collapses.
Decoherence

\[ |\psi\rangle = c_1 |v_1\rangle + c_2 |v_2\rangle \]

Any interaction with the environment can destroy the linear superposition and collapse the quantum state.

The products,

\[ c_2^* c_1 \quad \text{and} \quad c_1^* c_2 \]

present a good measure of the degree of superposition in a quantum state.

These products are generated by the operators \( \hat{\sigma}_+ = |v_2\rangle \langle v_1| \) and \( \hat{\sigma}_- = |v_1\rangle \langle v_2| \):

\[ \langle \psi | \hat{\sigma}_- | \psi \rangle = c_1^* c_2 \quad \langle \psi | \hat{\sigma}_+ | \psi \rangle = c_2^* c_1 \]

One can expect that as time goes by, interaction with the environment can make these products go to zero:

\[ \langle \psi(t) | \hat{\sigma}_- | \psi(t) \rangle = c_1^*(t)c_2(t) \xrightarrow{t \to \infty} 0 \]
\[ \langle \psi(t) | \hat{\sigma}_+ | \psi(t) \rangle = c_2^*(t)c_1(t) \xrightarrow{t \to \infty} 0 \]

This phenomenon which results in the destruction of quantum mechanical superpositions is called quantum mechanical decoherence.
Suppose: \( |\psi\rangle = c_1 |v_1\rangle + c_2 |v_2\rangle \)

We said interaction with the environment tends to destroy linear superpositions

**Wait a minute!!**

Define a new basis: 

\[
|v_+\rangle = \frac{1}{\sqrt{2}} (|v_1\rangle + |v_2\rangle) \quad |v_-\rangle = \frac{1}{\sqrt{2}} (|v_1\rangle - |v_2\rangle)
\]

\[
|\psi\rangle = c_1 |v_1\rangle + c_2 |v_2\rangle = \frac{c_1 + c_2}{\sqrt{2}} |v_+\rangle + \frac{c_1 - c_2}{\sqrt{2}} |v_-\rangle
\]

**LESSON:** Whether or not a superposition is destroyed depends on the exact nature of interaction with the environment (i.e. on what information is extracted by the environment during the interaction)
Pure States and Statistical Mixtures

\[ \hat{O} |v_k\rangle = \lambda_k |v_k\rangle \]

Consider two very different sets of states:

**Set A**

- A large number of identical copies of:
  \[ |\psi\rangle = c_1 |v_1\rangle + c_2 |v_2\rangle \]
  
  Linear superposition states (pure states)

  Measurement of \( O \) over the entire set

  Mean value obtained:

  \[ \langle \psi | \hat{O} |\psi\rangle = \lambda_1 |c_1|^2 + \lambda_2 |c_2|^2 \]

**Set B**

- A large number of states \( |v_1\rangle \) and \( |v_2\rangle \) in the ratio
  \[ |c_1|^2 : |c_2|^2 \]
  
  Statistical mixture of states

  Measurement of \( O \) over the entire set

  Mean value obtained:

  \[ \lambda_1 |c_1|^2 + \lambda_2 |c_2|^2 \]

How does one represent and/or distinguish these two states then??
Density Operator in Quantum Mechanics

Density operators are a useful way to represent quantum states. Most generally, a quantum state is not represented by a state vector $|\psi\rangle$ but by a density operator $\hat{\rho}$.

**Density Operator for Pure States (Set A):**

For pure states $|\psi\rangle$ the density operator is:

$$\hat{\rho} = |\psi\rangle \langle \psi|$$

Example:

$$|\psi\rangle = c_1 |v_1\rangle + c_2 |v_2\rangle$$

$$\Rightarrow \hat{\rho} = |\psi\rangle \langle \psi| = |c_1|^2 |v_1\rangle \langle v_1| + |c_2|^2 |v_2\rangle \langle v_2| + c_1^* c_2 |v_2\rangle \langle v_1| + c_2^* c_1 |v_1\rangle \langle v_2|$$

In matrix representation:

$$|v_1\rangle \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad |v_2\rangle \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\hat{\rho} = \begin{bmatrix} |c_1|^2 & c_2^* c_1 \\ c_1^* c_2 & |c_2|^2 \end{bmatrix}$$

The diagonal elements indicate the occupation probabilities, and the off-diagonal elements represent coherences.
Density Operator for Pure States

For pure states \( |\psi\rangle \) the density operator is: \( \hat{\rho} = |\psi\rangle \langle \psi | \)

The mean value of an observable is defined for a pure state as:

\[
\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle
\]

How do we use the density operator to calculate mean values of observables?

\[
\langle \hat{A} \rangle = \text{Trace} \{ \hat{\rho} \hat{A} \}
\]

\[
\text{Trace} \{ |\psi\rangle \langle \psi | \hat{A} \} = \sum_n \langle \psi | \hat{A} | \psi \rangle = \sum_n \langle \hat{A} | \psi \rangle \langle \psi | \psi \rangle
\]

\[
= \langle \psi | \hat{A} \left( \sum_n |\psi \rangle \langle \psi | \right) | \psi \rangle = \langle \psi | \hat{A} | \psi \rangle
\]

\[
= |c_1|^2 \langle \psi | \hat{A} | \psi \rangle + |c_2|^2 \langle \psi | \hat{A} | \psi \rangle + c_2^* c_1 \langle \psi | \hat{A} \rangle + c_1^* c_2 \langle \psi | \hat{A} \rangle
\]

\[
= \langle \psi | \hat{A} | \psi \rangle
\]
Density Operator for Statistical Mixtures

Density operators for the statistical mixture (set B) is defined as:

\[ \hat{\rho} = |c_1|^2 |v_1\rangle\langle v_1| + |c_2|^2 |v_2\rangle\langle v_2| \]

In matrix representation:

\[ |v_1\rangle \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |v_2\rangle \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

\[ \hat{\rho} = \begin{bmatrix} |c_1|^2 & 0 \\ 0 & |c_2|^2 \end{bmatrix} \]

Off-diagonal elements are zero \(\Rightarrow\) no coherences!

How do we use the density operator to calculate mean values of observables?

\[ \langle \hat{A} \rangle = \text{Trace} \{ \hat{\rho} \hat{A} \} \]

\[ = \text{Trace} \left\{ \left( |c_1|^2 |v_1\rangle\langle v_1| + |c_2|^2 |v_2\rangle\langle v_2| \right) \hat{A} \right\} \]

\[ = |c_1|^2 \langle v_1| \hat{A} |v_1\rangle + |c_2|^2 \langle v_2| \hat{A} |v_2\rangle \]
Decoherence makes the off-diagonal components of the density matrix go to zero with time!
Density Operator for General Mixed States

Suppose we have a statistical mixture of two states:

$$|v_+\rangle = \frac{1}{\sqrt{2}} (|v_1\rangle + |v_2\rangle) \quad \Leftrightarrow \quad p_+$$

$$|v_-\rangle = \frac{1}{\sqrt{2}} (|v_1\rangle - |v_2\rangle) \quad \Leftrightarrow \quad p_-$$

The density operator is:

$$\hat{\rho} = p_+ |v_+\rangle \langle v_+| + p_- |v_-\rangle \langle v_-|$$

$$= \frac{1}{2} |v_1\rangle \langle v_1| + \frac{1}{2} |v_2\rangle \langle v_2|$$

$$+ \frac{p_+ - p_-}{2} |v_1\rangle \langle v_2| + \frac{p_+ - p_-}{2} |v_2\rangle \langle v_1|$$

In matrix representation:

$$|v_+\rangle \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad |v_-\rangle \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\hat{\rho} = \begin{bmatrix} p_+ & 0 \\ 0 & p_- \end{bmatrix}$$

Whether or not a density operator has off-diagonal components depends on the choice of basis!
**Time Development: Schrödinger and Heisenberg Pictures for Density Operators**

Given a state $|\psi(t = 0)\rangle$ the mean value of an observable at time ‘$t$’ is:

$$\langle \hat{A}(t) \rangle = \langle \psi(t) | \hat{A} | \psi(t) \rangle \rightarrow \text{Schrödinger Picture}$$

$$= \langle \psi(t = 0) | \hat{A}(t) | \psi(t = 0) \rangle \rightarrow \text{Heisenberg Picture}$$

Given a state $\rho(t = 0)$ the mean value of an observable at time ‘$t$’ is:

$$\langle \hat{A}(t) \rangle = \text{Trace} \{ \rho(t) \hat{A}(t = 0) \} = \text{Trace} \{ \rho(t) \hat{A} \} \rightarrow \text{Schrödinger Picture}$$

$$= \text{Trace} \{ \rho(t = 0) \hat{A}(t) \} \rightarrow \text{Heisenberg Picture}$$
Time Evolution of Density Operators in the Schrödinger Picture

Heisenberg operators obey:

\[ \hat{A}(t) = e^{i\frac{\hat{H}}{\hbar}t} \hat{A} e^{-i\frac{\hat{H}}{\hbar}t} \]

\[ \text{Trace}\{\hat{\rho}(t=0)\hat{A}(t)\} = \text{Trace}\left\{\hat{\rho}(t=0) e^{i\frac{\hat{H}}{\hbar}t} \hat{A} e^{-i\frac{\hat{H}}{\hbar}t}\right\} \]

\[ = \text{Trace}\left\{e^{-i\frac{\hat{H}}{\hbar}t} \hat{\rho}(t=0) e^{i\frac{\hat{H}}{\hbar}t} \hat{A}\right\} \]

\[ = \text{Trace}\{\hat{\rho}(t)\hat{A}\} \]

Where:

\[ \hat{\rho}(t) = e^{-i\frac{\hat{H}}{\hbar}t} \hat{\rho}(t=0) e^{i\frac{\hat{H}}{\hbar}t} \]

Differentiate w.r.t. time to get:

\[ i\hbar \frac{\partial \hat{\rho}(t)}{\partial t} = [\hat{H}, \hat{\rho}(t)] \quad \text{Equation for the time development of the density operator in the Schrodinger picture} \]
Time Evolution of Density Operators in the Schrodinger Picture

Given a state \( \hat{\rho}(t = 0) \) solve:

\[
i \hbar \frac{\partial \hat{\rho}(t)}{\partial t} = [\hat{H}, \hat{\rho}(t)]
\]

And then calculate:

\[
\langle \hat{A}(t) \rangle = \text{Trace}\{\hat{\rho}(t) \hat{A}(t = 0)\} = \text{Trace}\{\hat{\rho}(t) \hat{A}\}
\]

This is not the same as the Heisenberg equation for other operators:

\[
i \hbar \frac{\partial \hat{A}(t)}{\partial t} = [\hat{A}(t), \hat{H}]
\]

Quantum Mechanical Correlations of Observables:

An advantage of the Heisenberg picture is that one can calculate correlations:

\[
\langle \hat{A}(t_1) \hat{B}(t_2) \rangle = \text{Trace}\{\hat{\rho}(t = 0) \hat{A}(t_1) \hat{B}(t_2)\}
\]
Dynamics of a Two Level System via the Density Matrix in the Schrodinger Picture

\[ \hat{H} = \varepsilon |e_1\rangle\langle e_1| + \varepsilon |e_2\rangle\langle e_2| - U |e_1\rangle\langle e_2| - U |e_2\rangle\langle e_1| \]

Suppose: \[ |\psi(t = 0)\rangle = |e_1\rangle \]
\[ \hat{\rho}(t = 0) = |\psi(t = 0)\rangle\langle \psi(t = 0)| = |e_1\rangle\langle e_1| \]
\[ \hat{\rho}(t = 0) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \]

The goal is to find:

\[ \langle e_1 | \psi(t) \rangle^2 = \langle \psi(t) | e_1 \rangle \langle e_1 | \psi(t) \rangle = \text{Trace}\{\hat{\rho}(t) e_1 \rangle \langle e_1 | \}
\]
\[ = \langle e_1 | \{\hat{\rho}(t) e_1 \rangle \langle e_1 | \} e_1 \rangle + \langle e_2 | \hat{\rho}(t) e_1 \rangle \langle e_1 | \} e_2 \rangle \]
\[ = \rho_{11}(t) \]

Start from:

\[ i\hbar \frac{\partial \hat{\rho}(t)}{\partial t} = \hat{H} \hat{\rho}(t) - \hat{\rho}(t) \hat{H} \]
\[ \hat{\rho}(t) = \begin{bmatrix} \rho_{11}(t) & \rho_{12}(t) \\ \rho_{21}(t) & \rho_{22}(t) \end{bmatrix} \]
Dynamics of a Two Level System via the Density Matrix

\[ \hat{H} = \varepsilon |e_1 \rangle \langle e_1 | + \varepsilon |e_2 \rangle \langle e_2 | - U |e_1 \rangle \langle e_2 | - U |e_2 \rangle \langle e_1 | \]

\[ i\hbar \frac{\partial \hat{\rho}(t)}{\partial t} = \hat{H} \hat{\rho}(t) - \hat{\rho}(t) \hat{H} \]

\[ \hat{\rho}(t) = \begin{bmatrix} \rho_{11}(t) & \rho_{12}(t) \\ \rho_{21}(t) & \rho_{22}(t) \end{bmatrix} \]

Take the 11 matrix element of the above equation by multiplying from the left by \( \langle e_1 | \) and from the right by \( | e_1 \rangle \)

\[ i\hbar \frac{d}{dt} \rho_{11}(t) = U \left[ \rho_{12}(t) - \rho_{21}(t) \right] \]

Note:

\[ \langle e_1 | \hat{H} \hat{\rho}(t) | e_1 \rangle = \langle e_1 | \left[ \varepsilon |e_1 \rangle \langle e_1 | + \varepsilon |e_2 \rangle \langle e_2 | - U |e_1 \rangle \langle e_2 | - U |e_2 \rangle \langle e_1 | \right] \hat{\rho}(t) | e_1 \rangle = \varepsilon \rho_{11}(t) - U \rho_{21}(t) \]

\[ \langle e_1 | \hat{H} \hat{\rho}(t) | e_1 \rangle = \langle e_1 | \hat{\rho}(t) \left[ \varepsilon |e_1 \rangle \langle e_1 | + \varepsilon |e_2 \rangle \langle e_2 | - U |e_1 \rangle \langle e_2 | - U |e_2 \rangle \langle e_1 | \right] | e_1 \rangle = \varepsilon \rho_{11}(t) - U \rho_{12}(t) \]
Dynamics of a Two Level System via the Density Matrix

\[ i\hbar \frac{\partial \hat{\rho}(t)}{\partial t} = \hat{H} \hat{\rho}(t) - \hat{\rho}(t) \hat{H} \]

\[ \hat{\rho}(t) = \begin{bmatrix} \rho_{11}(t) & \rho_{12}(t) \\ \rho_{21}(t) & \rho_{22}(t) \end{bmatrix} \]

One can obtain equations for all the diagonal and off-diagonal elements:

\[ i\hbar \frac{d}{dt} \rho_{11}(t) = U [\rho_{12}(t) - \rho_{21}(t)] \]

\[ i\hbar \frac{d}{dt} \rho_{22}(t) = -U [\rho_{12}(t) - \rho_{21}(t)] = -i\hbar \frac{d}{dt} \rho_{11}(t) \]

\[ i\hbar \frac{d}{dt} \rho_{12}(t) = -U [\rho_{22}(t) - \rho_{11}(t)] \]

\[ i\hbar \frac{d}{dt} \rho_{21}(t) = U [\rho_{22}(t) - \rho_{11}(t)] = -i\hbar \frac{d}{dt} \rho_{12}(t) \]

\[ \frac{d}{dt} [\rho_{11}(t) + \rho_{22}(t)] = 0 \]

\[ \Rightarrow \rho_{11}(t) + \rho_{22}(t) = \rho_{11}(t = 0) + \rho_{22}(t = 0) = 1 \]

\[ \rho_{12}(t) = \rho_{21}^*(t) \]
Dynamics of a Two Level System via the Density Matrix

\[ \rho_d(t) = \rho_{22}(t) - \rho_{11}(t) \]
\[ \rho_s(t) = \rho_{12}(t) - \rho_{21}(t) \]

It follows that:
\[ \frac{d}{dt} \rho_d(t) = \frac{2iU}{\hbar} \rho_s(t) \]
\[ \frac{d}{dt} \rho_s(t) = \frac{2iU}{\hbar} \rho_d(t) \]

And then one obtains:
\[ \frac{d^2}{dt^2} \rho_d(t) = -\left(\frac{2U}{\hbar}\right)^2 \rho_d(t) \]

Boundary conditions:
\[ \rho_d(t = 0) = \rho_{22}(t = 0) - \rho_{11}(t = 0) = -1 \]
\[ \frac{d}{dt} \rho_d(t) \bigg|_{t=0} = \frac{2iU}{\hbar} \rho_s(t = 0) \]
\[ = \frac{2iU}{\hbar} [\rho_{12}(t = 0) - \rho_{21}(t = 0)] \]
\[ = 0 \]
Dynamics of a Two Level System via the Density Matrix

Solution is:

\[ \rho_d(t) = -\cos\left(\frac{2U}{\hbar} t\right) \]

\[ \Rightarrow \rho_{11}(t) = \frac{[\rho_{11}(t) + \rho_{22}(t)] - [\rho_d(t)]}{2} = \frac{1 + \cos\left(\frac{2U}{\hbar} t\right)}{2} = \cos^2\left(\frac{Ut}{\hbar}\right) \]

\[ \Rightarrow \rho_{22}(t) = \sin^2\left(\frac{Ut}{\hbar}\right) \]

And therefore:

\[ \langle e_1 | \psi(t) \rangle^2 = \text{Trace}\{\hat{\rho}(t)e_1\langle e_1 |} = \rho_{11}(t) = \cos^2\left(\frac{Ut}{\hbar}\right) \]

Same as obtained earlier!
Dynamics via the Density Matrix with Decoherence

The diagonal components are unaffected by decoherence:

\[
\frac{d}{dt} \rho_{11}(t) = -\frac{i}{\hbar} U [\rho_{12}(t) - \rho_{21}(t)]
\]

\[
\frac{d}{dt} \rho_{22}(t) = \frac{i}{\hbar} U [\rho_{12}(t) - \rho_{21}(t)] = -i\hbar \frac{d}{dt} \rho_{11}(t)
\]

The off-diagonal components are assumed to decay due to decoherence:

\[
\frac{d}{dt} \rho_{12}(t) = -\gamma \rho_{12}(t) + i \frac{U}{\hbar} [\rho_{22}(t) - \rho_{11}(t)]
\]

\[
\frac{d}{dt} \rho_{21}(t) = -\gamma \rho_{21}(t) - i \frac{U}{\hbar} [\rho_{22}(t) - \rho_{11}(t)]
\]

The equation we get now is:

\[
\frac{d^2}{dt^2} \rho_d(t) + \gamma \frac{d}{dt} \rho_d(t) + \left(\frac{2U}{\hbar}\right)^2 \rho_d(t) = 0
\]
Dynamics via the Density Matrix with Decoherence

\[
\frac{d^2}{dt^2} \rho_d(t) + \gamma \frac{d}{dt} \rho_d(t) + \left( \frac{2U}{\hbar} \right)^2 \rho_d(t) = 0
\]

Boundary conditions:
\[
\rho_d(t = 0) = \rho_{22}(t = 0) - \rho_{11}(t = 0) = -1
\]

\[
\frac{d}{dt} \rho_d(t) \bigg|_{t=0} = 0
\]

Solution is:
\[
\rho_d(t) = -e^{-\frac{\gamma t}{2}} \left[ \cos \Omega t + \frac{\gamma}{2\Omega} \sin \Omega t \right]
\]

\[
\Omega = \sqrt{\left( \frac{2U}{\hbar} \right)^2 - \left( \frac{\gamma}{2} \right)^2}
\]

Therefore:
\[
\rho_{11}(t) = \frac{1}{2} \left\{ 1 + e^{-\frac{\gamma t}{2}} \left[ \cos \Omega t + \frac{\gamma}{2\Omega} \sin \Omega t \right] \right\}
\]

\[
\rho_{22}(t) = \frac{1}{2} \left\{ 1 - e^{-\frac{\gamma t}{2}} \left[ \cos \Omega t + \frac{\gamma}{2\Omega} \sin \Omega t \right] \right\}
\]
Dynamics via the Density Matrix with Decoherence

As time $t \to \infty$:

$$
\rho_{11}(t) \to \frac{1}{2} \quad \rho_{12}(t) \to 0 \\
\rho_{22}(t) \to \frac{1}{2} \quad \rho_{21}(t) \to 0
$$

Therefore as $t \to \infty$:

$$
\hat{\rho}(t) \xrightarrow{t \to \infty} \begin{bmatrix}
\frac{1}{2} & 0 \\
0 & \frac{1}{2}
\end{bmatrix}
$$
Dynamics of a Two Level System via the Density Matrix in the Heisenberg Picture

\[ \hat{H} = \varepsilon |e_1\rangle \langle e_1| + \varepsilon |e_2\rangle \langle e_2| - U |e_1\rangle \langle e_2| - U |e_2\rangle \langle e_1| \]

\[ \hat{N}_1 = |e_1\rangle \langle e_1| \quad \hat{\sigma}_- = |e_1\rangle \langle e_2| \]

\[ \hat{N}_2 = |e_2\rangle \langle e_2| \quad \hat{\sigma}_+ = |e_2\rangle \langle e_1| \]

\{ \hat{\sigma}^- = \hat{\sigma}^+ \}

The Hamiltonian is:

\[ \hat{H} = \varepsilon (\hat{N}_1 + \hat{N}_2) - U (\hat{\sigma}_- + \hat{\sigma}_+) \]

The Heisenberg equations are:

\[ i\hbar \frac{d\hat{N}_1(t)}{dt} = [\hat{N}_1(t), \hat{H}] = U [\hat{\sigma}_+(t) - \hat{\sigma}_-(t)] = -i\hbar \frac{d\hat{N}_2(t)}{dt} \]

\[ i\hbar \frac{d\hat{\sigma}_-(t)}{dt} = [\hat{\sigma}_-(t), \hat{H}] = U [\hat{N}_2(t) - \hat{N}_1(t)] = -i\hbar \frac{d\hat{\sigma}_+(t)}{dt} \]
Dynamics of a Two Level System via the Density Matrix in the Heisenberg Picture

\[ \epsilon \]

The goal is to find:

\[ \langle \hat{N}_1(t) \rangle = \text{Trace}\{\hat{\rho}(t)\hat{N}_1\} = \text{Trace}\{\hat{\rho}(t = 0)\hat{N}_1(t)\} \]

When:

\[ \hat{\rho}(t = 0) = |\psi(t = 0)\rangle \langle \psi(t = 0)| = |e_1\rangle \langle e_1| \]

Solution of the Heisenberg equation is:

\[
\hat{N}_1(t) = \frac{[\hat{N}_1(t) + \hat{N}_2(t)] - \hat{N}_d(t)}{2} = \frac{(\hat{N}_1 + \hat{N}_2) - \hat{N}_d(t)}{2}
\]

\[
= \hat{N}_1 \cos^2 \left(\frac{Ut}{\hbar}\right) + \hat{N}_2 \sin^2 \left(\frac{Ut}{\hbar}\right) - \frac{i}{2} (\hat{\sigma}_+ - \hat{\sigma}_-) \sin \left(\frac{2U}{\hbar} t\right)
\]

The result is:

\[ \langle \hat{N}_1(t) \rangle = \text{Trace}\{\hat{\rho}(t = 0)\hat{N}_1(t)\} = \cos^2 \left(\frac{Ut}{\hbar}\right) \]

Same as obtained earlier!
Dynamics of a Two Level System via the Density Matrix in the Heisenberg Picture with Decoherence

Consider the Heisenberg operator equations:

\[
\frac{d\hat{\sigma}_-}{dt} = -\frac{i}{\hbar} U [\hat{N}_2(t) - \hat{N}_1(t)]
\]

\[
\frac{d\hat{\sigma}_+}{dt} = \frac{i}{\hbar} U [\hat{N}_2(t) - \hat{N}_1(t)]
\]

Their average w.r.t. the density operator will yield the familiar equations:

\[
\frac{d}{dt} \rho_{21}(t) = -\frac{i}{\hbar} U [\rho_{22}(t) - \rho_{11}(t)]
\]

\[
\frac{d}{dt} \rho_{12}(t) = \frac{i}{\hbar} U [\rho_{22}(t) - \rho_{11}(t)]
\]

In the presence of decoherence the above equations got modified:

\[
\frac{d}{dt} \rho_{12}(t) = -\gamma \rho_{12}(t) + i \frac{U}{\hbar} [\rho_{22}(t) - \rho_{11}(t)]
\]

\[
\frac{d}{dt} \rho_{21}(t) = -\gamma \rho_{21}(t) - i \frac{U}{\hbar} [\rho_{22}(t) - \rho_{11}(t)]
\]
Dynamics of a Two Level System via the Density Matrix in the Heisenberg Picture with Decoherence

\[ \varepsilon \]

It would be tempting to introduce decoherence the following way:

\[
\frac{d\hat{\sigma}_-(t)}{dt} = -\gamma\hat{\sigma}_-(t) - \frac{i}{\hbar}U[\hat{N}_2(t) - \hat{N}_1(t)]
\]

\[
\frac{d\hat{\sigma}_+(t)}{dt} = -\gamma\hat{\sigma}_+(t) + \frac{i}{\hbar}U[\hat{N}_2(t) - \hat{N}_1(t)]
\]

**PROBLEM:**
Commutation rules are always satisfied provided operator time evolution equations obey the Heisenberg equation:

\[
[\hat{A}, \hat{B}] = \hat{C} \Rightarrow \quad [\hat{A}(t), \hat{B}(t)] = e^{\frac{i}{\hbar}\hat{H}t}[\hat{A}, \hat{B}]e^{\frac{-i}{\hbar}\hat{H}t} = e^{\frac{i}{\hbar}\hat{H}t}\hat{C}e^{\frac{-i}{\hbar}\hat{H}t} = \hat{C}(t)
\]

Commutation rules will get violated when we add decoherence terms arbitrarily as done above!!

Later we will introduce quantum Langevin equations to get over this problem
Joint or Product Hilbert Spaces

The Hilbert space of two independent quantum systems is obtained by “sticking” together the Hilbert spaces of the individual systems:

$$|\psi\rangle = |\phi\rangle_a \otimes |\chi\rangle_b$$

Or simply:

$$|\psi\rangle = |\phi\rangle_a |\chi\rangle_b$$

Inner Product:

$$|\psi\rangle = |\phi\rangle_a \otimes |\chi\rangle_b$$

$$|\omega\rangle = |\theta\rangle_a \otimes |\eta\rangle_b$$

$$\langle \omega | \psi \rangle = a \langle \theta | \phi\rangle_a \ b \langle \eta | \chi\rangle_b$$

Operators:

An operator in this enlarged Hilbert space is written as a tensor product:

$$\hat{A} \otimes \hat{B}$$

Each operator acts in its own Hilbert space:

$$\hat{A} \otimes \hat{B} \ |\phi\rangle_a \otimes |\chi\rangle_b = \hat{A} \ |\phi\rangle_a \otimes \hat{B} |\chi\rangle_b$$
Joint or Product Hilbert Spaces

Full Hilbert Space:

Consider two different two-level systems “a” and “b”. The full Hilbert space consists of the following four states which form a complete set:

\[ |1\rangle = |e_1\rangle_a \otimes |e_1\rangle_b \]
\[ |2\rangle = |e_2\rangle_a \otimes |e_1\rangle_b \]
\[ |3\rangle = |e_1\rangle_a \otimes |e_2\rangle_b \]
\[ |4\rangle = |e_2\rangle_a \otimes |e_2\rangle_b \]

Completeness:

\[
\sum_{k=1}^{4} |k\rangle\langle k| = \left[ |e_1\rangle_a e_1 + |e_2\rangle_a e_2 \right] \otimes \left[ |e_1\rangle_b e_1 + |e_2\rangle_b e_2 \right] = \hat{1}_a \otimes \hat{1}_b = \hat{1}
\]
The Hamiltonian for two different two-level systems is:

$$\hat{H} = \hat{H}_a \otimes \hat{1}_b + \hat{1}_a \otimes \hat{H}_b$$

Where:

$$\hat{H}_a = \varepsilon_1 \langle e_1 |_a \langle e_1 | + \varepsilon_2 | e_2 |_a \langle e_1 |$$

$$\hat{H}_b = \varepsilon_1 | e_1 |_b \langle e_1 | + \varepsilon_2 | e_2 |_b \langle e_1 |$$

An eigenstate of the combined system is, for example: $| e_1 |_a \otimes | e_2 |_b$

$$\hat{H} | e_1 |_a \otimes | e_2 |_b = \hat{H}_a \otimes \hat{1}_b \{ | e_1 |_a \otimes | e_2 |_b \} + \hat{1}_a \otimes \hat{H}_b \{ | e_1 |_a \otimes | e_2 |_b \}$$

$$= \hat{H}_a | e_1 |_a \otimes 1_b | e_2 |_b + 1_a | e_1 |_a \otimes \hat{H}_b | e_2 |_b$$

$$= \varepsilon_1 \{ | e_1 |_a \otimes | e_2 |_b \} + \varepsilon_2 \{ | e_1 |_a \otimes | e_2 |_b \}$$

$$= (\varepsilon_1 + \varepsilon_2) | e_1 |_a \otimes | e_2 |_b$$

Or with some abuse of notation:

$$\hat{H} = \hat{H}_a + \hat{H}_b$$

Eigenvalue
Unentangled States

States belonging to a combined Hilbert space of two systems, “a” and “b”, are of two types:

1) Unentangled states
2) Entangled states

Unentangled States:

These states can be written as:

\[ |a\text{ unique state of system }"a"\rangle_a \otimes |a\text{ unique state of system }"b"\rangle_b \]

Examples:

i) \( |\psi\rangle = |e_1\rangle_a \otimes |e_2\rangle_b \)

ii) \( |\psi\rangle = \left[ \frac{1}{\sqrt{2}} \left( |e_1\rangle_a + |e_2\rangle_a \right) \right] \otimes |e_1\rangle_b = \frac{1}{\sqrt{2}} \left\{ |e_1\rangle_a \otimes |e_1\rangle_b + |e_2\rangle_a \otimes |e_1\rangle_b \right\} \)

iii) \( |\psi\rangle = |e_1\rangle_a \otimes \left[ \frac{1}{\sqrt{2}} \left( |e_1\rangle_b - |e_2\rangle_b \right) \right] = \frac{1}{\sqrt{2}} \left\{ |e_1\rangle_a \otimes |e_1\rangle_b - |e_1\rangle_a \otimes |e_2\rangle_b \right\} \)

iv) \( |\psi\rangle = \left[ \frac{1}{\sqrt{2}} \left( |e_1\rangle_a + |e_2\rangle_a \right) \right] \otimes \left[ \frac{1}{\sqrt{2}} \left( |e_1\rangle_b + |e_2\rangle_b \right) \right] \)
Entangled States

Entangled States:

Entangled states cannot be factorized or separated in the same fashion, for example:

\[ \frac{1}{\sqrt{2}} \left[ |e_1\rangle_a \otimes |e_2\rangle_b - |e_2\rangle_a \otimes |e_1\rangle_b \right] \]

is an entangled state and it cannot be written as:

\[ |\phi\rangle_a \otimes |x\rangle_b \]
Unentangled States and Measurements

Consider the complicated un-entangled state of two different two-level systems:

\[
|\psi\rangle = \left[\left(\frac{\sqrt{3}}{2}|e_1\rangle_a + \frac{1}{2}|e_2\rangle_a\right)\right] \otimes \left[\frac{1}{\sqrt{2}}(|e_1\rangle_b + |e_2\rangle_b)\right]
\]

LESSON: Measurement of system b does not affect the subsequent measurement results for system a
Entangled States and Measurements

Consider the entangled state of two different two-level systems:

\[
\frac{\sqrt{3}}{2} |e_1\rangle_a \otimes |e_2\rangle_b - \frac{1}{2} |e_2\rangle_a \otimes |e_1\rangle_b
\]

LESSON: Measurement of system b affects the subsequent measurement results for system a (EPR Paradox, 1935)
Density Operators for Joint Hilbert Spaces

Unentangled States:

If the quantum state of a system consisting of two subsystems “a” and “b” is an unentangled state:

\[ |\psi\rangle = |\phi\rangle_a \otimes |x\rangle_b \]

then the density operator is:

\[ \hat{\rho} = |\psi\rangle \langle \psi| = \{ |\phi\rangle_a \otimes |x\rangle_b \} \{ a \langle \phi | \otimes _b a \langle x | \} \]

\[ = |\phi\rangle_a \langle \phi | \otimes _b b \langle x | \]

\[ = \hat{\rho}_a \otimes \hat{\rho}_b \]

Therefore, the density operator can be written as a tensor product of the density operators of the subsystems.

Example:

\[ |\psi\rangle = \left[ \frac{1}{\sqrt{2}} (|e_1\rangle_a + |e_2\rangle_a) \right] \otimes \left[ \frac{1}{\sqrt{2}} (|e_1\rangle_b - |e_2\rangle_b) \right] \]

\[ \hat{\rho} = \hat{\rho}_a \otimes \hat{\rho}_b \]

Where:

\[ \hat{\rho}_a = \frac{1}{2} \{ |e_1\rangle_a a \langle e_1| + |e_1\rangle_a a \langle e_2| + |e_2\rangle_a a \langle e_1| + |e_2\rangle_a a \langle e_2| \} \]

\[ \hat{\rho}_b = \frac{1}{2} \{ |e_1\rangle_b b \langle e_1| - |e_1\rangle_b b \langle e_2| - |e_2\rangle_b b \langle e_1| + |e_2\rangle_b b \langle e_2| \} \]
Density Operators for Joint Hilbert Spaces

Entangled States:

Consider the entangled state:

\[ |\psi\rangle = \frac{1}{\sqrt{2}} \left( |e_1\rangle_a \otimes |e_2\rangle_b - |e_2\rangle_a \otimes |e_1\rangle_b \right) \]

The density operator is:

\[ \hat{\rho} = |\psi\rangle \langle \psi | = \frac{1}{2} \left\{ |e_1\rangle_a a \langle e_1| \otimes |e_2\rangle_b b \langle e_2| + |e_2\rangle_a a \langle e_2| \otimes |e_1\rangle_b b \langle e_1| \\
- |e_1\rangle_a a \langle e_2| \otimes |e_2\rangle_b b \langle e_1| - |e_2\rangle_a a \langle e_1| \otimes |e_1\rangle_b b \langle e_2| \right\} \]

The density operator for entangled states cannot be written as a tensor product of the density operators of the subsystems:

\[ \hat{\rho} \neq \hat{\rho}_a \otimes \hat{\rho}_b \]
Density Operators for Joint Hilbert Spaces

Example:

Suppose: \( |\psi\rangle = |e_1\rangle_a \otimes |e_2\rangle_b \)

\[ \Rightarrow \hat{\rho} = |\psi\rangle \langle \psi| = |e_1\rangle_a \langle e_1| \otimes |e_2\rangle_b \langle e_2| \]

Calculation of average energy of the state:

\[ \langle \hat{H}_a + \hat{H}_b \rangle = \text{Trace}\{\hat{\rho} \left( \hat{H}_a + \hat{H}_b \right) \} \]

\[ = \text{Trace}\left\{ |e_1\rangle_a \langle e_1| \otimes |e_2\rangle_b \langle e_2| \left( \hat{H}_a + \hat{H}_b \right) \right\} \]

\[ = \sum_{k=1}^{4} \langle k | \left( |e_1\rangle_a \langle e_1| \otimes |e_2\rangle_b \langle e_2| \right) \left( \hat{H}_a + \hat{H}_b \right) |k\rangle \]

\[ = a \langle e_1| \otimes b \langle e_1| \left( \langle e_1|_a \langle e_1| \otimes |e_2\rangle_b \langle e_2| \right) \left( \hat{H}_a + \hat{H}_b \right) |e_1\rangle_a \otimes |e_1\rangle_b \]

\[ + a \langle e_2| \otimes b \langle e_1| \left( \langle e_1|_a \langle e_1| \otimes |e_2\rangle_b \langle e_2| \right) \left( \hat{H}_a + \hat{H}_b \right) |e_2\rangle_a \otimes |e_1\rangle_b \]

\[ + a \langle e_1| \otimes b \langle e_2| \left( \langle e_1|_a \langle e_1| \otimes |e_2\rangle_b \langle e_2| \right) \left( \hat{H}_a + \hat{H}_b \right) |e_1\rangle_a \otimes |e_2\rangle_b \]

\[ + a \langle e_2| \otimes b \langle e_2| \left( \langle e_1|_a \langle e_1| \otimes |e_2\rangle_b \langle e_2| \right) \left( \hat{H}_a + \hat{H}_b \right) |e_2\rangle_a \otimes |e_2\rangle_b \]

\[ = \text{only the third line above gives a nonzero answer equal to} \ (\varepsilon_1 + \varepsilon_2) \]
Density Operators of Subsystems: Partial Traces

Sometimes a density operator for two (or more) systems contains too much information.

If one is interested in only system “a” but has the joint density operator for system “a” and system “b”, then one needs to “extract” a density operator for system “a”:

$$\hat{\rho}_a = \text{Trace}_b \{ \hat{\rho} \}$$

$$= \langle e_1 | \hat{\rho} | e_1 \rangle_b + \langle e_2 | \hat{\rho} | e_2 \rangle_b$$

Unentangled States: For unentangled states, we know that:

$$\hat{\rho} = \hat{\rho}_a \otimes \hat{\rho}_b$$

$$\text{Trace}_b \{ \hat{\rho} \} = \langle e_1 | \hat{\rho}_a \otimes \hat{\rho}_b | e_1 \rangle_b + \langle e_2 | \hat{\rho}_a \otimes \hat{\rho}_b | e_2 \rangle_b$$

$$= \hat{\rho}_a \langle e_1 | \hat{\rho}_b | e_1 \rangle_b + \hat{\rho}_a \langle e_2 | \hat{\rho}_b | e_2 \rangle_b$$

$$= \hat{\rho}_a \left[ \langle e_1 | \hat{\rho}_b | e_1 \rangle_b + \langle e_2 | \hat{\rho}_b | e_2 \rangle_b \right]$$

$$= \hat{\rho}_a \text{Trace}_b \{ \hat{\rho}_b \}$$

$$= \hat{\rho}_a \quad \text{And this is exactly what we expected to get!}$$
Density Operators of Subsystems: Partial Traces

Entangled States: Consider the entangled state:

$$|\psi\rangle = \left\{ \frac{\sqrt{3}}{2} |e_1\rangle_a \otimes |e_2\rangle_b + \frac{1}{2} |e_2\rangle_a \otimes |e_1\rangle_b \right\}$$

$$\hat{\rho} = |\psi\rangle \langle \psi| = \frac{1}{4} \left\{ 3 |e_1\rangle_a a \langle e_1| \otimes |e_2\rangle_b b \langle e_2| + \sqrt{3} |e_1\rangle_a a \langle e_2| \otimes |e_2\rangle_b b \langle e_1| ight. $$

$$+ \sqrt{3} |e_2\rangle_a a \langle e_1| \otimes |e_1\rangle_b b \langle e_2| + |e_2\rangle_a a \langle e_2| \otimes |e_1\rangle_b b \langle e_1| \right\}$$

Density operator of the subsystem ‘a’ is obtained as follows:

$$\hat{\rho}_a = \text{Trace}_p \{ \hat{\rho} \}$$

$$= b \langle e_1| \hat{\rho} |e_1\rangle_b + b \langle e_2| \hat{\rho} |e_2\rangle_b$$

$$= \left\{ \frac{3}{4} |e_1\rangle_a a \langle e_1| + \frac{1}{4} |e_2\rangle_a a \langle e_2| \right\}$$

$$\hat{\rho}_a = \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$$

Therefore, $\hat{\rho}_a$ is a statistical mixture of states $|e_1\rangle_a$ and $|e_2\rangle_a$. 
Entanglement and Decoherence

There is an intimate connection between entanglement and decoherence.

A Brief Review:

\[ |\psi\rangle = c_1 |v_1\rangle + c_2 |v_2\rangle \]

Decoherence makes the off-diagonal components of the density matrix go to zero with time!

\[ \hat{\rho} = \begin{bmatrix} |c_1|^2 & c_2^* c_1 \\ c_1^* c_2 & |c_2|^2 \end{bmatrix} \]

\[ \hat{\rho} = \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix} \]
Entanglement and Decoherence

First, we need to make a model of the environment. Suppose the (mutually orthogonal) environment states are:

\[ |E_0\rangle, |E_1\rangle, |E_2\rangle \]

The initial quantum state of the system is:

\[ |\psi\rangle = c_1 |v_1\rangle + c_2 |v_2\rangle \]

The initial joint state of the “system + environment” is:

\[ |\phi(t = 0)\rangle = |\psi\rangle \otimes |E_0\rangle = c_1 |v_1\rangle \otimes |E_0\rangle + c_2 |v_2\rangle \otimes |E_0\rangle \]

Now we find the density operator for the system by taking the partial trace of the full density operator:

\[ \hat{\rho}_{\text{full}}(t) = |\phi(t)\rangle \langle \phi(t)| \]

\[ \Rightarrow \hat{\rho}_{\text{sys}}(t) = \text{Trace}\{ \hat{\rho}_{\text{full}}(t) \} = \langle E_0 | \hat{\rho}_{\text{full}}(t) | E_0 \rangle + \langle E_1 | \hat{\rho}_{\text{full}}(t) | E_1 \rangle + \langle E_2 | \hat{\rho}_{\text{full}}(t) | E_2 \rangle \]

\[ = |c_1|^2 |v_1\rangle \langle v_1| + |c_2|^2 |v_2\rangle \langle v_2| + c_1^* c_2 |v_2\rangle \langle v_1| + c_2^* c_1 |v_1\rangle \langle v_2| = |\psi(t)\rangle \langle \psi(t)| \]

\[ = \begin{bmatrix} |c_1|^2 & c_2^* c_1 \\ c_1^* c_2 & |c_2|^2 \end{bmatrix} \]

Displays full coherence!
Entanglement and Decoherence

The initial joint state of the “system + environment” is:

$$\phi(t = 0) = \psi \otimes E_0 = c_1 |v_1 \otimes E_0 + c_2 |v_2 \otimes E_0$$

At some later time $t$:

$$\phi(t) = b_1 |v_1 \otimes E_1 + b_2 |v_2 \otimes E_2$$

Interaction with the environment

The environment “measures” the state of the system such that at some later time the environment state reflects the system state as follows:

$$\phi(t) = c_1 |v_1 \otimes E_1 + c_2 |v_2 \otimes E_2$$

This is an entangled state
Entanglement and Decoherence

\[ |\phi(t=0)\rangle = |\psi\rangle \otimes |E_0\rangle \]

Interaction with the environment

\[ |\phi(t)\rangle = b_1 |v_1\rangle \otimes |E_1\rangle + b_2 |v_2\rangle \otimes |E_2\rangle \]

There is no state collapse!

Now we find the density operator for the system by taking the partial trace of the full density operator:

\[ \hat{\rho}_{full}(t) = |\phi(t)\rangle \langle \phi(t)| \]

\[ \Rightarrow \hat{\rho}_{sys}(t) = \text{Trace}\{ \hat{\rho}_{full}(t) \} = \langle E_0 | \hat{\rho}_{full}(t) | E_0 \rangle + \langle E_1 | \hat{\rho}_{full}(t) | E_1 \rangle + \langle E_2 | \hat{\rho}_{full}(t) | E_2 \rangle \]

\[ = |b_1|^2 |v_1\rangle \langle v_1| + |b_2|^2 |v_2\rangle \langle v_2| \]

\[ \hat{\rho} = \begin{bmatrix} |c_1|^2 & c_2^* c_1 \\ c_1^* c_2 & |c_2|^2 \end{bmatrix} \]

\[ \hat{\rho} = \begin{bmatrix} |b_1|^2 & 0 \\ 0 & |b_2|^2 \end{bmatrix} \]

Statistical mixture!

Interaction with the environment made the off-diagonal components of the system density operator go to zero.