ECE 5310: Quantum Optics

Fall 2019

Homework 9 **Due on Nov. 11 at 5:00 PM (self-grade)**

Problem 9.1 (Gauge Transformations for the Electric-Dipole Hamiltonian)

Consider the *N*-particle Hamiltonian in the Coulomb gauge,

$$\hat{H} = \sum_{\alpha=1}^{N} \frac{\left[\hat{\vec{p}}_\alpha - q\hat{\vec{A}}_T(\vec{r}_{\alpha o})\right]^2}{2m} + qV_\alpha(\hat{\vec{r}}_\alpha) + \int d^3\vec{r}\left\{\frac{1}{2}\varepsilon_o \hat{\vec{E}}_T(\vec{r}).\hat{\vec{E}}_T(\vec{r}) + \frac{1}{2}\mu_o \hat{\vec{H}}(\vec{r}).\hat{\vec{H}}(\vec{r})\right\}$$

where the subscript α stands for the α-th particle, $\vec{r}_{\alpha o}$ is the fixed center coordinate of the α-th atom (or two-level system) in which the α-th particle resides, $V_\alpha\left(\hat{\vec{r}}_\alpha\right)$ is the atomic potential of the α-th atom experienced by the α-th particle. **Note carefully what are operators and what are just variables**.

The electric field is:

$$\hat{\vec{E}}_T(\vec{r},t) = V\int \frac{d^3\vec{k}}{(2\pi)^3}\sum_j i\sqrt{\frac{\hbar\omega_k}{2\varepsilon_o}}\left[\hat{a}_j\left(\vec{k},t\right) - \hat{a}_j^+\left(-\vec{k},t\right)\right]\frac{e^{i\vec{k}.\vec{r}}}{\sqrt{V}}\hat{\varepsilon}_j\left(\vec{k}\right)$$

The above collection of two-level systems essentially consists of dipoles whose polarization density can be written as,

$$\hat{\vec{P}}(\vec{r},t) = \sum_{\alpha=1}^{N} \delta^3(\vec{r}-\vec{r}_{\alpha o})q\left(\hat{\vec{r}}_\alpha(t) - \vec{r}_{\alpha o}\right)$$

If we want to select the transverse component of the polarization density we can use the transverse delta function:

$$\hat{\vec{P}}_T(\vec{r},t) = \int d^3\vec{r}\,'\,\overline{\overline{\delta}}_T(\vec{r}-\vec{r}\,')\bullet\hat{\vec{P}}(\vec{r}\,',t) = \sum_{\alpha=1}^{N} q\,\overline{\overline{\delta}}_T(\vec{r}-\vec{r}_{\alpha o})\bullet\left(\hat{\vec{r}}_\alpha(t) - \vec{r}_{\alpha o}\right)$$

One can then also write the transverse D-field as,

$$\hat{\vec{D}}_T(\vec{r},t) = \varepsilon_o\hat{\vec{E}}_T(\vec{r},t) + \hat{\vec{P}}_T(\vec{r},t)$$

Now we will see how these quantities transform under a gauge transformation that will convert the above Hamiltonian into the electric-dipole Hamiltonian.

In what follows, everything will be done in the Schrodinger picture (so no time dependence anywhere).

a) Consider a gauge transformation implemented by the operator,

$$\hat{T} = e^{-\frac{i}{\hbar}q\sum_{\alpha=1}^{N}\hat{\vec{A}}_T(\vec{r}_{\alpha o})\bullet\left(\hat{\vec{r}}_\alpha - \vec{r}_{\alpha o}\right)}$$

And show that under the above gauge transformation,

$$\hat{\vec{E}}_{T\,new}(\vec{r}) = \hat{T}\hat{\vec{E}}_T(\vec{r})\hat{T}^+ = \hat{\vec{E}}_T(\vec{r}) - \frac{\hat{\vec{P}}_T(\vec{r})}{\varepsilon_o}$$

b) Find out how the transverse D-field behaves under the above gauge transformation,

$$\hat{\vec{D}}_{T\,new}(\vec{r}) = \hat{T}\hat{\vec{D}}_T(\vec{r})\hat{T}^+ = ??$$

c) Find out how the kinetic momentum behaves under the above gauge transformation,

$$m\hat{\vec{v}}_{\alpha\,new} = \hat{T}\, m\hat{\vec{v}}_\alpha\, \hat{T}^+ = \hat{T}\left[\hat{\vec{p}}_\alpha - q\hat{\vec{A}}_T(\vec{r}_{\alpha o})\right]\hat{T}^+ = ??$$

c) Find out how the Hamiltonian transforms under the above gauge transformation,

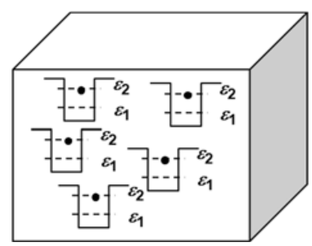
$$\hat{H}_{new}(\vec{r}) = \hat{T}\hat{H}\hat{T}^+ = ??$$

Hint: It will have terms of the form,

$$\int d^3\vec{r}\,\frac{\hat{\vec{P}}_T(\vec{r})\cdot\hat{\vec{P}}_T(\vec{r})}{2\varepsilon_o} \text{ and } -\int d^3\vec{r}\,\frac{\hat{\vec{P}}_T(\vec{r})\cdot\hat{\vec{D}}_T(\vec{r})}{\varepsilon_o}$$

Problem 9.2 (Cooperative Phenomena in Cavity Quantum Optics and Purcell Enhancement)

Consider a cavity with N two-level systems. The Hamiltonian is:



$$\hat{H} = \frac{\Delta\varepsilon}{2}\hat{J}_z + \left(k\,\hat{J}_+\,\hat{a} + k^*\hat{a}^+\hat{J}_-\right) + \hbar\omega_o\,\hat{a}^+\hat{a}$$

$$\hat{J}_z = \sum_{j=1}^{N}\sigma_{zj} \qquad \hat{J}_+ = \sum_{j=1}^{N}\hat{\sigma}_{+j} \qquad \hat{J}_- = \sum_{j=1}^{N}\hat{\sigma}_{-j}$$

$$\hat{J}^2 = \hat{J}_z^2 + \hat{J}_x^2 + \hat{J}_y^2 = \hat{J}_z^2 + 2\left(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+\right)$$

a) Show that:

$$\left[\hat{J}_+,\hat{J}_-\right] = \hat{J}_z$$

$$\left[\hat{J}_\pm,\hat{J}_z\right] = \mp 2\hat{J}_z$$

$$\left[\hat{J}_z,\hat{J}^2\right] = \left[\hat{J}_\pm,\hat{J}^2\right] = 0$$

b) Consider the state, $|N,N\rangle = |\uparrow,\uparrow,\uparrow,.....\uparrow\rangle$. Clearly, it is an eigenstate of $\hat{J}_z$,

$$\hat{J}_z|N,N\rangle = N|N,N\rangle$$

Also, clearly,

$$\hat{J}_+|N,N\rangle = 0$$

Then show that,

$$\hat{J}^2|N,N\rangle = N(N+2)|N,N\rangle$$

Therefore, $|N,N\rangle = |\uparrow,\uparrow,\uparrow,.....\uparrow\rangle$ is a simultaneous eigenstate of $\hat{J}^2$ and $\hat{J}_z$ (they commute so they can have simultaneous eigenstates).

c) Given an eigenstate $|M,N\rangle$ of $\hat{J}^2$ and $\hat{J}_z$,

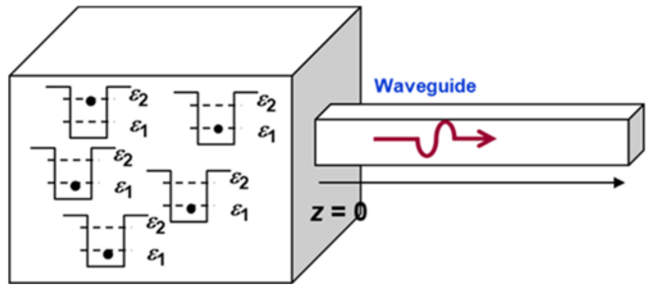
$$\hat{J}^2 |M,N\rangle = N(N+2)|M,N\rangle$$
$$\hat{J}_z |M,N\rangle = M|M,N\rangle$$

Show that the application of the ladder operators can generate eigenstates with all smaller and all larger eigenvalues of $\hat{J}_z$ and argue that they all have the same eigenvalue for $\hat{J}^2$,

$$J_{\pm}|M,N\rangle = \frac{\sqrt{N(N+2)-M(M\pm 2)}}{2}|M\pm 2,N\rangle$$

Note: $\hat{J}_{\pm}|\pm N,N\rangle = 0$

d) Suppose the cavity is now connected to a waveguide and the photon lifetime τ_p in the cavity is much smaller than the vacuum Rabi frequency.



Suppose the initial state is:

$$|\psi(t=0)\rangle = |\uparrow,\downarrow,\downarrow,\downarrow,\downarrow,\ldots\ldots\ \downarrow\rangle \otimes |0\rangle$$

(i.e. only one particular atom is in the exited state)

Find the cavity enhanced spontaneous emission rate at time *t*~0 ?

e) Same as part (d) above, but now,

$$|\psi(t=0)\rangle = |-N+2,N\rangle \otimes |0\rangle$$
$$= \frac{1}{\sqrt{N}}\Big[|\uparrow,\downarrow,\downarrow,\ldots\ldots\ \downarrow\rangle + |\downarrow,\uparrow,\downarrow,\ldots\ldots\ \downarrow\rangle + |\downarrow,\downarrow,\uparrow,\ldots\ldots\ \downarrow\rangle + \ldots.. + |\downarrow,\downarrow,\downarrow,\ldots\ldots\ \uparrow\rangle + \Big] \otimes |0\rangle$$

(still only one atom is in the exited state but you don't know which one)

Find the cavity enhanced spontaneous emission rate at time *t*~0 and compare your results with part (d)?

f) **Explain, as best as you can, why do you see the difference in the spontaneous emission rate in parts (d) and (e) and what can possibly be causing this difference?**