Department of Electrical and Computer Engineering, Cornell University

ECE 5310: Quantum Optics

Fall 2019

Homework 6 **Due on Oct. 21, 2019 (self-grade)**

Problem 6.1 (Phase Noise in Electronic/Optical Oscillators)

This problem is meant to prepare you for the phase properties of laser light.

For this problem, the following definitions will be used.

White Noise (WN):
If any signal $F(t)$ has the properties that,

$$\langle F(t) \rangle = 0$$
$$\langle F(t_1) F(t_2) \rangle = A\ \delta(t_1 - t_2)$$

then it is called white noise. It is called "noise" because its average is zero. It is called "white" because its spectral density $S_{FF}(\omega)$ is non-zero for all frequencies (or has all the frequency components), just like white light.

Gaussian White Noise (GWN):
Gaussian white noise is white noise with the additional property that all higher order correlation functions can be written as a sum of products of second order correlation functions.

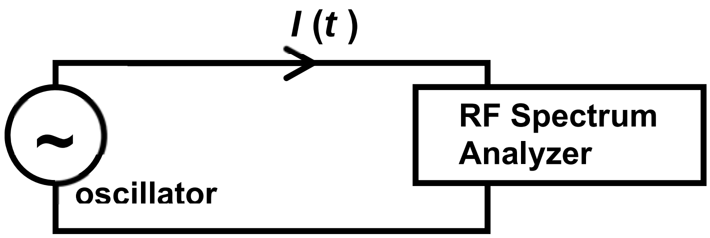
$$\langle F(t) \rangle = 0$$
$$\langle F(t_1) F(t_2) \rangle = A\ \delta(t_1 - t_2)$$

and
$\langle F(t_1) F(t_2) F(t_3) \ldots\ldots F(t_n) \rangle = 0$ for "n" odd, and for "n" even the correlation functions breaks down into a sum of products of all possible pairs that can be formed, e.g.

$$\langle F(t_1) F(t_2) F(t_3) F(t_4) \rangle = \langle F(t_1) F(t_2) \rangle \langle F(t_3) F(t_4) \rangle + \langle F(t_1) F(t_3) \rangle \langle F(t_2) F(t_4) \rangle + \langle F(t_1) F(t_4) \rangle \langle F(t_2) F(t_3) \rangle$$
$$\langle F(t_1) F(t_2) F(t_3) F(t_4) F(t_5) F(t_6) \rangle =$$
$$\langle F(t_1) F(t_2) \rangle \langle F(t_3) F(t_4) \rangle \langle F(t_5) F(t_6) \rangle + \langle F(t_1) F(t_3) \rangle \langle F(t_2) F(t_4) \rangle \langle F(t_5) F(t_6) \rangle + \ldots\ldots$$

a) For "n" even, how many terms will be there be in the expansion of $\langle F(t_1) F(t_2) F(t_3) \ldots\ldots F(t_n) \rangle$?

For example, as shown above, there are 3 terms when n = 4, and you can convince yourself their will be 15 terms when n = 6.
For many applications (e.g. wireless systems) it is important to have electronic oscillators that produce sinusoids of a desired frequency with great precision (i.e. no frequency errors). In this problem you will analyze such an oscillator. Similar analysis will give the phase noise of a laser later in the course. ALL stand alone oscillators (whether electronic or optical or atomic clocks) have phase noise.



Suppose the oscillator output is corrupted by phase noise so that the output signal is,
$I(t) = I_o \cos[\omega_o t + \theta(t)]$
The phase is assumed to be randomly kicked around by some noise inside the oscillator and obeys the Langevin equation,
$$\frac{d\,\theta(t)}{dt} = W(t)$$
where $W(t)$ is **Gaussian** white noise and obeys,
$\langle W(t) \rangle = 0$
$\langle W(t_1) W(t_2) \rangle = \gamma\, \delta(t_1 - t_2)$
The question now is what is measured by the RF spectrum analyzer and how precise and accurate is the frequency produced by the oscillator. Spectrum analyzers determine the spectral density of the incoming signal and display that on the screen. In this problem you will calculate the spectral density of the current signal produced by the oscillator.

b) Show that: $\left\langle [\theta(t_1) - \theta(t_2)]^2 \right\rangle = \gamma\, |t_1 - t_2|$.

c) Show that: $\left\langle [\theta(t_1) - \theta(t_2)]^n \right\rangle = \dfrac{n!}{(n/2)!\ 2^{n/2}}\, \gamma^{n/2}\, |t_1 - t_2|^{n/2}$ for "n" even, and,

$\left\langle [\theta(t_1) - \theta(t_2)]^n \right\rangle = 0$ for "n" odd.

d) Using the results in parts (b) and (c), show that: $\langle\ \exp[i\ \theta(t_1) - i\ \theta(t_2)]\ \rangle = \exp\left[-\gamma \dfrac{|t_1 - t_2|}{2}\right]$

e) Calculate the correlation function $\langle I(t_1) I(t_2) \rangle$ of the signal $I(t) = I_o \cos[\omega_o t + \theta(t)]$ produced by the oscillator. Hint: convert the cosines to exponentials in the correlation function and then discard the terms that contain factors $\pm \omega_o (t_1 + t_2)$ since these go to zero on time averaging performed inside the spectrum analyzer.

f) Fourier transform the correlation function obtained in part (e) to get the spectral density $S_{II}(\omega)$. Your answer should look like a sum of two Lorentzians centered at $\pm \omega_o$. What is the **linewidth** of the frequency produced by the oscillator (linewidth is defined as the FWHM of the spectral density near $\pm \omega_o$)?

g) If there were no phase noise what then would be $S_{II}(\omega)$?

h) If the input resistance of the RF spectrum analyzer is R, what is the total power $R \int_{-\infty}^{\infty} S_{II}(\omega) \dfrac{d\omega}{2\pi}$ produced by the oscillator in parts (f) and (g)? Does it correspond to your intuition just looking at the form of the signal $I(t) = I_o \cos[\omega_o t + \theta(t)]$?

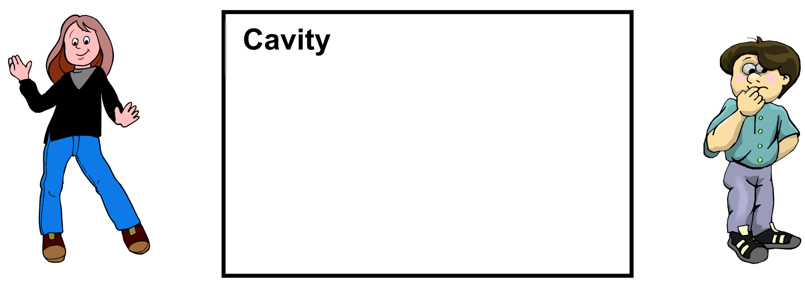
NOTE ADDED: Any stand alone oscillator can have no absolute internal phase reference, and so it cannot completely correct for phase errors generated in the output. How does a stand alone oscillator know its phase is not the right phase? It can't because there is no "right" phase. So the best an oscillator can do is to try and maintain a constant phase. How does an oscillator determine that its phase has changed and a correction is in order to get back the original phase? It can probably use a sophisticated feedback mechanism so that whenever there is a phase jump detected, the feedback mechanism tries to nullify the phase jump. But then what about the noise generated inside the feedback mechanism (or feedback circuitry) itself? It is fundamentally impossible for an oscillator to be completely free from phase noise. Large phase errors (or phase jumps) can generally be detected and corrected but small phase errors accumulate over time (this is called phase drift), and phase noise at small enough frequencies (or equivalently, at large enough time scales) will always appear to do a random walk. This is universal behavior. EVERY stand-alone electronic and/or optical oscillator shows this behavior. This cannot be avoided. These phase jumps are responsible for the linewidth of most oscillators (including lasers – as we will see).

Problem 6.2 (Coherent or not coherent, that is the question)

Your friends Alice and Bob are playing with their cavity and their photodetector. Suppose their cavity supports only a single mode, and the Hamiltonian is (ignoring the vacuum energy part and also ignoring the mode number subscripts from operators and states since we are dealing with a single mode),

$$\hat{H} = \hbar\omega_m \hat{a}^+ \hat{a}$$

Bob is trying to develop a technique to prepare the cavity in a coherent state $|\alpha\rangle$.



a) Write the density matrix $\hat{\rho}$ corresponding to the coherent state $|\alpha\rangle$ in **number state basis**. What is the probability $\rho(n)$ of obtaining "n" photons as a result when making a photon number measurement on a coherent state $|\alpha\rangle$.

In order to make sure his preparation technique yields coherent states $|\alpha\rangle$, Bob decides to make photon number measurements on the states he prepares. Bob prepares his states many times and makes photon number measurements on the states he prepares. After a lot of hard work he plots a histogram of his results and obtains a distribution that looks exactly Poissonian with a mean equal to $|\alpha|^2$. Satisfied with his efforts, Bob goes to Alice and reports his success.

Much to Bob's displeasure, Alice discards Bob's results. Alice tells Bob that the states that Bob prepares using his technique are not coherent states $|\alpha\rangle$ but a statistical mixture of number states described by the following density matrix,

$$\hat{\rho}_A = \sum_{n=0}^{\infty} \exp\left(-|\alpha|^2\right) \frac{\left(|\alpha|^2\right)^n}{n!} |n\rangle \langle n|$$

NOTE ADDED: What Alice is really saying is that each time Bob performs his experiment and uses his technique to prepare the coherent state $|\alpha\rangle$, he ends up preparing a number state instead, and the probability of ending up with the number state $|n\rangle$ in a particular experiment is given by $\exp\left(-|\alpha|^2\right)\left(|\alpha|^2\right)^n / n!$. Therefore, the density matrix for Bob's states is $\hat{\rho}_A$ given above.

c) What is the probability $\rho(n)$ of obtaining "n" photons as a result when making a photon number measurement for Alice's density matrix?

d) How does Alice's density matrix $\hat{\rho}_A$ differ from what you found in part (a) for a coherent state $|\alpha\rangle$? Is there any experimental way that relies **only on photon number measurements** and that can distinguish between Alice's density matrix $\hat{\rho}_A$ and the density matrix that corresponds to the coherent state $|\alpha\rangle$?

e) Is there any experimental method (**not necessarily relying only on photon number measurements**) that can distinguish between Alice's density matrix $\hat{\rho}_A$ and the density matrix that corresponds to the coherent state $|\alpha\rangle$?

After hearing Alice's criticism Bob is depressed. He was pretty sure he was preparing the coherent state $|\alpha\rangle$ with his technique. After going through his apparatus very carefully, Bob discovers that each time he tries to prepare the coherent state $|\alpha\rangle$ he gets the magnitude right but he has no control over the phase of the coherent state. Recall that α is a complex number with a magnitude and a phase, i.e. $\alpha = |\alpha| \exp(i\phi)$. Bob concludes that each time he tries to prepare the coherent state $|\alpha\rangle$, he ends up preparing a coherent state with a magnitudes equal to $|\alpha|$ but with a random phase (i.e. the phase generated in each experiment is arbitrary and any phase value between 0 and 2π has an equal probability of being generated). Consequently, Bob writes the density matrix for the states he generates as an average over the phases,

$$\hat{\rho}_B = \int_0^{2\pi} \frac{d\phi}{2\pi} \; |\alpha\rangle \langle \alpha| \qquad \text{where } \alpha = |\alpha| \exp(i\phi)$$

f) Write Bob's density matrix $\hat{\rho}_B$ in number state basis and do the phase integral to get your final answer.

g) Compare Alice's and Bob's density matrices (i.e. $\hat{\rho}_A$ and $\hat{\rho}_B$). Is there any experimental method that relies **only on photon number measurements** and that can determine whether Alice is right (i.e. Bob produces a number state each time he tries to generate the coherent state $|\alpha\rangle$) or whether Bob is right (i.e. Bob produces a coherent state with a random phase each time he tries to generate the coherent state $|\alpha\rangle$)?

h) Follow up of part (g): is there any experimental method (**not necessarily relying only on photon number measurements**) that is able to determine whether Alice is right or whether Bob is right?

NOTE ADDED: After this course is over, read the paper: **Phys. Rev. A, 55, 3195–3203 (1997)**. If you read it before the course is over, it might prove a bit disturbing. Also try, **Physics Letters A, 333, 378–381 (2004)**.

Problem 6.3 (Two-Mode Squeezing and Continuous Variable Entanglement)

You might have hear of the famous EPR paradox. In the original 1935 paper, EPR considered an entangled state composed of variables that took a continuous set of values. In particular they considered entangled states of two particles with opposite momenta,

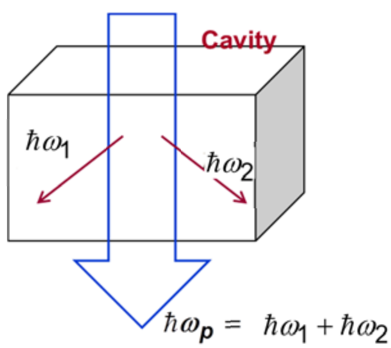
$$\int d^3\vec{p}\, \psi(\vec{p})\, \frac{1}{\sqrt{2}}\left(\left|\vec{p},-\vec{p}\right\rangle + \left|-\vec{p},\vec{p}\right\rangle\right)$$

In the above state, whereas the difference momentum can take on any values, the total momentum is zero. So the momentum values are perfectly correlated in the above entangled state. Such entangled states are distinct from states composed of variables that take on only discrete values. For example, the spin up-down entangled singlet/triplet states of two electrons,

$$\frac{1}{\sqrt{2}}\left(\left|\uparrow,\downarrow\right\rangle \pm \left|\downarrow,\uparrow\right\rangle\right)$$

are a good example of discrete variable entanglement. In this problem you will generate continuous variable (quadrature) entangled states of two radiation modes in a cavity.

Consider two optical modes in a cavity. One can use non-degenerate parametric down conversion, to generate photons in these two modes as follows:



The Hamiltonian is,

$$\hat{H}(t) = \hbar\omega_1\, \hat{a}_1^{+}\, \hat{a}_1 + \hbar\omega_2\, \hat{a}_2^{+}\, \hat{a}_2 + i\hbar\frac{|\kappa|}{2}\left(e^{i\omega_p t - i2\theta}\,\hat{a}_1\hat{a}_2 - e^{-i\omega_p t + i2\theta}\,\hat{a}_2^{+}\hat{a}_1^{+}\right)$$

a) Suppose $\left|\psi(t=0)\right\rangle = \left|0\right\rangle$. Show that,

$$\left|\psi(t)\right\rangle = e^{\frac{|\kappa t|}{2}\left(e^{i\omega_p t - i2\theta}\hat{a}_1\hat{a}_2 - e^{-i\omega_p t + i2\theta}\hat{a}_2^{+}\hat{a}_1^{+}\right)}\left|0\right\rangle$$

The above state is a complicated entangled state of the two cavity modes.

b) Define four quadrature operators in the usual way,

$$\hat{x}_{\theta}(t)=\frac{\hat{a}_1(t)e^{-i\theta}\,e^{i\omega_1 t}+\hat{a}_1^{+}(t)e^{i\theta}\,e^{-i\omega_1 t}}{2}$$

$$\hat{x}_{\theta+\pi/2}(t)=\frac{\hat{a}_1(t)e^{-i\theta}\,e^{i\omega_1 t}-\hat{a}_1^{+}(t)e^{i\theta}\,e^{-i\omega_1 t}}{2i}$$

$$\hat{y}_{\theta}(t)=\frac{\hat{a}_2(t)e^{-i\theta}\,e^{i\omega_2 t}+\hat{a}_2^{+}(t)e^{i\theta}\,e^{-i\omega_2 t}}{2}$$

$$\hat{y}_{\theta+\pi/2}(t)=\frac{\hat{a}_2(t)e^{-i\theta}\,e^{i\omega_2 t}-\hat{a}_2^{+}(t)e^{i\theta}\,e^{-i\omega_2 t}}{2i}$$

Find the four quadrature operators at time t in terms of the same four quadrature operators at time zero and show that at all times the mean values of all the above quadratures are zero.

c) At time t, find the values of these four operators in terms of the quadrature operators at time zero.

$$\hat{x}_{\theta}(t)+\hat{y}_{\theta}(t)$$
$$\hat{x}_{\theta}(t)-\hat{y}_{\theta}(t)$$
$$\hat{x}_{\theta+\pi/2}(t)+\hat{y}_{\theta+\pi/2}(t)$$
$$\hat{x}_{\theta+\pi/2}(t)-\hat{y}_{\theta+\pi/2}(t)$$

Hint: some will grow exponentially and some will decay exponentially.

d) From the results obtained in part c, argue that as time goes on the values of the quadratures become perfectly correlated between the two modes in the cavity.

For fun, see:

[1] "Realization of the Einstein-Podolsky-Rosen Paradox for Continuous Variables,“ Phys. Rev. Lett. 68, 3663 (1992).

[2] "Continuous-Variable Entanglement on a Chip," Nature Photonics volume 9, pages 316–319 (2015).

[3] "Teleportation of Continuous Quantum Variables," Phys. Rev. Lett. 80, 869 – Published 26 January 1998

e) If you are feeling brave show that (if you are not feeling brave, don't worry about it),

$$\left|\psi(t)\right\rangle = e^{\frac{|\kappa t|}{2}\left(e^{i\omega_p t-i2\theta}\hat{a}_1\hat{a}_2-e^{-i\omega_p t+i2\theta}\hat{a}_2^{+}\hat{a}_1^{+}\right)}\left|0\right\rangle$$

$$=\frac{1}{\cosh\left(\frac{|\kappa t|}{2}\right)}\sum_{n=0}^{\infty}\left(-e^{-i\omega_p t+i2\theta}\tanh\left(\frac{|\kappa t|}{2}\right)\right)^{n}\left|n,n\right\rangle$$

Other than the quadratures, there is something else strongly correlated among the two field modes. Can you guess?

f) Suppose one is only able to measure one of the two cavity modes. Find the reduced density operator that describes, say mode 1, and show that the density operator represents a thermal state (Bose-Einstein distribution) and find the mean photon number in mode 1 from your results.