ECE 5310: Quantum Optics Fall 2019

Homework 4 **Due on Oct. 07, 2019 (self-grade)**

Midterm: Midterm will take place in October after the Fall break.

Problem 4.1: (Explorations in quantum optics)

This problem will explore the concept of a "photon" in more detail. You will learn how the probability distribution of the field strengths can be calculated for different photonic states. This problem will tie together much of what you learned about basic quantum mechanics earlier in the course and bring together many fundamental concepts. Most of the calculations are simple and can be done in one line provided you understand the concepts.

The operator equations for a simple harmonic oscillator (particle in a quadratic potential well) are,

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega_o^2\,\hat{x}^2 = \hbar\omega_o\left(\hat{a}^+\hat{a} + \frac{1}{2}\right)$$

where,

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega_o}}\left[\omega_o\,\hat{x} + i\,\hat{p}\right] \qquad \hat{a}^+ = \frac{1}{\sqrt{2\hbar\omega_o}}\left[\omega_o\,\hat{x} - i\,\hat{p}\right]$$

$$\left[\hat{x},\hat{p}\right] = i\hbar \qquad \left[\hat{a},\hat{a}^+\right] = 1$$

Just from the commutation relation $\left[\hat{x},\hat{p}\right] = i\hbar$, and the fact that operators $\hat{x}$ and $\hat{p}$ have eigenkets $|x\rangle$ and $|p\rangle$ that satisfy the completeness relations, we were able to transform the Shrodinger equation in the operator form,

$$i\hbar\frac{\partial|\psi(t)\rangle}{\partial t} = \hat{H}\,|\psi(t)\rangle$$

into a differential equation of the form ,

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2}\frac{\partial^2\psi(x,t)}{\partial x^2} + \frac{1}{2}\omega_o^2\,x^2\psi(x,t)$$

The time-independent form of the above equation yielded the eigenenergies and eigenfunctions of a simple harmonic oscillator.

$$-\frac{\hbar^2}{2}\frac{\partial^2\phi_n(x)}{\partial x^2} + \frac{1}{2}\omega_o^2\,x^2\phi_n(x) = E_n\,\phi_n(x)$$

The eigenfunctions $\phi_n(x)$ (for n=0,1,2,3,…..) are Hermite Gaussians, and are given below,

$$\langle x|0\rangle = \phi_0(x) = \left(\frac{\omega_o}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{\omega_o x^2}{2\hbar}\right) \qquad \text{energy: } \frac{\hbar\omega_o}{2}$$

$$\langle x|1\rangle = \phi_1(x) = \left(\frac{4}{\pi}\left(\frac{\omega_o}{\hbar}\right)^3\right)^{1/4} x\,\exp\left(-\frac{\omega_o x^2}{2\hbar}\right) \qquad \text{energy: } \hbar\omega_o + \frac{\hbar\omega_o}{2}$$

$$\langle x|2\rangle = \phi_2(x) = \left(\frac{\omega_o}{4\pi\hbar}\right)^{1/4}\left[2\frac{\omega_o x^2}{\hbar} - 1\right]\exp\left(-\frac{\omega_o x^2}{2\hbar}\right) \qquad \text{energy: } 2\hbar\omega_o + \frac{\hbar\omega_o}{2}$$

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$$\langle x|n\rangle = \phi_n(x) \qquad \text{energy: } n\hbar\omega_o + \frac{\hbar\omega_o}{2}$$

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Now consider a single mode of electromagnetic field inside a cavity. **Since we are only considering a single mode of the field, I will drop the mode index "m" from all the operators and the states (but keep it for the mode frequency).** The Hamiltonian is,

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega_m^2\,\hat{q}^2 = \hbar\omega_m\left(\hat{a}^+\hat{a} + \frac{1}{2}\right)$$

where,

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega_m}}\left[\omega_m\,\hat{q} + i\,\hat{p}\right] \qquad \hat{a}^+ = \frac{1}{\sqrt{2\hbar\omega_m}}\left[\omega_m\,\hat{q} - i\,\hat{p}\right]$$

$$[\hat{q},\hat{p}] = i\hbar \qquad [\hat{a},\hat{a}^+] = 1$$

The electric and magnetic fields (in the Schrodinger picture) are related directly to the operators $\hat{p}$ and $\hat{q}$ as follows,

$$\hat{\vec{E}}(\vec{r}) = -\frac{\hat{p}}{\sqrt{\varepsilon_o\varepsilon_m}}\vec{U}_m(\vec{r}) \qquad \hat{\vec{H}}(\vec{r}) = \frac{\hat{q}}{\mu_o\sqrt{\varepsilon_o\varepsilon_m}}\nabla\times\vec{U}_m(\vec{r})$$

Since the electric and magnetic fields are physical observables, the operators $\hat{p}$ and $\hat{q}$ are also physical observables and are Hermitian. These operators represent the field amplitudes. They will have eigenstates $|q\rangle$ and $|p\rangle$ that will satisfy the completeness relations,

$$\int_{-\infty}^{\infty} dq\,|q\rangle\langle q| = \hat{1} \text{ and } \int_{-\infty}^{\infty} dp\,|p\rangle\langle p| = \hat{1}$$

a) Starting from the commutator for the field variables $[\hat{q},\hat{p}] = i\hbar$, and using the completeness relations for the eigenkets $|q\rangle$ and $|p\rangle$, show that $\langle p|q\rangle = \dfrac{\exp\left(-i\frac{p}{\hbar}q\right)}{\sqrt{2\pi\hbar}}$.

b) We can expand any quantum state of a **single mode electromagnetic field** in the $|q\rangle$ eigenkets as follows: $|\psi(t)\rangle = \int_{-\infty}^{\infty} dq\;\;\psi(q,t)|q\rangle$. Show that $\psi(q,t)$ satisfies the differential equation,

$$i\hbar\frac{\partial\psi(q,t)}{\partial t} = -\frac{\hbar^2}{2}\frac{\partial^2\psi(q,t)}{\partial q^2} + \frac{1}{2}\omega_o^2\,q^2\psi(q,t)$$

c) The vacuum state $|0\rangle$ of the electromagnetic field (associated with the mode under consideration) can be expanded in the $|q\rangle$ basis set. Find the coefficients of this expansion.

d) The number state $|2\rangle$ of the electromagnetic field can be expanded in the $|q\rangle$ basis set. Find the coefficients of this expansion.

e) Suppose we prepare the single photon state $|1\rangle$ inside the cavity. If a measurement is made at the location $\vec{r}$ to determine the magnetic field strength, given by the operator $\hat{\vec{H}}(r)$, what is the probability $P(w)$ of obtaining the result $\frac{w}{\mu_o\sqrt{\varepsilon_o\varepsilon_m}}\nabla\times\vec{U}_m(r)$? Your answer $P(w)$ must be normalized such that,

$\int_{-\infty}^{\infty} P(w)\,dw = 1$.

f) Suppose we prepare the number state $|2\rangle$ inside the cavity. If a measurement is made at the location $\vec{r}$ to determine the magnetic field strength, given by the operator $\hat{\vec{H}}(r)$, then what field strengths w, if any, can never be obtained as a result of this measurement.

g) Suppose we have a vacuum state $|0\rangle$ inside the cavity. If a measurement is made at the location $\vec{r}$ to determine the electric field strength, given by the operator $\hat{\vec{E}}(r)$, what is the probability $P(w)$ of obtaining the result $\left(-\frac{w}{\sqrt{\varepsilon_o\varepsilon_m}}\vec{U}_m(r)\right)$? Note that $P(w)$ must be normalized such that $\int_{-\infty}^{\infty} P(w)\,dw = 1$.

h) Suppose the field is in the state $|2\rangle$ (i.e. a state with a definite number of photons). A magnetic field measurement is made at the location $\vec{r}$ and the result $\frac{q}{\mu_o\sqrt{\varepsilon_o\varepsilon_m}}\nabla\times\vec{U}_m(r)$ is obtained. What is the state of the field immediately after the measurement? Write your answer as a linear superposition of photon number states $|n\rangle$. Try to explain where the other photons came from?

Problem 4.2: (Working with photon number states)

In this problem you will get some practice with different photon states. The Hamiltonian of the field in a cavity is,

$$\hat{H} = \sum_{m=1}^{\infty} \hbar\omega_m\left(\hat{a}_m^+\hat{a}_m + \frac{1}{2}\right)$$

where $\left[\hat{a}_m,\hat{a}_n^+\right] = \delta_{nm}$. The photon number operator for the mode "m" is $\hat{n}_m = \hat{a}_m^+\hat{a}_m$. And the photon number operator for all the modes is $\hat{N} = \sum_{m=1}^{\infty}\hat{n}_m$.

a) Suppose the state of the field is described by a state vector $|\psi\rangle = \frac{\sqrt{3}}{2}|n\rangle_m + \frac{1}{2}|n+2\rangle_m$ (*** recall that $|n\rangle_m$ means "n" photons in mode "m" AND zero photons in every other mode).

If a measurement is made to determine the number of photons in mode "m", what is going to be the mean result? What is going to be the standard deviation?

b) Suppose the state of the field is described by a density operator,

$$\hat{\rho} = \frac{1}{4}\left[\,|n\rangle_{m\,m}\langle n| + |n+2\rangle_{m\,m}\langle n+2| + |n-1\rangle_{p\,p}\langle n-1| + |n+2\rangle_{p\,p}\langle n+2|\right]$$

If a measurement is made to determine the number of photons in mode "m", what is going to be the mean result? What is going to be the standard deviation?

Hint: Here you will encounter the first case of complexity creeping in. When you take averages w.r.t. a density operator you do a trace operation and for that purpose you use any complete basis set. Here you have a problem that deals with two modes of the field so the appropriate complete basis set to use is not the following,

$$\sum_{n=0}^{\infty} |n\rangle_m \ {}_m\langle n| = \hat{1}$$

Since this only works in the Hilbert space of mode "m". You need to use a complete basis for the Hilbert space of the two modes,

$$\sum_{n=0}^{\infty} \sum_{q=0}^{\infty} \Big(|n\rangle_m \otimes |q\rangle_p \Big)\Big({}_m\langle n| \otimes {}_p\langle q| \Big) = \hat{1}$$

If I ever gave you a density operator that dealt with three modes, you should use a complete basis set in the Hilbert space of those three modes. Make sure you understand all the confusing indexing and labeling procedures before attempting the problem.

c) For the state of the field given in part (b), if a measurement is made to determine the number of photons in mode "p", what is going to be the mean result? What is going to be the standard deviation?

d) For the state given in part (a), if a measurement is made to determine the TOTAL number of photons in all the modes what is going to be the mean result? What is going to be the standard deviation?

e) For the state given in part (b), if a measurement is made to determine the TOTAL number of photons in all the modes what is going to be the mean result? What is going to be the standard deviation?

f) Suppose the state of the field is described by a density operator,

$$\hat{\rho} = \frac{1}{3}\Big[\Big(|n\rangle_m \otimes |n-1\rangle_p \Big)\Big({}_m\langle n| \otimes {}_p\langle n-1| \Big) + |n+2\rangle_m \ {}_m\langle n+2| + |n+2\rangle_p \ {}_p\langle n+2| \Big]$$

If a measurement is made to determine the TOTAL number of photons in all the modes what is going to be the mean result? What is going to be the standard deviation?

Points to ponder:
In real life, the way you will do a measurement of the number of photons in mode "m" (or "p") is by sticking a photodetector inside the cavity that will respond to only photons at frequency of mode "m" (or "p"). You can do a measurement of the TOTAL photon number by using a photodetector that responds to frequencies of both the modes, but then you may not get information about how many photons were in each mode since you made a total photon number measurement. Notice that the process of measurement destroys the photon (i.e. destroys the state we want to measure). Can you think of a photon number measurement scheme that would NOT destroy the photons it was trying to count?

Problem 4.3 (Field Heisenberg equations and some polarization algebra)

In the lecture handouts we found the following equal-time commutation relations between the fields:

$$\Big[\hat{E}_a(\vec{r},t), \hat{E}_b(\vec{r}\,',t)\Big] = 0$$

$$\Big[\hat{H}_a(\vec{r},t), \hat{H}_b(\vec{r}\,',t)\Big] = 0$$

$$\Big[\hat{E}_a(\vec{r},t), \hat{H}_b(\vec{r}\,',t)\Big] = -i\hbar c^2 \sum_c \varepsilon_{abc} \partial_c \delta^3(\vec{r} - \vec{r}\,')$$

a) Find the Heisenberg equations for the time evolution of the fields:

$$i\hbar \frac{\partial \hat{\vec{E}}(\vec{r},t)}{\partial t} = \left[\hat{\vec{E}}(\vec{r},t), \hat{H} \right] = ?$$

$$i\hbar \frac{\partial \hat{\vec{H}}(\vec{r},t)}{\partial t} = \left[\hat{\vec{H}}(\vec{r},t), \hat{H} \right] = ?$$

Hint: Might be easier to find the equation for one component first and then assemble the result in vector notation.

b) Prove the following 3 relations:

$$\sum_{j=1}^{2} \hat{\varepsilon}_j(\hat{k}).\hat{e}_a \quad \hat{\varepsilon}_j(\hat{k}).\hat{e}_b = \delta_{ab} - \frac{k_a k_b}{k^2}$$

$$\sum_{j=1}^{2} \left(\hat{k} \times \hat{\varepsilon}_j(\hat{k})\right).\hat{e}_a \quad \left(\hat{k} \times \hat{\varepsilon}_j(\hat{k})\right).\hat{e}_b = \delta_{ab} - \frac{k_a k_b}{k^2}$$

$$\sum_{j=1}^{2} \hat{\varepsilon}_j(\hat{k}).\hat{e}_a \quad \left(\hat{k} \times \hat{\varepsilon}_j(\hat{k})\right).\hat{e}_b = \sum_{c} \varepsilon_{abc} \frac{k_c}{k}$$

Problem 4.4 (The Puzzle of Photon Momentum)

There are two ways to express electromagnetic momentum in classical E&M:

$$\vec{P}_A = \varepsilon_o \mu_o \int d^3\vec{r}\ \vec{E}(\vec{r},t) \times \vec{H}(\vec{r},t) \qquad (1)$$

$$\vec{P}_M = \int d^3\vec{r}\ \vec{D}(\vec{r},t) \times \vec{B}(\vec{r},t) \qquad (2)$$

Of course, in free space both expressions are identical. But they can be very different in a dielectric medium. Suppose we have an infinite medium with constant refractive index n and dielectric constant ε such that,

$$n = \sqrt{\frac{\varepsilon}{\varepsilon_o}}$$

a) Suppose one quantizes the electromagnetic fields in this medium, and obtains the photon number and energy eigenstates $|n\rangle_{\vec{k},j}$ such that $\hat{H}|n\rangle_{\vec{k},j} = \left(n + \frac{1}{2}\right)\hbar\omega_k |n\rangle_{\vec{k},j}$. How is the eigenvalue ω_k related to the magnitude k of the wavevector?

b) Show that the states $|n\rangle_{\vec{k},j}$ are also eigenstates of both momentum operators $\hat{\vec{P}}_A$ and $\hat{\vec{P}}_M$ corresponding to the quantities $\vec{P}_A$ and $\vec{P}_M$ defined above, and this can be expressed as (for a single photon state),

$$\hat{\vec{P}}_M |1\rangle_{\vec{k},j} = \alpha_M\ \hbar \frac{\omega}{c} \hat{k} |1\rangle_{\vec{k},j}$$

$$\hat{\vec{P}}_A |1\rangle_{\vec{k},j} = \alpha_A\ \hbar \frac{\omega}{c} \hat{k} |1\rangle_{\vec{k},j}$$

Find expressions for the constants α_A and α_M.

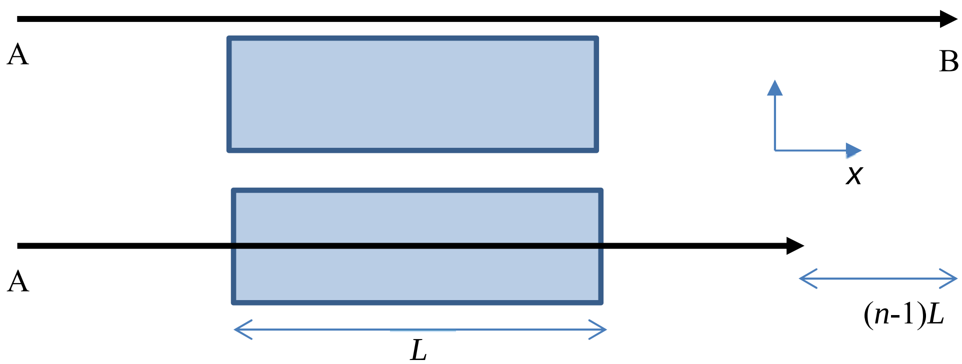
c) Consider a photon with energy $\hbar\omega$ and momentum p_{ph} (unknown) moving in a medium with refractive index n towards an atom of mass M sitting at rest inside that medium. After the atom has absorbed this photon, the atom starts moving with velocity v (in order to conserve momentum before and after the absorption event). Now to get full complete absorption, the atomic resonance must be carefully tuned to match the photon frequency. The problem is that the atom starts moving while it is still absorbing the photon so it sees the Doppler shifted frequency of the incoming photon. Therefore, we assume that the atomic resonance energy (energy level separation) $\Delta\varepsilon$ is related to the photon frequency as,

$$\Delta\varepsilon = \hbar\omega\left(1 - \frac{nv}{2c}\right)$$

The factor of 2 difference (inside the brackets) from the standard doppler shift expression is because $v/2$ is the average velocity of the atom during the absorption event. Now use energy and momentum conservation arguments to find the momentum p_{ph} of the photon before absorption. Does it equal

$\alpha_M \hbar\frac{\omega}{c}$ or $\alpha_A \hbar\frac{\omega}{c}$?

d)



Consider a photon of energy $\hbar\omega$ and momentum $\hbar\omega/c$ in free-space going from point A to point B in time T. The photon goes past a dielectric slab of mass M and length L and refractive index n (but it does not enter the slab, i.e. does not interact with the slab). If now the dielectric slab of length L is placed in the path of the photon, the photon will travel slower in the dielectric slab (velocity will get reduced to c/n) and so after the same time T it will travel an overall smaller distance by $(n\text{-}1)L$, as shown in the figure. There is a theorem in classical relativistic mechanics that says that the center-of-energy coordinate of a closed system must change at a uniform rate (without any accelerations) no matter what interactions are present within the closed system. This means that the reduction in the distance traveled by the photon (when the photon went through the slab) times the photon energy must equal a forward horizontal displacement Δx in the position of the slab times the energy of the slab (which can be taken to be Mc^2),

$$Mc^2\Delta x = (n-1)L\hbar\omega$$

What this means is that the slab must have had a non-zero momentum p_{sl} while the photon was travelling inside it. Find the average momentum p_{sl} of the dielectric slab while the photon was travelling inside it and express your answer completely in terms of $\hbar\omega/c$ and the refractive index n.

e) Same as part d. Use your result from part (d) and then use momentum conservation to find the photon momentum p_{ph} when the photon was inside the slab. Does it equal $\alpha_M \hbar\frac{\omega}{c}$ or $\alpha_A \hbar\frac{\omega}{c}$?

f) Based on the above analysis, can you say which expression for the photon momentum, $\alpha_M \hbar\frac{\omega}{c}$ or $\alpha_A \hbar\frac{\omega}{c}$, is the correct one? Explain your answer.