ECE 5310: Quantum Optics for Photonics and Optoelectronics

Fall 2019

Homework 3 **Due on Sep. 30, 2019 (self-grade)**

Problem 3.1: (EIT in the Dressed State Formalism)

In the lectures, we discussed EIT and derived the susceptibility working in the basis $|e_1\rangle$, $|e_2\rangle$ and $|e_3\rangle$. In this problem, you will work with the time-independent Hamiltonian, and work in the dressed basis $|e_1\rangle$, $|v_a\rangle$ and $|v_b\rangle$ and re-derive the expression for the susceptibility. The starting point is the Hamiltonian in the dressed basis,

$$\hat{H}_R = \begin{bmatrix} \varepsilon_3 - \Delta_{13} & -\frac{\hbar\Omega_{R13}}{\sqrt{8}} e^{+i\phi_{13} - i\phi_{23}/2} & \frac{\hbar\Omega_{R13}}{\sqrt{8}} e^{+i\phi_{13} - i\phi_{23}/2} \\ -\frac{\hbar\Omega_{R13}}{\sqrt{8}} e^{-i\phi_{13} + i\phi_{23}/2} & \lambda_a & 0 \\ \frac{\hbar\Omega_{R13}}{\sqrt{8}} e^{-i\phi_{13} + i\phi_{23}/2} & 0 & \lambda_b \end{bmatrix}$$

where,

$$|v_a\rangle = \frac{1}{\sqrt{2}}\left(e^{i\phi_{23}/2}|e_2\rangle + e^{-i\phi_{23}/2}|e_3\rangle\right) \qquad \lambda_a = \varepsilon_3 - \frac{\hbar\Omega_{R23}}{2}$$

$$|v_b\rangle = \frac{1}{\sqrt{2}}\left(e^{i\phi_{23}/2}|e_2\rangle - e^{-i\phi_{23}/2}|e_3\rangle\right) \qquad \lambda_b = \varepsilon_3 + \frac{\hbar\Omega_{R23}}{2}$$

In the lecture handout, when we used the time-dependent Hamiltonian for EIT, we wrote in steady state,

$$\rho_{31}(t) = \tilde{\rho}_{31} e^{-i\omega_{13}t - i\phi_{13}}$$

And then the susceptibility was given by the simple expression,

$$\chi(\omega_{13}) = \frac{qN\left(\vec{d}_{13}.\hat{n}\right)}{\varepsilon_o}\frac{2\tilde{\rho}_{31}}{E_{13}}$$

When using the time-independent Hamiltonian above, in steady state one would get,

$$\rho_{31}(t) = \tilde{\rho}_{31} e^{-i\phi_{13}} = \text{independent of time}$$

And then the susceptibility could be obtained using the same expression as given above.

a) Write the density matrix element $\rho_{31}(t)$ in terms of the elements $\rho_{a1}(t)$ and $\rho_{b1}(t)$ in the basis $|e_1\rangle$, $|v_a\rangle$ and $|v_b\rangle$. It should only depend on the difference $\rho_{a1}(t) - \rho_{b1}(t)$

b) Starting from the density matrix equation,

$$i\hbar\frac{d\hat{\rho}(t)}{dt} = \left[\hat{H}_R, \hat{\rho}(t)\right]$$

Drive equations for the elements $\rho_{a1}(t)$ and $\rho_{b1}(t)$. You can assume that the populations do not change when calculating linear response, and so if at time $t = 0$ the electron was in state $|e_1\rangle$ then assume that $\rho_{11} = 1$ and $\rho_{aa} = \rho_{bb} = 0$ for all time.

c) When modeling decoherence in the lecture handouts we assumed that,

$$\frac{d\rho_{31}(t)}{dt} \sim -\gamma_{13}\rho_{31}(t) + ...$$

$$\frac{d\rho_{21}(t)}{dt} \sim -\gamma_{12}\rho_{21}(t) + ...$$

And then we assumed that $\gamma_{12} = 0$. Working under the assumption that $\gamma_{12} = 0$, but $\gamma_{13} \neq 0$, how would you modify the equations you obtained in part (b) to include decoherence due to the fact that $\gamma_{13} \neq 0$. You need to add terms in the equations for $\rho_{a1}(t)$ and $\rho_{b1}(t)$ obtained in part (b), and these terms should depend on γ_{13}. Hint: results from part (a) will be helpful.

d) Define,

$$\rho_d(t) = \rho_{a1}(t) - \rho_{b1}(t)$$

$$\rho_s(t) = \rho_{a1}(t) + \rho_{b1}(t)$$

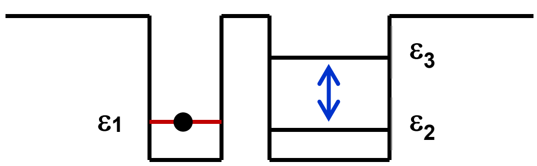
and, using results from part (c), write down uncoupled second order differential equations for $\rho_d(t)$ and $\rho_s(t)$.

e) Argue that $\rho_d(t)$ will have a well-defined steady state value, and the find that value.

f) Since the susceptibility only depends on the steady state value of $\rho_d(t)$ (see part (a) above), find the susceptibility and compare with the expression in the lecture handouts.

Problem 3.2: (Dressed State Formalism and AC Start Effects)

Consider the problem in which a state 1 in a potential well is coupled (via tunneling) to a state 2 in an adjacent potential well. The other well has two states and these two states are coupled by strong radiation that is tuned to the energy separation of levels 2 and 3 (**without detuning**).



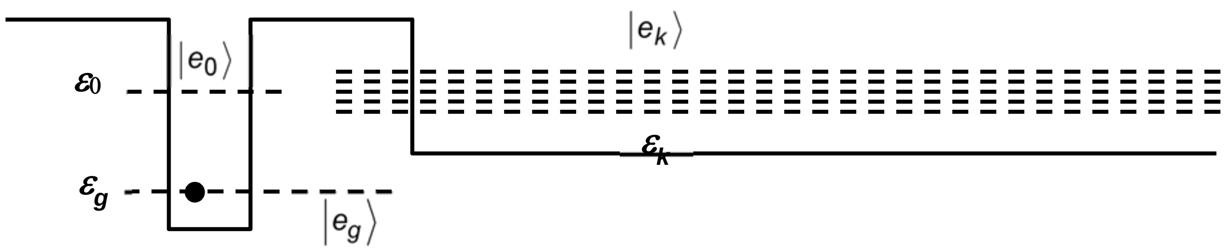
$$\hat{H}(t) = \varepsilon_1|e_1\rangle\langle e_1| + \varepsilon_2|e_2\rangle\langle e_2| + \varepsilon_3|e_3\rangle\langle e_3| - U\left[\,|e_1\rangle\langle e_2| + |e_2\rangle\langle e_1|\,\right]$$
$$-\frac{\hbar\Omega_{R23}}{2}\left[e^{i\omega_{23}t + i\phi_{23}}|e_2\rangle\langle e_3| + e^{-i\omega_{23}t - i\phi_{23}}|e_3\rangle\langle e_2|\,\right]$$

Suppose one can tune the energy ε_1 of the state 1 (by applying a DC electric field from outside or by applying strain if this was a semiconductor material). We need to find out what ought to be the optimal energy of the state 1 such that an electron sitting in that state can easily tunnel from one well into the other. Of course, the answer is easy and trivial if there was no radiation present and for tunneling we then should have $\varepsilon_1 = \varepsilon_2$. But the answer is not so simple if strong radiation is present (assume $\hbar\Omega_{R23} >> U$).

a) You will need to come up with a time-independent Hamiltonian $\hat{H}_R$, find the eigenstates and eigenenergies in the well with the two states, and then you can tune the energy of the state 1 to any one of these energy levels in order for tunneling to happen. Carry out this procedure and find the optimal energy of the state 1.

Problem 3.3: (Fano Resonances and Lineshapes in Optical Transitions)

Often in optical transitions the final upper state is not isolated but coupled to many other states. Consider the problem shown below. The upper state is coupled to a continuum.



Let the Hamiltonian be,

$$\hat{H}_o = \varepsilon_g |e_g\rangle\langle e_g| + \varepsilon_0 |e_0\rangle\langle e_0| + \sum_{k=1}^{\infty} \varepsilon_k |e_k\rangle\langle e_k| - U\sum_{k=1}^{\infty}\left[|e_0\rangle\langle e_k| + |e_k\rangle\langle e_0| \right]$$

This implies that for our Hilbert space,

$$|e_g\rangle\langle e_g| + |e_0\rangle\langle e_0| + \sum_{k=1}^{\infty} |e_k\rangle\langle e_k| = \hat{1}$$

Cleary, $|e_0\rangle$ is not an eigenstate of the full Hamiltonian because an electron placed in it will decay into the continuum on the right with a rate given by,

$$\Gamma = \frac{2\pi}{\hbar} \sum_{k=1}^{\infty} U^2 \,\delta(\varepsilon_k - \varepsilon_0) = \frac{2\pi}{\hbar} U^2 D$$

But $|e_g\rangle$ is an eigenstate of the full Hamiltonian. So we try to find the exact eigenstates of the full Hamiltonian for at least all the upper energy levels. Let these be $|E\rangle$ and the corresponding eigenvalues, E . The states $|E\rangle$ form a continuum. We assume that,

$$|e_g\rangle\langle e_g| + \int_{min}^{max} dE\, D\, |E\rangle\langle E| = \hat{1} \qquad \langle E|E'\rangle = \delta_{E,E'}$$

The density of states per unit energy interval, D , for the eigenstates is assumed to be the roughly same as those for states $|e_k\rangle$. The limits on the integral above include an appropriate energy interval in the vicinity of ε_0 .

a) Show that,

$$-U\frac{\langle e_0|E\rangle}{E-\varepsilon_k} = \langle e_k|E\rangle$$

b) Now use the result in part (a) to show that,

$$\sum_k \frac{U^2}{E-\varepsilon_k} = E - \varepsilon_0$$

We know that the state $\left|e_0\right\rangle$ decays, so it must be made up of a superposition of the energy eigenstates of the full Hamiltonian. Knowing its decay rate Γ one may write,

$$\left|e_0\right\rangle = \int_{\min}^{\max} dE\, D \sqrt{\frac{\left(\hbar\Gamma/2\pi D\right)}{\left(E-\varepsilon_o\right)^2+\left(\hbar\Gamma/2\right)^2}}\left|E\right\rangle$$

Although the above result can be shown to be true rigorously, a plausible validity of the above can be shown as follows. Suppose we prepare a particle in the initial state $\left|\psi\left(t=0\right)\right\rangle = \left|e_0\right\rangle$. Then at a later time we must have,

$$\left|\psi\left(t\right)\right\rangle = e^{-\frac{i}{\hbar}\hat{H}t}\left|\psi\left(t=0\right)\right\rangle = \int_{\min}^{\max} dE\, D \sqrt{\frac{\left(\hbar\Gamma/2\pi D\right)}{\left(E-\varepsilon_o\right)^2+\left(\hbar\Gamma/2\right)^2}}\, e^{-\frac{i}{\hbar}Et}\left|E\right\rangle$$

It follows that the probability of finding the particle in the initial state $\left|\psi\left(t=0\right)\right\rangle = \left|e_0\right\rangle$ at a later time can be found as follows,

$$\left\langle e_0\middle|\psi\left(t\right)\right\rangle = \int_{\min}^{\max} dE\, D \sqrt{\frac{\left(\hbar\Gamma/2\pi D\right)}{\left(E-\varepsilon_o\right)^2+\left(\hbar\Gamma/2\right)^2}}\, e^{-\frac{i}{\hbar}Et}\left\langle e_0\middle|E\right\rangle$$

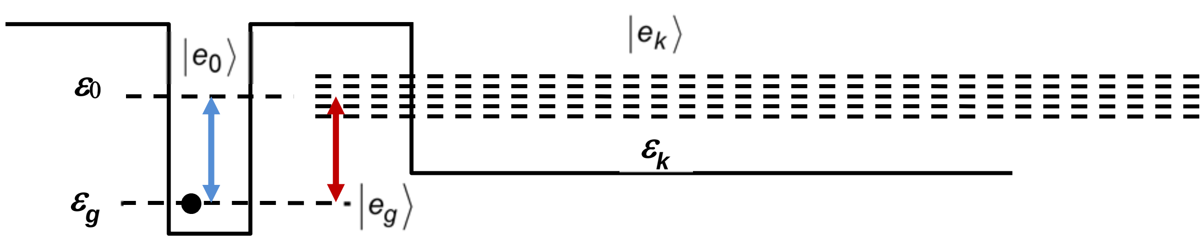
$$= \int_{\min}^{\max} dE\, D \frac{\left(\hbar\Gamma/2\pi D\right)}{\left(E-\varepsilon_o\right)^2+\left(\hbar\Gamma/2\right)^2} e^{-\frac{i}{\hbar}Et} = e^{-\frac{i}{\hbar}\varepsilon_o t}\, e^{-\frac{\Gamma}{2}t}$$

$$\Rightarrow \left|\left\langle e_0\middle|\psi\left(t\right)\right\rangle\right|^2 = e^{-\Gamma t}$$

which is as expected. This means,

$$\left\langle e_0\middle|E\right\rangle = \sqrt{\frac{\left(\hbar\Gamma/2\pi D\right)}{\left(E-\varepsilon_o\right)^2+\left(\hbar\Gamma/2\right)^2}}$$

Now we change the problem as follows.



We shine light on the system and we assume the Hamiltonian is,

$$\hat{H} = \varepsilon_g\left|e_g\right\rangle\left\langle e_g\right| + \varepsilon_0\left|e_0\right\rangle\left\langle e_0\right| + \sum_{k=1}^{\infty}\varepsilon_k\left|e_k\right\rangle\left\langle e_k\right| - U\sum_{k=1}^{\infty}\left[\left|e_0\right\rangle\left\langle e_k\right| + \left|e_k\right\rangle\left\langle e_0\right|\right]$$

$$-\frac{\hbar\Omega_{R0}}{2}\left[e^{-i\omega t}\left|e_0\right\rangle\left\langle e_g\right| + e^{i\omega t}\left|e_g\right\rangle\left\langle e_0\right|\right]$$

$$-\frac{\hbar\Omega_R}{2}\sum_k\left[e^{-i\omega t}\left|e_k\right\rangle\left\langle e_g\right| + e^{i\omega t}\left|e_g\right\rangle\left\langle e_k\right|\right]$$

The problem is now more complex than it seems. Suppose the electron was sitting initially in the ground state $\left|e_g\right\rangle$. The naïve expectation would be that optical transition leading to light absorption can occur in two ways shown by the two arrows in the figure above. But this naïve picture ignores the interference between these two paths and is therefore incorrect.

A better way to think about the optical transitions would be to consider the optical transition to be from the initial ground state $\left|e_g\right\rangle$ to the continuum of final states that are the exact eigenstates of the Hamiltonian $\hat{H}_o$. To calculate the transition rates one needs to rewrite the Hamiltonian $\hat{H}$ using the basis $\left|e_g\right\rangle$ and $\left|E\right\rangle$,

$$\hat{H} = \varepsilon_g \left|e_g\right\rangle\left\langle e_g\right| + \int_{\min}^{\max} dE\, D\; E\left|E\right\rangle\left\langle E\right| - \int_{\min}^{\max} dE\, D\, \frac{\hbar\Omega_R(E)}{2}\left[e^{-i\omega t}\left|E\right\rangle\left\langle e_g\right| + e^{i\omega t}\left|e_g\right\rangle\left\langle E\right|\right]$$

c) Find the energy-dependent Rabi energy function $\hbar\Omega_R(E)$, and show that it equals,

$$\hbar\Omega_R(E) = \frac{\left(\hbar\Gamma/2\pi D\right)^{1/2}\hbar\Omega_{R0} - \left(E-\varepsilon_o\right)\hbar\Omega_R}{\sqrt{\left(E-\varepsilon_o\right)^2 + \left(\hbar\Gamma/2\right)^2}}$$

d) Show that the optical transition rate (due to stimulated absorption) then equals,

$$R_{st-ab} = \frac{\pi}{2\hbar}\left[\frac{\sqrt{\hbar\Gamma/2\pi D}\;\hbar\Omega_{R0} - \left(\varepsilon_g + \hbar\omega - \varepsilon_o\right)\hbar\Omega_R}{\sqrt{\left(\varepsilon_g + \hbar\omega - \varepsilon_o\right)^2 + \left(\hbar\Gamma/2\right)^2}}\right]^2 D$$

If $\hbar\Omega_R$ were zero, i.e. red-arrow transitions in the figure are not allowed, the above reduces to the expected result for optical transition rate in a simple two-level system where the decoherence rate is $\Gamma/2$. But more interesting things happen when $\hbar\Omega_R$ is not zero.

e) Plot the absorption spectra, as a function of the dimensionless variable,

$$\left(\varepsilon_g + \hbar\omega - \varepsilon_0\right)/\left(\hbar\Gamma/2\right)$$

for three different values of the ratio:

$$\frac{\hbar\Omega_{R0}}{\hbar\Omega_R}\sqrt{\frac{2}{\hbar\Gamma\pi D}}$$

Choose values of 0.2, 1.0, and 5.0 and show (qualitatively) how the absorption lineshape varies. You will see some interesting behavior.

Problem 3.4 (Optical Linear Response: Propagating Fields) OPTIONAL (do it if you have never seen this material before)

A two level system interacting with a classical E&M field is described by the Hamiltonian,

$$\hat{H} = \varepsilon_1\left|e_1\right\rangle\left\langle e_1\right| + \varepsilon_2\left|e_2\right\rangle\left\langle e_2\right| - \frac{\hbar\Omega_R}{2}\left[\left|e_1\right\rangle\left\langle e_2\right| + \left|e_2\right\rangle\left\langle e_1\right|\right]\left(e^{i\omega t} + e^{-i\omega t}\right)$$

where $\hbar\Omega_R = qE_o\left\langle e_2\right|\hat{\vec{r}}.\hat{n}\left|e_1\right\rangle = qE_o\left\langle e_1\right|\hat{\vec{r}}.\hat{n}\left|e_2\right\rangle$. Consider a medium made up of two level systems with N systems per unit volume. In the lecture notes, you have seen the derivation for the macroscopic polarization density $\vec{P}(t)$. This polarization, which is in fact produced by the E&M field, will in turn affect the E&M field that produced it, and the goal of this and the next problem is to study this "back reaction" since this forms the basis of optical gain and loss in the semi-classical approximation. We assume wave-propagation in the +z-direction. E&M wave propagation in the presence of a space dependent polarization density $\vec{P}(z,t)$ is described by the Maxwell's wave equation,

$$\nabla \times \nabla \times \vec{E}(z,t) = -\mu_o \frac{\partial^2 \vec{D}(z,t)}{\partial t^2} = -\frac{1}{c^2}\frac{\partial^2 \vec{E}(z,t)}{\partial t^2} - \mu_o \frac{\partial^2 \vec{P}(z,t)}{\partial t^2}$$

$$\text{Note that}: \vec{D}(z,t) = \varepsilon_o \vec{E}(z,t) + \vec{P}(z,t)$$

$$\Rightarrow \qquad \nabla^2 \vec{E}(z,t) - \frac{1}{c^2}\frac{\partial^2 \vec{E}(z,t)}{\partial t^2} = \mu_o \frac{\partial^2 \vec{P}(z,t)}{\partial t^2} \qquad \textbf{(1)}$$

In free space, where the polarization density is zero, i.e. $\vec{P}(z,t) = 0$, the solution is,

$$\vec{E}(z,t) = \hat{n}E_o \cos(kz - \omega t) = \hat{n}\frac{E_o}{2}\exp(i\,kz - i\,\omega t) + \hat{n}\frac{E_o}{2}\exp(-i\,kz + i\,\omega t)$$

In the following parts suppose an E&M wave enters a medium of length L which contains our two level systems.

Assume that the amplitude of the E&M wave is so small that it does not cause a significant change in the populations of the two-level systems as it propagates through them. So the populations remain constant. Of course, this is only an approximation but it is not a particularly bad one in the linear response regime.

You need to solve equation (1) above in the presence of $\vec{P}(z,t)$. To do this, assume a solution in which the amplitude of the E&M wave is slowly changing in space (slowly varying envelope approximation in space),

$$\vec{E}(z,t) = \hat{n}\frac{E_o(z)}{2}\exp(i\,kz - i\,\omega t) + \hat{n}\frac{E_o^*(z)}{2}\exp(-i\,kz + i\,\omega t) \qquad \textbf{(2)}$$

This amplitude change is caused by the material polarization density. The polarization density $\vec{P}(z,t)$ is related to the electric field $\vec{E}(z,t)$ at the same location through $\chi(\omega)$ as shown below,

$$\vec{P}(z,t) = \varepsilon_o \chi(\omega)\hat{n}\frac{E_o(z)}{2}\exp(i\,kz - i\,\omega t) \; + \; \varepsilon_o \chi^*(\omega)\hat{n}\frac{E_o^*(z)}{2}\exp(-i\,kz + i\,\omega t) \qquad \textbf{(3)}$$

The expression for $\chi(\omega)$ is the same as that calculated in the lecture notes (**without making the rotating wave approximation**), and the populations are those calculated in part (a) of this problem. Since the amplitude of the E&M wave is changing slowly, one may make the following approximation,

$$\nabla^2 \vec{E}(z,t) \approx -k^2 \vec{E}(z,t) + ik\hat{n}\frac{\partial E_o(z)}{\partial z}\exp(i\,kz - i\,\omega t) - ik\hat{n}\frac{\partial E_o^*(z)}{\partial z}\exp(-i\,kz + i\,\omega t) \quad \textbf{(4)}$$

where the terms that have second space derivatives of the amplitudes $E_o(z)$ have been ignored.

a) Substitute equations (2) and (3), along with the approximation in equation (4), in equation (1) and derive simple first order differential equations for the amplitude of the E&M wave. Your answer should look like,

$$\frac{\partial E_o(z)}{\partial z} = ??$$

$$\frac{\partial E_o^*(z)}{\partial z} = ??$$

Hint: After making all the substitutions, match the time dependencies given by the exponentials on both sides of equation (1) to project out the desired equations. No need to do any integrations.

b) What does the real part of $\chi(\omega)$ do to the complex E&M wave amplitude? Does it change the value of the propagation vector k ?

c) What does the imaginary part of $\chi(\omega)$ do to the complex E&M wave amplitude?

NOTE: You should have seen that the E&M wave amplitude changes as it propagates. This must mean that photons are either taken away from the wave or added to the wave as it propagates. And this in turn means that the populations in the two-level-system medium must change if the medium emits or absorbs photons. So our assumption that the population remains constant is not entirely correct.

Problem 3.5 (Optical Linear Response: Cavity Fields) OPTIONAL (do it if you have never seen this material before)

You solved for the effect of the medium on the E&M field when the field was a propagating wave. One can do the same for fields confined in cavities, and this you will do here. Suppose the field is confined within a cavity filled with a medium that contains our two-level systems as shown below.

The electric field inside the cavity satisfies,

$$\nabla^2 \vec{E}(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2} = \mu_o \frac{\partial^2 \vec{P}(\vec{r},t)}{\partial t^2} \qquad \textbf{(1)}$$

When the polarization density $\vec{P}(\vec{r},t)=0$, the solution for the field amplitude of the *m*-th cavity mode is,

$$E(\vec{r},t) = E_m \vec{U}_m(\vec{r})\cos(\omega_m t)$$

Only the *m*-th mode is assumed to be close to resonance with the two-level-system medium and, therefore, all other modes can be safely ignored. When $\vec{P}(\vec{r},t) \neq 0$, assume a solution in which the amplitude of the *m*-th mode of the E&M field is slowly changing in time (slowly varying envelope approximation in time),

$$\vec{E}(\vec{r},t) = \frac{E_m(t)}{2}\vec{U}_m(\vec{r})\exp(-i\omega_m t) + \frac{E_m^*(t)}{2}\vec{U}_m(\vec{r})\exp(i\omega_m t)$$

Since the amplitude is assumed to be changing slowly in time, our previous analysis of two level systems excited with time-independent field amplitudes remains valid, and the polarization density can be approximated as,

$$\vec{P}(\vec{r},t) = \varepsilon_o\ \chi(\omega_m)\frac{E_m(t)}{2}\vec{U}_m(\vec{r})\exp(-i\omega_m t) + \varepsilon_o\ \chi^*(\omega_m)\frac{E_m^*(t)}{2}\vec{U}_m(\vec{r})\exp(i\omega_m t)$$

As before, assume that the populations do not change with time. Since the field amplitude is slowly changing in time, make the following approximation,

$$\frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2} \approx -\omega_m{}^2 \vec{E}(\vec{r},t) - i\omega_m \frac{\partial E_m(t)}{\partial t} \vec{U}_m(\vec{r}) \exp(-i\,\omega_m t) + i\omega_m \frac{\partial E_m^*(t)}{\partial t} \vec{U}_m(\vec{r}) \exp(i\,\omega_m t)$$

where the terms that have second order time derivatives of the field amplitudes $E_m(t)$ have been ignored. Also, for the polarization one may assume that,

$$\frac{\partial^2 \vec{P}(\vec{r},t)}{\partial t^2} \approx -\omega_m{}^2 \vec{P}(\vec{r},t)$$

where even terms that have first order time derivatives of the field amplitudes $E_m(t)$ have been ignored. This is justified since keeping these terms would not change the final answer by much within the level of our approximations.

a) With the approximations mentioned above, use equation (1) and obtain simple first order differential equations for the field amplitudes. Your answers should look like,

$$\frac{\partial E_m(t)}{\partial t} = ??$$

$$\frac{\partial E_m^*(t)}{\partial t} = ??$$

Hint: After making all the substitutions, match the time dependencies given by the exponentials on both sides of equation (1) to project out the desired equations. No need to do any integrations.

b) What does the real part of $\chi(\omega_m)$ do to the complex E&M field amplitude as time progresses? Does it effectively modify the mode frequency ω_m? If yes, does it move it closer to the resonance frequency of the two level systems or further away from it?

This effect is called "micro-cavity frequency pulling". The mode frequency of a "loaded" cavity (that is filled with medium that interacts with the mode) is not the same as that of a "bare" cavity. Does the frequency pulling depend on the population difference $\rho_{22} - \rho_{11}$?

c) What does the imaginary part of $\chi(\omega_m)$ do to the E&M field amplitude as time progresses?

NOTE: You should have seen that the E&M wave amplitude changes in time. This must mean that photons are either taken away from the wave or added to the wave as time goes on. And this in turn means that the populations in the two-level-system medium must change if the medium emits or absorbs photons. So our assumption that the population remains constant is not entirely correct.