

ECE 5310: Quantum Optics

Fall 2019

Homework 2

Due on Sep. 23, 2019 (self-grade)

OPTIONAL READING: A good (short but nice) description of density operators can be found in compliment E<sub>III</sub> of chapter 3 of “Quantum Mechanics: Volume I” by Claude Cohen-Tanoudji.

**Problem 2.1: (Fermi’s Golden Rule for Time-Dependent Problems)**

In the class we saw that in systems time dependent behavior can take the form of periodic oscillations or just a one-shot decay (or a one-shot transition) described by Fermi’s Golden Rule. And which of the above happens depends upon whether the final possible state(s) is a single state or a large number of closely spaced states (i.e. a continuum of states). In this problem you will derive the Fermi’s Golden Rule for transitions caused by electromagnetic radiation.

In the last homework set you saw that the time *dependent* Hamiltonian that described the interaction between a two level system and a time dependent electric field is,

$$\hat{H}(t) = \varepsilon_1 |e_1\rangle\langle e_1| + \varepsilon_2 |e_2\rangle\langle e_2| - \frac{\hbar\Omega_R}{2} [\exp(i\omega t) |e_1\rangle\langle e_2| + \exp(-i\omega t) |e_2\rangle\langle e_1|]$$

where the parameter  $\hbar\Omega_R$  is related to the electric field strength.

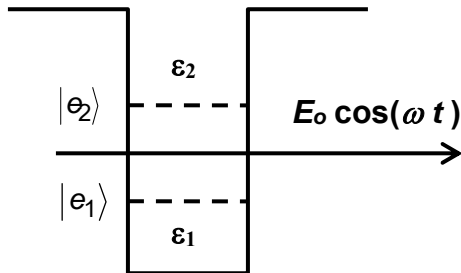


Fig. p2.1a: A driven two level system

Here the problem is changed a little. The upper state  $|e_2\rangle$  is replaced by a large collection of states  $|e_k\rangle$ , with energies  $\varepsilon_k$ , that form a continuum characterized with a density of states  $D(\varepsilon)$  (i.e. the number of states within an energy interval  $d\varepsilon$  is  $D(\varepsilon) d\varepsilon$ ) as shown in the figure below.

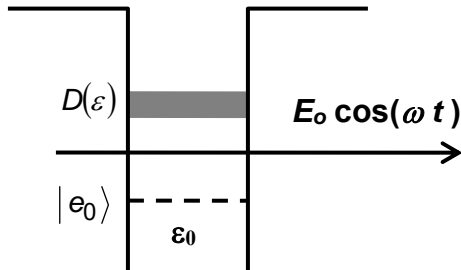


Fig. p2.1b: The upper level is replaced with a continuum of states.

The Hamiltonian of the problem becomes,

$$\hat{H}(t) = \varepsilon_0 |e_0\rangle\langle e_0| + \sum_k \varepsilon_k |e_k\rangle\langle e_k| - \sum_k \frac{\hbar\Omega_{Rk}}{2} [\exp(i\omega t) |e_0\rangle\langle e_k| + \exp(-i\omega t) |e_k\rangle\langle e_0|]$$

Suppose the initial quantum state of the particle at  $t = 0$  is,

$$|\psi(t=0)\rangle = |e_0\rangle$$

The state of the particle at time  $t$  can be written as,

$$|\psi(t)\rangle = c_0(t) |e_0\rangle + \sum_k c_k(t) |e_k\rangle$$

a) Following the method used in the problem of particle decay from a well given in the lecture notes show that the probability  $P(t)$  that the particle is in the initial state decays approximately as,

$$\frac{dP(t)}{dt} \approx -\gamma P(t)$$

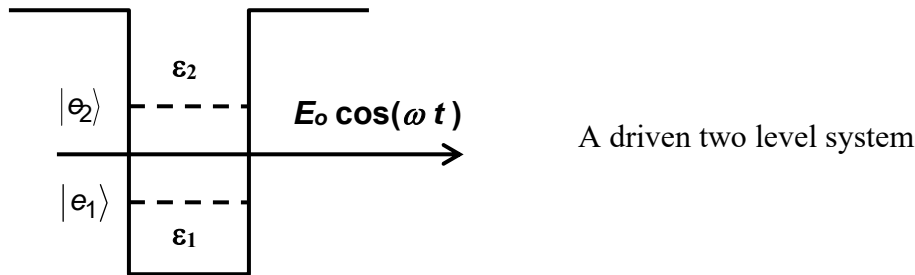
where the “transition rate”  $\gamma$  from the lower state to the upper states is given by the Fermi’s Golden Rule,

$$\begin{aligned} \gamma &= \frac{2\pi}{\hbar} \sum_k \left( \frac{\hbar\Omega_{Rk}}{2} \right)^2 \delta(\varepsilon_k - \varepsilon_0 - \hbar\omega) = \frac{2\pi}{\hbar} \int_{-\infty}^{\infty} d\varepsilon \left( \frac{\hbar\Omega_R(\varepsilon)}{2} \right)^2 D(\varepsilon) \delta(\varepsilon - \varepsilon_0 - \hbar\omega) \\ &= \frac{2\pi}{\hbar} \left[ \frac{\hbar\Omega_R(\varepsilon_0 + \hbar\omega)}{2} \right]^2 D(\varepsilon_0 + \hbar\omega) \end{aligned}$$

## Problem 2.2: (Density operators and optical Bloch equation)

In this problem you will continue to explore the Bloch sphere dynamics with **decoherence** and **relaxation** BOTH included in the model. Consider the time dependent two-level system where a time-dependent electric field interacts with a two level system,

$$\hat{H}(t) = \varepsilon_1 |e_1\rangle\langle e_1| + \varepsilon_2 |e_2\rangle\langle e_2| - \frac{\hbar\Omega_R}{2} [\exp(i\omega t) |e_1\rangle\langle e_2| + \exp(-i\omega t) |e_2\rangle\langle e_1|]$$



The detuning is given by  $\Delta = \varepsilon_2 - (\varepsilon_1 + \hbar\omega)$ . The equations for the components,  $V_x(t)$ ,  $V_y(t)$ , and  $V_z(t)$  are now as follows,

$$\frac{dV_x(t)}{dt} = -\frac{1}{T_2} V_x(t) - \frac{\Delta}{\hbar} V_y(t)$$

$$\frac{dV_y(t)}{dt} = -\frac{1}{T_2} V_y(t) + \frac{\Delta}{\hbar} V_x(t) + \Omega_R V_z(t)$$

$$\frac{dV_z(t)}{dt} = -\frac{(V_z(t)+1)}{T_1} - \Omega_R V_y(t)$$

**Parts (a)-(e):** Suppose  $|\psi(t=0)\rangle = |e_1\rangle$ , and at time  $t=0$  an electromagnetic field with **zero detuning** is turned on.

a) Derive the uncoupled second order differential equations for each of the components  $V_y(t)$  and  $V_z(t)$ . Since the detuning is zero, and the initial quantum state is such that  $V_x(t=0) = 0$ ,  $V_x(t)$  will remain zero for all time  $t \geq 0$ , and all the action is in the y-z plane.

b) Solve the differential equations derived in part (a) with initial conditions corresponding to the initial quantum state  $|\psi(t=0)\rangle = |e_1\rangle$ , and obtain  $V_y(t)$  and  $V_z(t)$  as functions of time. Find the time period of one complete oscillation (i.e. the time it takes the vector  $\vec{V}(t)$  to go once around the sphere and come back to the starting location)?

c) What is  $\vec{V}(t)$  at time  $t = \infty$ ? Sketch the motion of the tip of the vector  $\vec{V}(t)$  in the y-z plane.

d) Can you describe the final state by a state vector  $|\psi(t = \infty)\rangle$ ? If so, what is  $|\psi(t = \infty)\rangle$ ? If not, why not?

e) Can you describe the final state by a 2x2 density matrix  $\hat{\rho}(t = \infty)$ ? If so, what is  $\hat{\rho}(t = \infty)$ ? If not, why not?

### Problem 2.3: (Atomic clocks: everything that you always wanted to know about them but were afraid to ask)

In this problem you will analyze the timing/frequency performance of an atomic clock. Consider a Cesium atomic clock, as discussed in the lecture handouts. Assume no decoherence or relaxation. The Ramsey Fringes are shown below.

In lecture handouts it is mentioned that the occupancy of the upper level at the end of the second  $\pi/2$  pulse is,

$$\rho_{22} = \frac{1}{2} \left[ 1 + \cos\left(\frac{\Delta}{\hbar} T\right) \right]$$

a) Prove the relation above.

b) Suppose the quantum state of an atom at the of the second pulse is written as,

$$|\psi\rangle = c_1|e_1\rangle + c_2|e_2\rangle$$

What are the values of  $|c_1|^2$  and  $|c_2|^2$ ?

c) Suppose at the end of the second pulse a measurement is made on a single atom to determine the occupancy of the upper level. The average obtained as a result of this measurement is obviously  $|c_1|^2$ . What is the uncertainty or variance in the measurement result? Hint: Need to find,

$$\langle (\Delta \hat{N}_2^2) \rangle = \langle \hat{N}_2^2 \rangle - \langle \hat{N}_2 \rangle^2$$

d) In an actual experiment many atoms, say  $N_a$ , are used and not a single atom, and the upper level occupancy for all the atoms is determined jointly. We define an operator for such as a measurement as follows,

$$\hat{N}_2^{all} = \frac{\sum_{k=1}^{N_a} \hat{N}_2^k}{N_a}$$

where the summation is over the number operators for all the  $N_a$  atoms. What is the average (or expectation) value of  $\hat{N}_2^{all}$  at the end of the second pulse?

e) What is the uncertainty or variance in  $\hat{N}_2^{all}$ ? Hint: Need to find,

$$\langle (\Delta \hat{N}_2^{all})^2 \rangle$$

f) Is it beneficial to make a joint measurement on all atoms or a measurement on just a single atom to determine the upper level occupancy at the end of the second pulse?

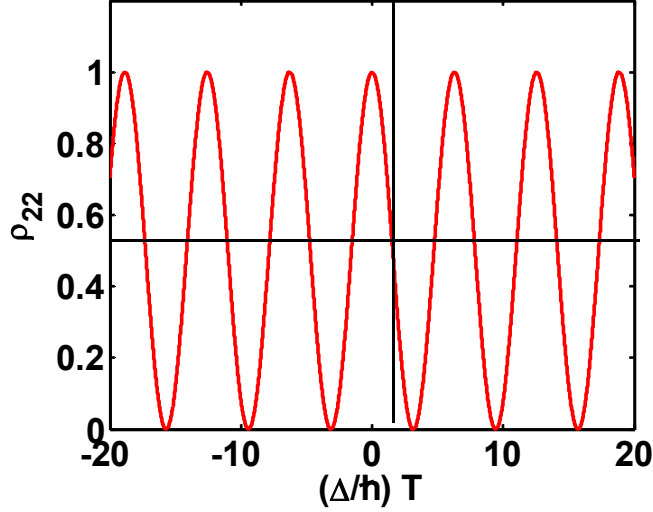
The center frequency of the two pulses is slightly detuned from the atomic resonance such that the average upper level occupancy  $\langle \hat{N}_2^{all} \rangle$  at the end of the second pulse is not 1 but 0.5, as shown by the intersection of the two lines in the Figure. This means that the ideal detuning is such that,

$$\Delta = \Delta\varepsilon - \hbar\omega = \frac{\pi\hbar}{2T}$$

This detuning determines the ideal center frequency  $\omega_0$  of the pulses,

$$\omega_0 = \frac{\Delta\varepsilon}{\hbar} - \frac{\pi}{2T}$$

The idea behind using the Cesium atoms is to ensure that the actual frequency  $\omega$  of the pulses remain locked as closely as possible to the value  $\omega_0$ . The reason for choosing  $\omega_0$  in such a way (as opposed to, say, just  $\Delta\varepsilon/\hbar$ ) is because for this value any slight variation in the actual frequency  $\omega$  from the ideal frequency  $\omega_0$  will generate a large change in the measured value of  $N_2^{all}$  and then this feedback can be used to adjust the frequency of the pulses and move it back to  $\omega_0$ .



g) Show that if a small variation in frequency  $\Delta\omega = \omega - \omega_0$  results in a value for  $N_2^{all}$  different from 0.5 at the end of the second pulse then the frequency error  $\Delta\omega$  can be determined from,

$$\Delta\omega = \omega - \omega_0 = \frac{2}{T} (N_2^{all} - 0.5)$$

h) There is some intrinsic noise associated with determining the value of  $N_2^{all}$  (part (d) of this problem). This will translate into an intrinsic noise in determining  $\Delta\omega$ . To see this clearly, make the observed frequency variation an operator related to the operator  $\hat{N}_2^{all}$  as follows,

$$\Delta\hat{\omega} = \frac{2}{T} (\hat{N}_2^{all} - 0.5) = \frac{2}{T} \Delta\hat{N}_2^{all}$$

Find the uncertainty  $\langle \Delta\hat{\omega}^2 \rangle$  in determining the frequency error? This answer tells you that the frequency  $\omega$ , or equivalently the frequency difference  $\Delta\omega$ , cannot be determined with arbitrary accuracy.

The cycle time  $T_c$  is the time taken to complete one frequency measurement (which involves preparing the Cesium atoms in the right initial state, subjecting them to the two pulses, and then measuring the upper level occupancy). Suppose  $\omega[n]$  is the frequency of the RF pulses during the  $n$ -th cycle. The frequency error  $\omega[n] - \omega_0$  was determined and the frequency was adjusted in the  $(n+1)$ -th cycle according to the rule,

$$\omega[n+1] = \omega[n] - \gamma(\omega[n] - \omega_0)$$

where,  $\gamma$  is a feedback parameter, typically much less than unity. The quantity in the brackets is the error signal. Equivalently we can write,

$$\Delta\omega[n+1] = \Delta\omega[n] - \gamma(\Delta\omega[n])$$

The problem with the above relation is that it does not capture the fact that the error  $\Delta\omega[n] = \omega[n] - \omega_0$  cannot be determined with infinite accuracy in a measurement. To incorporate this feature into the above equation we write,

$$\Delta\omega[n+1] = \Delta\omega[n] - \gamma(\Delta\omega[n] + F[n])$$

where,  $F[n]$  is a classical variable that models the intrinsic noise in determining the frequency error.

From part (g) we can write,

$$\begin{aligned}\langle F[n] \rangle &= 0 \\ \langle F^2[n] \rangle &= \frac{4}{T^2} \langle (\Delta \hat{N}_2^{all})^2 \rangle \\ \langle F[n] F[p] \rangle &= 0 \quad \text{if } n \neq p\end{aligned}$$

The averaging above implies averaging with respect to different cycles. The last equation says that the noise in determining the frequency error is completely uncorrelated from cycle to cycle, as one would expect. So finally we have,

$$\Delta\omega[n+1] = (1-\gamma)\Delta\omega[n] - \gamma F[n]$$

The errors  $\Delta\omega[n] = \omega[n] - \omega_o$  are correlated from cycle to cycle. This happens because  $\gamma$  is small and the feedback mechanism does not completely correct for the error in just one cycle (otherwise the feedback can cause instability).

i) Show that,

$$\sum_{p=-\infty}^{\infty} \langle \Delta\omega[n] \Delta\omega[n+p] \rangle = \frac{4}{T^2} \langle (\Delta \hat{N}_2^{all})^2 \rangle$$

In practice, the frequency  $\omega[n]$  is averaged over many cycles and the output RF oscillator of the atomic clock is tuned to this average value. So we define an average frequency (averaged over  $N$  cycles) as,

$$\Omega = \frac{\sum_{n=-N/2}^{N/2} \omega[n]}{N} \quad \Delta\Omega = \frac{\sum_{n=-N/2}^{N/2} \omega[n]}{N} - \omega_o = \frac{\sum_{n=-N/2}^{N/2} \Delta\omega[n]}{N}$$

j) Argue from results obtained in previous parts, that if  $N$  is large enough then,

$$\langle \Delta\Omega^2 \rangle \approx \frac{4}{T^2 N} \langle (\Delta \hat{N}_2^{all})^2 \rangle$$

The relative frequency stability is defined as,

$$\sigma = \sqrt{\frac{\langle \Delta\Omega^2 \rangle}{\omega_o^2}}$$

k) The integration time  $\tau$  is defined in terms of the cycle time and the number  $N$  of cycles used in averaging as,  $\tau = NT_c$ . Show that,

$$\sigma \approx \frac{1}{\omega_o T} \sqrt{\frac{T_c}{\tau}} \sqrt{\frac{1}{N_a}} = \frac{1}{\pi Q} \sqrt{\frac{T_c}{\tau}} \sqrt{\frac{1}{N_a}}$$

where the quality factor  $Q$  of the fringes is defined as,

$$Q = \omega_o \frac{T}{\pi}$$

Compare your results with those in: **Physical Review Letters, 82, 4619 (1999)**. What did we miss?

l) Consider a Cesium atomic clock with the following parameters:

$$T = 1 \text{ s}$$

$$T_c = 1 \text{ s}$$

$$N_a = 10^6 \text{ atoms/cycle}$$

$$\tau = 10^4 \text{ s}$$

$$\omega_0/2\pi = 9.192 \text{ GHz}$$

Find the relative frequency stability?