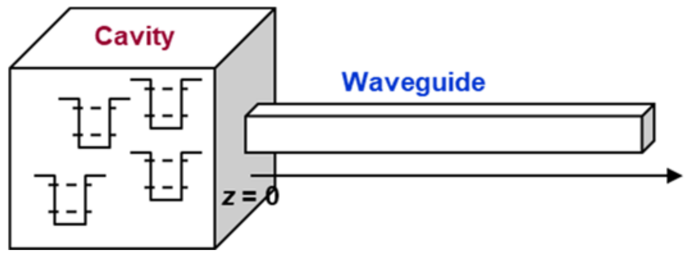
ECE 5310: Quantum Optics

Fall 2019

Homework 11 **Due on Nov.25 at 5:00 PM (self-grade)**

Problem 11.1 (A Laser below Threshold)

In this problem, you will look at the laser photon number and photon flux noise below threshold.



Consider a laser structure discussed in the lecture handouts. The equations for the field operator and the photon number are:

$$\frac{d\hat{a}(t)}{dt} = \left(-i\omega_o - \frac{1}{2\tau_p}\right)\hat{a}(t) + g_d\,\hat{N}_d(t)\hat{a}(t) + e^{-i\omega_o t}\,\hat{F}_{sp}(t) + \sqrt{\frac{1}{\tau_p}}\,\hat{S}_{in}(t)e^{-i\omega_o t}$$

And

$$\frac{d\hat{n}(t)}{dt} = -\frac{\hat{n}(t)}{\tau_p} + 2g_d\hat{N}_d(t)\hat{n}(t) + \left\{e^{-i\omega_o t}\,\hat{a}^+(t)\hat{F}_{sp}(t) + e^{i\omega_o t}\,\hat{F}_{sp}^+(t)\hat{a}(t)\right\}$$
$$+\sqrt{\frac{1}{\tau_p}}\left\{\hat{a}^+(t)\hat{S}_{in}(t)e^{-i\omega_o t} + e^{i\omega_o t}\hat{S}_{in}^+(t)\hat{a}(t)\right\}$$

Below threshold, a decent (not accurate, but decent) approximation when studying fluctuations is to assume that the operator $\hat{N}_d(t)$ is replaced by its average value N_d and we ignore any fluctuations in the difference population. The reason is that below threshold the steady state photon number is very small and if expand the gain term in the photon number equation as follows,

$$2g_d\hat{N}_d(t)\hat{n}(t) = 2g_d\left[N_d + \Delta\hat{N}_d(t)\right]\left[n + \Delta\hat{n}(t)\right] \approx 2g_d N_d n + 2g_d N_d \Delta\hat{n}(t) + 2g_d n\Delta\hat{N}_d(t)$$

Then the last term on RHS above can be ignored. In this problem you will need to

a) Find the spectral density of the photon number noise $S_{\Delta n,\Delta n}(\omega)$ in the laser below threshold.

b) Find the (normalized) variance of the photon number noise $\left\langle \Delta\hat{n}^2(t)\right\rangle / n$ in the laser below threshold.

c) Find the spectral density of the photon flux noise $S_{\Delta F,\Delta F}(\omega)$ in the laser output below threshold and normalize it to the average flux n/τ_p.

d) As the laser approaches threshold, and $2g_d N_d \to 1/\tau_p$, what happens to the laser photon number noise and the laser output flux noise?

e) Find the field correlation function (assuming $t \to \infty$),

$$\left\langle \hat{a}^+(t)a(t+\tau)\right\rangle$$

(that gives the laser linewidth (below threshold)) by direct integration of,

$$\frac{d\hat{a}(t)}{dt} = \left(-i\omega_o - \frac{1}{2\tau_p}\right)\hat{a}(t) + g_d N_d \hat{a}(t) + e^{-i\omega_o t}\hat{F}_{sp}(t) + \sqrt{\frac{1}{\tau_p}}\hat{S}_{in}(t)e^{-i\omega_o t}$$

and explicitly show the contributions from the two noise sources $\hat{F}_{sp}(t)$ and $\hat{S}_{in}(t)$. Do both, or just one, contribute to the correlation function and to the laser linewidth?

f) Find the field correlation function (assuming $t \to \infty$),

$$\left\langle a(t+\tau)\hat{a}^+(t)\right\rangle$$

(that can also give the laser linewidth (below threshold)) by direct integration of,

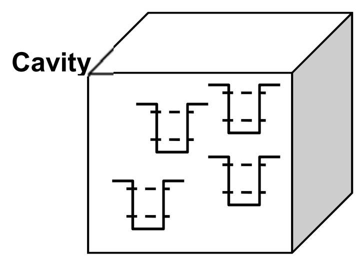
$$\frac{d\hat{a}(t)}{dt} = \left(-i\omega_o - \frac{1}{2\tau_p}\right)\hat{a}(t) + g_d N_d \hat{a}(t) + e^{-i\omega_o t}\hat{F}_{sp}(t) + \sqrt{\frac{1}{\tau_p}}\hat{S}_{in}(t)e^{-i\omega_o t}$$

and explicitly show the contributions from the two noise sources $\hat{F}_{sp}(t)$ and $\hat{S}_{in}(t)$. Do both, or just one, contribute to the correlation function?

g) Based on your answers to (e) and (f) explain if vacuum noise contributes to the field correlation function and to the laser linewidth or not.

Problem 11.2 (A Generic Amplifier)

Suppose you would like to amplify a photon number state $|m\rangle$. You do this in a cavity amplifier, as shown in the figure. Assume that the population inversion is perfect and that $n_{sp} = 1$. Assume,



$$|\psi(t=0)\rangle = |m\rangle$$

The amplification process proceeds for time "t" and the field operator satisfies the equations,

$$\hat{a}(t) = \hat{a}\sqrt{G}e^{-i\omega_o t} + \hat{F}e^{-i\omega_o t}$$

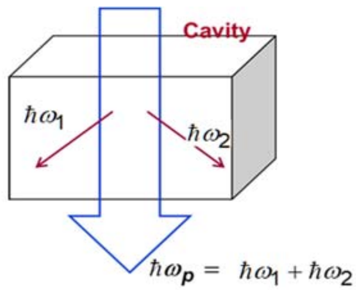
$$\hat{a}^+(t) = \hat{a}^+\sqrt{G}e^{i\omega_o t} + \hat{F}^+e^{-i\omega_o t}$$

where $G = e^{2gt}$.

a) Find the probability that the cavity has "p" photons $(p \geq m)$ at time "t".

Problem 11.3 (A Noise Free Amplifier (or Not?))

A friend tells you that using non-degenerate parametric down conversion, one can build a noise-free phase INSENSITIVE amplifier.



He starts from the standard Hamiltonian,

$$\hat{H}(t) = \hbar\omega_1 \hat{a}_1^+ \hat{a}_1 + \hbar\omega_2 \hat{a}_2^+ \hat{a}_2 + i\hbar\frac{|\kappa|}{2}\left(e^{i\omega_p t - i2\theta}\hat{a}_1\hat{a}_2 - e^{-i\omega_p t + i2\theta}\hat{a}_2^+\hat{a}_1^+\right)$$

and then configures the down-conversion scheme to realize a phase insensitive amplifier. He then tells you that average photon number can be amplified, as well as the average value of ANY field quadrature can amplified, without adding any spontaneously generated photons, as follows:

$$\langle \hat{n}(t) \rangle = \langle \hat{n}(t=0) \rangle G$$

$$\langle \hat{x}_\phi(t) \rangle = \langle \hat{x}_\phi(t=0) \rangle \sqrt{G}$$

You don't believe him. For a phase INSENSITIVE amplifier based on population inversion the relation obviously is quite different,

$$\langle \hat{n}(t) \rangle = \langle \hat{n}(t=0) \rangle G + n_{sp}(G-1)$$

a) See if the non-degenerate parametric down conversion scheme be used to get the following results,

$$\langle \hat{n}(t) \rangle = \langle \hat{n}(t=0) \rangle G$$

$$\langle \hat{x}_\phi(t) \rangle = \langle \hat{x}_\phi(t=0) \rangle \sqrt{G}$$

and find the value of G.

b) You, of course, still don't believe your friend and proceed to find the variance in the photon number. A truly noise-free photon number amplifier would give,

$$\langle \Delta\hat{n}^2(t) \rangle = \langle \Delta\hat{n}^2(t=0) \rangle G^2$$

$$\langle \Delta\hat{x}_\phi^2(t) \rangle = \langle \Delta\hat{x}_\phi^2(t=0) \rangle G$$

What do you get?