Problem 4.1 (Energy bands for a hexagonal lattice in 2D)

a) The K-point \((0, \frac{4\pi}{3a})\) is strongly coupled to two other equivalent K-points in the FBZ, as shown in the picture above (all three K-points have been labeled with subscripts 1, 2, and 3 for convenience). The equivalent three K-points are related to each other by a reciprocal lattice vector, and the free-electron states with these wavevectors have the same energy. The wavevectors of the other two K-points (2 and 3) are \((\frac{2\pi}{\sqrt{3}a}, -\frac{2\pi}{3a})\) and \((-\frac{2\pi}{\sqrt{3}a}, -\frac{2\pi}{3a})\).

b) \(\psi_{K_1} = c_1 \phi_{K_1} + c_2 \phi_{K_2} + c_3 \phi_{K_3}\)

After plugging into the solution into the Schrodinger equation and multiplying by the bras corresponding to the three plane wave states one by one we obtain three equations which in the matrix form are:

\[
\begin{bmatrix}
    e(K) & V_1/2 & V_1/2 \\
    V_1/2 & e(K) & V_1/2 \\
    V_1/2 & V_1/2 & e(K)
\end{bmatrix}
\begin{bmatrix}
    c_1 \\
    c_2 \\
    c_3
\end{bmatrix}
= E_{K_1}
\begin{bmatrix}
    c_1 \\
    c_2 \\
    c_3
\end{bmatrix}
\]

Where \(e(K)\) equals the free electron energy at (all the equivalent) K-points. Also note that the vector \((\vec{b}_1 + \vec{b}_2)\) points from \(K_3\) to \(K_2\) and so in the expression for the potential the term with the wavevector equal to this vector couples points \(K_3\) and \(K_2\).

The energies of the 3 bands at the \(K_1\)-point are found to be:
\[
E_1(K_1) = E_2(K_1) = e(K) - V_1/2
\]
\[
E_3(K_1) = e(K) + V_1
\]
Note that two bands are still degenerate and one has a higher energy.

The corresponding wavefunctions are:

\[ E_1(K_1) \Rightarrow |\psi_{K_1}\rangle = \frac{|\phi_{K_1}\rangle - |\phi_{K_2}\rangle}{\sqrt{2}} \]
\[ E_2(K_1) \Rightarrow |\psi_{K_1}\rangle = \frac{|\phi_{K_1}\rangle - |\phi_{K_3}\rangle}{\sqrt{2}} \]
\[ E_3(K_1) \Rightarrow |\psi_{K_1}\rangle = \frac{|\phi_{K_1}\rangle + |\phi_{K_2}\rangle + |\phi_{K_3}\rangle}{\sqrt{3}} \]

c) The M₁-point \(2\pi/\sqrt{3} \ a,0\) is coupled to the equivalent M₂-point \((-2\pi/\sqrt{3} \ a,0)\). The subscripts 1 and 2 have been added for clarity.

d) \(|\psi_{M_1}\rangle = c_1 \ |\phi_{M_1}\rangle + c_2 \ |\phi_{M_2}\rangle\)

After plugging into the solution into the Schrodinger equation and multiplying by the bras corresponding to the two plane wave states one by one we obtain two equations which in the matrix form are:

\[
\begin{bmatrix}
e(M) & V_1/2 \\
V_1/2 & e(M)
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
= E(M_1)
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
\]

The energies of the 2 bands at the M₁-point are found to be:

\[ E_1(M_1) = e(M) - V_1/2 \]
\[ E_2(M_1) = e(M) + V_1/2 \]

The corresponding wavefunctions are:

\[ E_1(M_1) \Rightarrow |\psi_{M_1}\rangle = \frac{|\phi_{M_1}\rangle - |\phi_{M_2}\rangle}{\sqrt{2}} \]
\[ E_2(M_1) \Rightarrow |\psi_{M_1}\rangle = \frac{|\phi_{M_1}\rangle + |\phi_{M_2}\rangle}{\sqrt{2}} \]

e) Notice the points where the bandgaps open up (M-point) and where they don’t (K-point) between the first and the second energy bands. Note also that the exact shape of the bands comes from my imagination – so don’t pay too much to it except to note where the bandgaps open up and where they don’t.
f) and g) and h) Solution is obvious if you take the expression for the potential and write the vectors $\mathbf{b}_1, \mathbf{b}_2$ in terms of their x- and y-components.

i) $|\psi_{K_1}\rangle = c_1 |\phi_{K_1}\rangle + c_2 |\phi_{K_2}\rangle + c_3 |\phi_{K_3}\rangle$

After plugging into the solution into the Schrodinger equation and multiplying by the bras corresponding to the three plane wave states one by one we obtain three equations which in the matrix form are:

$$
\begin{bmatrix}
\bar{e}(K) & V_1/2 & V_1/2 \\
V_1/2 & \bar{e}(K) & i V_1/2 \\
V_1/2 & -i V_1/2 & \bar{e}(K)
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix}
= E(\bar{K}_1)
\begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix}
$$

Notice how the entries (2,3) and (3,2) of the matrix have changed. The energies of the 3 bands at the K-point are found to be:

$E_1(K_1) = \bar{e}(K) - \sqrt{3} V_1/2$

$E_2(K_1) = \bar{e}(K)$

$E_3(K_1) = \bar{e}(K) + \sqrt{3} V_1/2$

Note that all bands are now non-degenerate at the K-point.

The corresponding wavefunctions are:
\[ E_1(K_1) \Rightarrow |\psi_{K_1}\rangle = \frac{-e^{-i \pi/6} |\phi_{K_1}\rangle + e^{-i \pi/3} |\phi_{K_2}\rangle + |\phi_{K_3}\rangle}{\sqrt{3}} \]

\[ E_2(K_1) \Rightarrow |\psi_{K_1}\rangle = \frac{|\phi_{K_1}\rangle + |\phi_{K_2}\rangle - |\phi_{K_3}\rangle}{\sqrt{3}} \]

\[ E_3(K_1) \Rightarrow |\psi_{K_1}\rangle = \frac{e^{i \pi/6} |\phi_{K_1}\rangle + e^{i \pi/3} |\phi_{K_2}\rangle + |\phi_{K_3}\rangle}{\sqrt{3}} \]

Problem 4.2 (Free electron energy bands for a FCC lattice in 3D)

![Diagram of a FCC lattice](image)

a) and b)
Problem 4.3 (Ethene (or Ethylene) molecule: LCAO)

a) As discussed in the lectures, the pz-orbitals (the ones that stick out of the plane of the molecule) are anti-symmetric with respect to reflections about the x-y plane. On the other hand, px-orbitals, py-orbitals, and s-orbitals are all symmetric with respect to reflections about the x-y plane. Consequently, the energy matrix elements of pz-orbitals with all other orbitals will be zero.

b) Each carbon atoms is sp2 hybridized.

c) The trial solution could be:

\[
|\psi(\vec{r})| = \frac{4}{\sum c_j} \phi_{1s}^{H}(\vec{r} - \vec{r}_j) + \frac{6}{c_m} \phi_{2s}^{C}(\vec{r} - \vec{r}_m) + \frac{c_{m+2}}{2} \phi_{2px}^{C}(\vec{r} - \vec{r}_m) + \frac{c_{m+4}}{2} \phi_{2py}^{C}(\vec{r} - \vec{r}_m)
\]

or in terms of the carbon sp2 orbitals:

\[
|\psi(\vec{r})| = \frac{4}{\sum c_j} \phi_{1s}^{H}(\vec{r} - \vec{r}_j) + \frac{6}{c_m} \phi_{2s}^{C}(\vec{r} - \vec{r}_m) + \frac{c_{m+2}}{2} \phi_{2px}^{C}(\vec{r} - \vec{r}_m) + \frac{c_{m+4}}{2} \phi_{2py}^{C}(\vec{r} - \vec{r}_m)
\]

d) Let the energy of the s-orbital of the hydrogen atom be \(E_{1s}\) and let the energies of the 2s and 2p orbitals of the carbon atom be \(E_{2s}\) and \(E_{2p}\), respectively. The average energy of the carbon sp2 orbital is given by the expression:

\[
E_{sp2} = \left( \frac{E_{2s} + 2E_{2p}}{3} \right)
\]

Each hydrogen atoms will make a sigma bond with the sp2-orbital of the carbon atom pointing towards it. This will result in a sigma bonding orbital and a sigma anti-bonding orbital with energies equal to:

\[
E_{CH}(A) = \left( E_{1s} + E_{sp2} \right) + \sqrt{\left( \frac{E_{1s} - E_{sp2}}{2} \right)^2 + \eta^2}
\]

\[
E_{CH}(B) = \left( E_{1s} + E_{sp2} \right) - \sqrt{\left( \frac{E_{1s} - E_{sp2}}{2} \right)^2 + \eta^2}
\]

where:

\[
\eta = \frac{V_{sp\sigma} + \sqrt{2}V_{pp\sigma}}{\sqrt{3}}
\]

The sp2-orbitals of the two carbon atoms will make a sigma bond. This will result in a sigma bonding orbital and a sigma anti-bonding orbital with energies equal to:

\[
E_{CC}(A) = E_{sp2} + |\eta|
\]

degeneracy = 1
\[ E_{CC}(B) = E_{sp2} - |\lambda| \]  
where:  
\[ \lambda = \left( \frac{V_{ss\sigma} + 2V_{pp\sigma} + 2\sqrt{2} V_{sp\sigma}}{3} \right) \]

e) The solution is:
\[ |\psi(\vec{r})| = b_1 |\phi_{2pz}^C (\vec{r} - \vec{r}_5)| + b_2 |\phi_{2pz}^C (\vec{r} - \vec{r}_6)| \]

f) The pz-orbitals on carbon atoms make a \( \pi \)-bond with each other. The corresponding energies of the bonding and anti-bonding orbitals are:
\[ E_{CC}(A) = E_{2p} + |V_{pp\pi}| \quad \text{degeneracy} = 1 \]
\[ E_{CC}(B) = E_{2p} - |V_{pp\pi}| \quad \text{degeneracy} = 1 \]

g) The energy level diagram is shown below. The specific alignments are rough – my best guesses.