ECE Homework #1 Solutions. (Farhan Rana)

1.1.

a) $\vec{E}=0 \Rightarrow m\frac{dV_x}{dt} = -eB_0V_y$ & $m\frac{dV_y}{dt} = eB_0V_x$

$\Rightarrow \frac{d^2V_x}{dt^2} = -\omega_c^2 V_x$ & $\frac{d^2V_y}{dt^2} = -\omega_c^2 V_y$ $\quad \omega_c = \frac{eB_0}{m}$

$\Rightarrow V_x = A\cos\omega_c t$ & $V_y = A\sin\omega_c t$

b) $\frac{dU_x}{dt} = V_x = A\cos\omega_c t \Rightarrow U_x = \frac{A}{\omega_c}\sin\omega_c t$

$\frac{dU_y}{dt} = V_y = A\sin\omega_c t \Rightarrow U_y = -\frac{A}{\omega_c}\cos\omega_c t + \frac{A}{\omega_c}$

Position vector of the electron is $= \vec{r}(t) = \frac{A}{\omega_c}\hat{y} + \frac{A}{\omega_c}\left[\sin\omega_c t\,\hat{x} - \cos\omega_c t\,\hat{y}\right]$

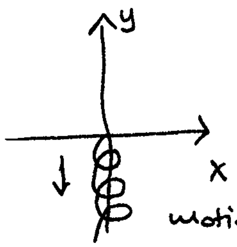
$\Rightarrow$ circular motion is centered at $\frac{A}{\omega_c}\hat{y}$ and has radius $\frac{A}{\omega_c}$.

c) $m\frac{dV_x}{dt} = -eE_x - eB_0V_y$ $\quad m\frac{dV_y}{dt} = eB_0V_x$

$\frac{d^2V_y}{dt^2} = -\omega_c^2 V_y - \frac{\omega_c e}{m}E_x$ & $\frac{d^2V_x}{dt^2} = -\omega_c^2 V_x$

$\Rightarrow V_x = A\cos\omega_c t$ & $V_y = A\sin\omega_c t - \frac{E_x}{B_0}$

$\Rightarrow$ There is a constant velocity component in the $-\hat{y}$ direction!



motion is helical!

d) $m\frac{dV_x}{dt} = -eE_x - eB_0V_y - \frac{mV_x}{\tau} = 0$

$m\frac{dV_y}{dt} = -eE_y + eB_0V_x - \frac{mV_y}{\tau} = 0$

$$V_x = \frac{\omega_c^2\frac{E_y}{B_0} - \frac{\mu}{\tau^2}E_x}{\omega_c^2 + 1/\tau^2} \qquad V_y = \frac{-\omega_c^2\frac{E_x}{B_0} - \frac{\mu}{\tau^2}E_y}{\omega_c^2 + 1/\tau^2} \qquad \left\{ \mu = \frac{e\tau}{m} \right.$$

e) $J_x = n(-e)V_x$ $\qquad$ $J_y = n(-e)V_y$

$$J_x = n\left\{ \frac{-e\omega_c^2 \frac{E_y}{B_0} + \frac{e\mu}{\tau^2} E_x}{\omega_c^2 + 1/\tau^2} \right\} \qquad J_y = n\left\{ \frac{e\omega_c^2 \frac{E_x}{B_0} + \frac{e\mu}{\tau^2} E_y}{\omega_c^2 + 1/\tau^2} \right\}.$$

f) $J_y = 0 \Rightarrow E_y = -\frac{(\omega_c\tau)^2}{\mu}\frac{E_x}{B_0} = B_0 V_x.$

g) $\rho_H = \frac{E_y}{J_x} = \frac{B_0 V_x}{n(-e)V_x} = -\frac{B_0}{ne} \Rightarrow$ ρ_H is -ve for electrons.

1.2

a) $\sigma = \frac{1}{2.2\times10^{-8}}$ S/m $\qquad \sigma = \frac{ne^2\tau}{m} \Rightarrow n = 5.39\times10^{28}\ /m^3$

$\Rightarrow \omega_p = \sqrt{\frac{ne^2}{\varepsilon_0 m}} = 1.3\times10^{16}$ rad/s $= 2\times10^{15}$ Hz $\Rightarrow$ UV range

b) $\omega_p = \sqrt{\frac{ne^2}{\varepsilon_0 m}} = 1.78\times10^{13}$ rad/s $= 2.83\times10^{12}$ Hz $\Rightarrow$ IR range

1.3 $\quad |\psi\rangle = C_1|\phi_1\rangle + C_2|\phi_2\rangle \Rightarrow$ Since $H|\psi\rangle = E|\psi\rangle$

(1) $\langle\phi_1|\hat{H}|\psi\rangle = E\langle\phi_1|\psi\rangle \Rightarrow C_1 E_0 + tC_2 = EC_1$

(2) $\langle\phi_2|\hat{H}|\psi\rangle = E\langle\phi_2|\psi\rangle \Rightarrow C_1 t + E_0 C_2 = EC_2.$

a) Writing in matrix form: $\begin{bmatrix} E_0 & t \\ t & E_0 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = E\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$ eigenvalue equation!

To find the eigenvalues set $\det\begin{bmatrix} E_0 - E & t \\ t & E_0 - E \end{bmatrix} = 0$

$\Rightarrow (E_0 - E)^2 - t^2 = 0 \Rightarrow E = E_0 \pm t \Rightarrow E_1 = E_0 - t,\ E_2 = E_0 + t$

b) Need to find the eigenstates corresponding to the eigenvalues E_1 and E_2. These are $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\Rightarrow |\psi_1\rangle = \frac{1}{\sqrt{2}}\left[|\phi_1\rangle - |\phi_2\rangle\right]$ and $|\psi_2\rangle = \frac{1}{\sqrt{2}}\left[|\phi_1\rangle + |\phi_2\rangle\right]$

1.4

a) $f(t) = \sum_n \delta(t - nT)$

$$f(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t} \sum_n \delta(t - nT) = \sum_n e^{i\omega n T} = \frac{2\pi}{T}\sum_n \delta\left(\omega - n\frac{2\pi}{T}\right)$$

b) $g(\omega) = \int_{-W/2}^{W/2} dt\ e^{i\omega t} = W \dfrac{\sin\left(\omega \frac{W}{2}\right)}{\left(\frac{\omega W}{2}\right)}.$

c) $h(t) = f(t) \otimes g(t) \Rightarrow h(\omega) = f(\omega)\, g(\omega)$

$$\Rightarrow h(\omega) = W \frac{\sin\left(\frac{\omega W}{2}\right)}{\left(\frac{\omega W}{2}\right)} \sum_n \frac{2\pi}{T} \delta\left(\omega - n\frac{2\pi}{T}\right)$$

$$= \frac{2\pi W}{T} \sum_n \delta\left(\omega - n\frac{2\pi}{T}\right) \frac{\sin\left(n\frac{\pi W}{T}\right)}{\left(\frac{n\pi W}{T}\right)}.$$

d) $h(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} h(\omega) = \sum_n \frac{W}{T} \dfrac{\sin\left(\frac{n\pi W}{T}\right)}{\left(\frac{n\pi W}{T}\right)} e^{-in\frac{2\pi}{T}t}$

e) $h(\vec{k}) = \int d^3\vec{r}\ e^{-i\vec{k}\cdot\vec{r}}\, h(\vec{r})$

$$= L_x \frac{\sin\left(\frac{k_x L_x}{2}\right)}{\left(\frac{k_x L_x}{2}\right)} L_y \frac{\sin\left(\frac{k_y L_y}{2}\right)}{\left(\frac{k_y L_y}{2}\right)} \cdot L_z \frac{\sin\left(\frac{k_z L_z}{2}\right)}{\left(\frac{k_z L_z}{2}\right)}.$$