*Department of Electrical and Computer Engineering, Cornell University*

**ECE 4070: Physics of Semiconductor and Nanostructures**

**Spring 2014**

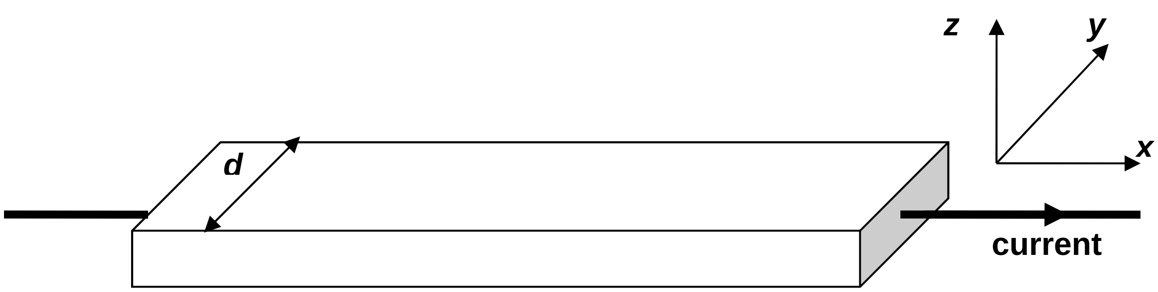
**Homework 1** **Due on Feb. 04, 2014 at 5:00 PM**

## Suggested Readings:

a) Revise Fourier transforms from your favorite book(s).
b) Lecture notes

## Problem 1.1: (Drude Model: Motion in Magnetic Fields and the Hall Effect)

In the lecture notes, we considered electron motion in electric fields. In this problem we will include the magnetic field as well. Consider the metallic sample shown below.




The metal has an electron density equal to $n$, and an electron scattering time $\tau$. A uniform magnetic field in the z-direction, given by $\vec{B} = B_o \hat{z}$, is applied to the sample. In addition a uniform electric field in the x-direction, given by $E_x$, is also applied by connecting the sample to an external voltage/current source via leads, as shown. In the presence of the fields, the total force on the electrons is given by the Lorentz expression:

$$\vec{F} = -e\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

And the electron average velocity satisfies the equation:

$$m\frac{d\vec{v}}{dt} = -e\left(\vec{E} + \vec{v} \times \vec{B}\right) - m\frac{\vec{v}}{\tau}$$

a) Suppose the scattering rate is zero (i.e. ignore scattering). And also assume that the electric field is zero. Now solve the equation given above and find $v_x(t)$ and $v_y(t)$, the components of the electron average velocity, assuming the initial conditions that $v_x(t=0) = A$ and $v_y(t=0) = 0$. **Hint:** It would be easiest to break the vector equation into its $x$ and $y$ components. And then you will get two coupled linear differential equations for the two components of the electron average velocity.

b) The components of the electron average displacement, $u_x(t)$ and $u_y(t)$ are related to the average velocities by the relations:

$$\frac{du_x(t)}{dt} = v_x(t) \qquad \frac{du_y(t)}{dt} = v_y(t)$$

Once you have obtained $v_x(t)$ and $v_y(t)$, integrate them once more, assuming the initial conditions that $u_x(t=0) = u_y(t=0) = 0$, to get the components of the electron average displacement. You will see that the electron motion is oscillatory with an angular frequency given by $\omega_c = eB_o/m$. This frequency $\omega_c$ is called the electron cyclotron frequency. You will also find that the electron moves in a circular path. What is the radius of the electron orbit? Looking down on the sample from the top, is the electron motion clockwise or counter clockwise?

c) Now assume that $E_x$ is not zero. Repeat parts (a) and (b). **Be very careful**. The presence of $E_x$ (a driving term) will require that you add a particular solution to the homogenous solution of the coupled differential equations found in parts (a) and (b). Once you have found a complete solution, plug it back into the coupled differential equations to make sure your solution satisfies the differential equations. You will find that the electron motion is helical. In addition to the circular motion, the electron moves with a uniform average velocity in a certain direction. What is this direction? **Hint:** Your intuition might deceive you here.

d) Now assume that both $E_x$ and $E_y$ are not zero. Although there is no E-field applied in the y-direction, you will see that a field in the y-direction must exist in the sample. Also assume that the scattering term is present. This is the most general and realistic situation. We will not attempt a full blown time-dependent solution, but try to find only the steady state solution. A well defined time-independent steady state solution will exist because of the damping introduced by the presence of the scattering term. In steady state, the left hand side of the equation,

$$m\frac{d\vec{v}}{dt} = -e\left(\vec{E} + \vec{v}\times\vec{B}\right) - m\frac{\vec{v}}{\tau}$$

will be zero. Find the time-independent components, $v_x$ and $v_y$, of the electron average velocity in steady state and relate them to the components of the electric and magnetic fields.

e) Find the components, $J_x$ and $J_y$, of the electron current density in the $x$ and $y$ directions in steady state. You will find a non-zero value for $J_y$.

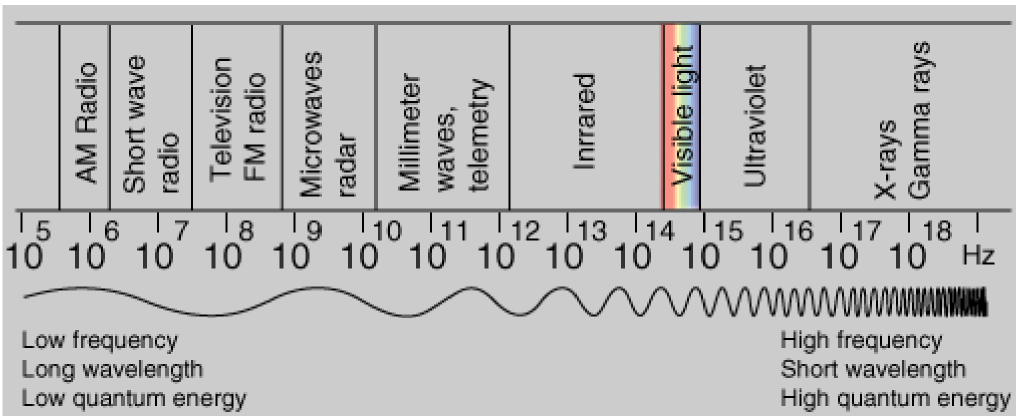
f) Since the sample is finite in the y-direction, and is not connected by any leads on its sides, there cannot be steady state current flowing in the y-direction inside the sample. The only way to have $J_y = 0$ is to have a non-zero field $E_y$ in the y-direction. From the condition $J_y = 0$, find the field $E_y$ (magnitude and sign).

g) A non-zero field in the y-direction can be measured experimentally by putting voltage probes on the two sides of the sample that are spaced apart by $d$ in the figure. The ratio $E_y/J_x$ is called the Hall resistivity $\rho_H$. Find $\rho_H$ (magnitude and sign).

**Experimental Importance:** The Hall resistivity is the easiest and the best way to measure the carrier density $n$ as well as the sign of the charge of the carriers. If the charge carriers are electrons (-vely charged) the Hall resistivity will come out negative (as you must have found out in part (g) above). If the charge carriers are holes (+vely charged) then the Hall resistivity will come out positive (to see this, let $e$ go to $-e$ in your expression for the Hall resistivity in part (g)).

## Problem 1.2: (Plasma frequency)

a) In gold, the electron scattering time $\tau$ is around 30 fs (or $30 \times 10^{-15}$ sec). You can find the DC conductivity (or the DC resistivity, which is the inverse of conductivity) of gold from the website: http://www.webelements.com/. Find the carrier density $n$ of electrons in gold. Find the plasma frequency $\omega_p$ of gold, convert it into Hertz, and then using the figure below find in which part of the electromagnetic radiation spectrum (i.e. am radio, infrared, etc) does it fall. **Be careful that you don't mix up the units.**




b) In a doped semiconductor, the electron density is $10^{17}\ 1/\mathrm{cm}^3$. Assume the dielectric constant of the semiconductor to be $\varepsilon_o$. Find the plasma frequency $\omega_p$ of this semiconductor, convert it into Hertz, and then using the figure below find in which part of the electromagnetic radiation spectrum (i.e. am radio, infrared, etc) does it fall. **Be careful that you don't mix up the units.**

## Problem 1.3: (Basic Quantum Mechanics Review: Finite Basis Expansions)

Consider a quantum system with the Hamiltonian operator $\hat{H}_o$. The Hamiltonian has two orthogonal eigenstates $|\phi_1\rangle$ and $|\phi_2\rangle$ with the same eigenenergy $E_o$, so that: $\hat{H}_o|\phi_1\rangle = E_o|\phi_1\rangle$ and $\hat{H}_o|\phi_2\rangle = E_o|\phi_2\rangle$ and $\langle\phi_n|\phi_m\rangle = \delta_{nm}$. The eigenstates are degenerate (i.e. they have the same energy).

Suppose that a perturbing term $\hat{H}'$ is added to the Hamiltonian so that the new Hamiltonian $\hat{H}$ is:

$$\hat{H} = \hat{H}_o + \hat{H}'$$

The matrix elements of the perturbing term $\hat{H}'$ are as follows:

$$\langle\phi_2|\hat{H}'|\phi_1\rangle = \langle\phi_1|\hat{H}'|\phi_2\rangle = t$$

Seek a trial solution for the eigenstate of the full new Hamiltonian $\hat{H}$ such that the trial solution is a superposition of the eigenstates of $\hat{H}_o$:

$$|\psi\rangle = c_1\,|\phi_1\rangle + c_2\,|\phi_2\rangle$$

a) Using the trial solution given above, find the eigenenergies, $E_1$ and $E_2$, of the new Hamiltonian $\hat{H}$.

b) Find the corresponding eigenstates, $|\psi_1\rangle$ and $|\psi_2\rangle$, of the new Hamiltonian $\hat{H}$, in terms of the states $|\phi_1\rangle$ and $|\phi_2\rangle$.

Hint: We seek a solution such that:

$$\hat{H}|\psi\rangle = E|\psi\rangle$$

Plug in the trial solution in the above equation, and then multiply the resulting equation from the left hand side first by the bra $\langle\phi_1|$ and then by the bra $\langle\phi_2|$. You will get a set of two equations that can be cast in the standard form of a matrix eigenvalue problem.

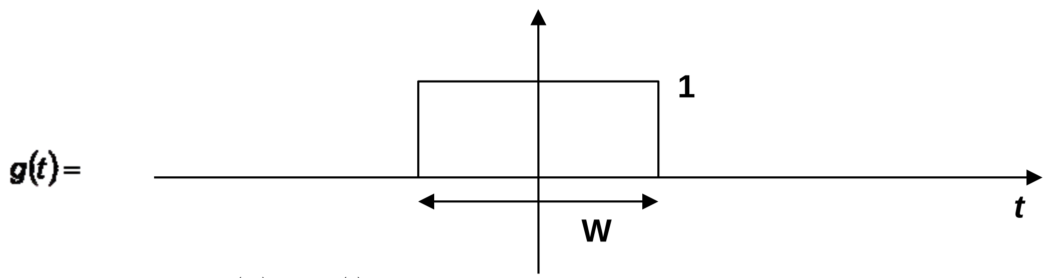
## Problem 1.4: (Basic Fourier Transforms)

a) Consider a periodic function that consists of a train of delta functions of equal weights separated in time by $T$:

$$f(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

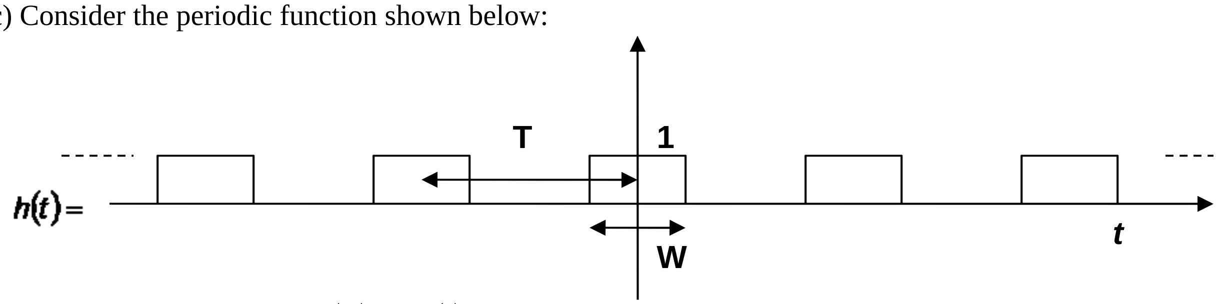
Find the Fourier transform $f(\omega)$ of $f(t)$ and show that it also consists of a train of delta functions of equal weight in the frequency domain. If you don't know how to do this problem then consult your favorite book on Fourier transforms.

b) Consider the box function shown below:




Find the Fourier transform $g(\omega)$ of $g(t)$.

c) Consider the periodic function shown below:




Find the Fourier transform $h(\omega)$ of $h(t)$ and show that it consists of a train of delta functions with unequal weights. **Hint:** write $h(t)$ as a convolution of the functions in parts (a) and (b). **Lesson:** The Fourier transform of a periodic function consists of only certain discrete frequencies.

d) Show that $h(t)$ in part (c) can be written as Fourier series: $h(t) = \sum_{n=-\infty}^{\infty} h_n \, e^{-i \frac{2\pi n}{T} t}$ and find the coefficients $h_n$. **Hint:** Use your results in part (c). **Lesson:** A periodic function can be expanded in a Fourier series.

e) Consider the following function $h(x,y,z)$ in 3-dimenions. The function $h(x,y,z)$ is equal to unity inside a cube centered at the origin and of dimensions shown in the figure below, and is equal to zero outside the cube. Find the Fourier transform $h(k_x, k_y, k_z)$ of the function $h(x,y,z)$.