# Lecture 9

# Quantum Dynamics, Measurement, and State Collapse

**In this lecture you will learn:**

- **Time evolution in quantum physics**
- **Measurement of observables in quantum physics**
- **Collapse of the quantum state post-measurement**

# Time Evolution of Quantum States

**Schrödinger equation is:**

$$i\hbar \frac{\partial}{\partial t}\left|\psi(t)\right\rangle = \hat{H}\left|\psi(t)\right\rangle$$

**A formal solution of the above equation (for time-independent Hamiltonian) is:**

$$\left|\psi(t)\right\rangle = e^{-i\frac{\hat{H}}{\hbar}t}\left|\psi(t=0)\right\rangle$$

**(The proof is by direct substitution)**

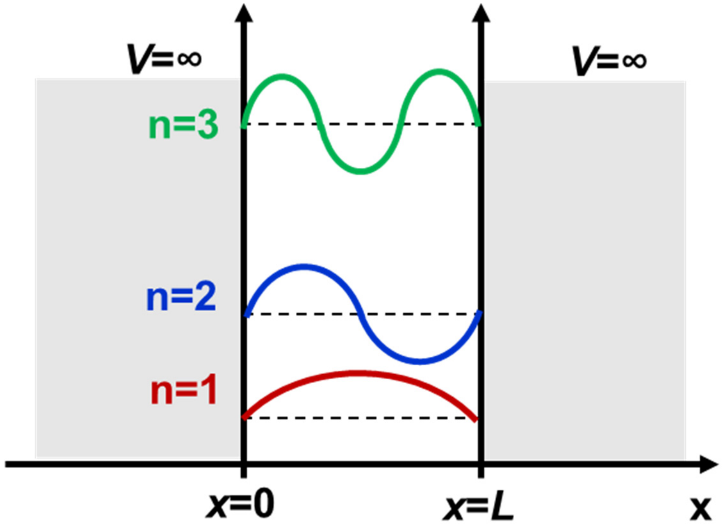
**The operator exponential (just like matrix exponentials in linear algebra) is to be conceptually understood in terms of its Taylor series expansion:**

$$e^{-i\frac{\hat{H}}{\hbar}t} = 1 + \left(-i\frac{\hat{H}}{\hbar}t\right) + \frac{1}{2!}\left(-i\frac{\hat{H}}{\hbar}t\right)^2 + \frac{1}{3!}\left(-i\frac{\hat{H}}{\hbar}t\right)^3 + ...........$$

**The expansion above is not the best way to solve problems!**

# Time Evolution of Quantum States: Infinite Well Problem I

**Consider an electron in an infinite potential well:**




**We had solved the following eigenvalue equation:**

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x)\right]\phi_n(x)=E_n\phi_n(x)$$

$$\Rightarrow \langle x|\hat{H}|\phi_n\rangle = E_n\langle x|\phi_n\rangle$$

$$\Rightarrow \hat{H}|\phi_n\rangle = E_n|\phi_n\rangle$$

$$\begin{cases} i\hbar\dfrac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle \\ |\psi(t)\rangle = e^{-i\frac{\hat{H}}{\hbar}t}|\psi(t=0)\rangle \end{cases}$$

**Suppose:** $|\psi(t=0)\rangle = |\phi_n\rangle$

**Need to find:** $|\psi(t)\rangle$

**We get:**

$$|\psi(t)\rangle = e^{-i\frac{\hat{H}}{\hbar}t}|\psi(t=0)\rangle = e^{-i\frac{\hat{H}}{\hbar}t}|\phi_n\rangle = e^{-i\frac{E_n}{\hbar}t}|\phi_n\rangle$$

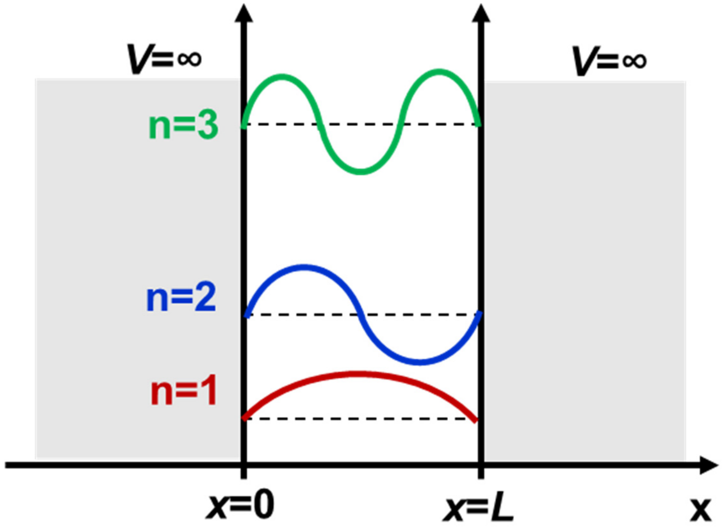
# Time Evolution of Quantum States: Infinite Well Problem II

Consider an electron in an infinite potential well and now consider an arbitrary initial state:

$$\left|\psi(t=0)\right\rangle$$

Need to find:

$$\left|\psi(t)\right\rangle$$




**STEP 1: Express the initial state as a superposition of the energy eigenstates:**

$$\left|\psi(t=0)\right\rangle = \sum_j c_j \left|\phi_j\right\rangle$$

This can be done for any initial state since the energy eigenkets form a complete set:

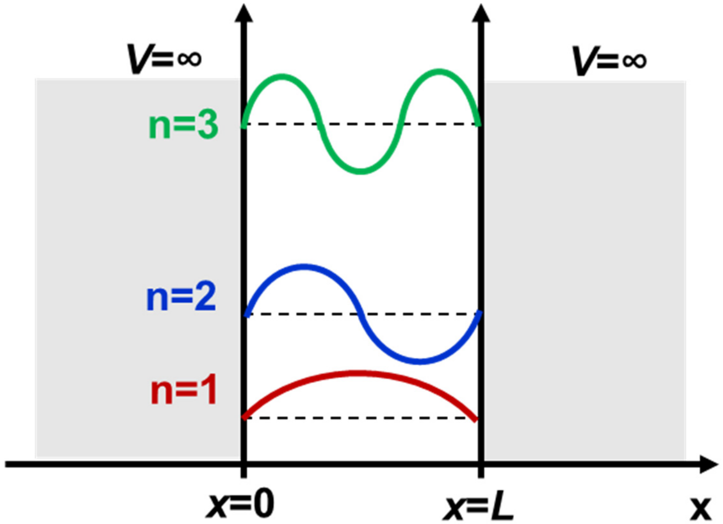
$$\left|\psi(t=0)\right\rangle = \hat{1}\left|\psi(t=0)\right\rangle = \left(\sum_j \left|\phi_j\right\rangle\left\langle\phi_j\right|\right)\left|\psi(t=0)\right\rangle = \sum_j \underbrace{\left\langle\phi_j\middle|\psi(t=0)\right\rangle}_{c_j}\left|\phi_j\right\rangle$$

$$= \sum_j c_j \left|\phi_j\right\rangle \qquad \left\{c_j = \left\langle\phi_j\middle|\psi(t=0)\right\rangle\right.$$

# Time Evolution of Quantum States: Infinite Well Problem III

**STEP 1: Express the initial state as a superposition of the energy eigenstates:**

$$\left|\psi(t=0)\right\rangle = \sum_j c_j \left|\phi_j\right\rangle$$




**STEP 2: We then find the quantum state at any later time *t* >0 as follows:**

$$\left|\psi(t)\right\rangle = e^{-i\frac{\hat{H}}{\hbar}t}\left|\psi(t=0)\right\rangle$$

$$= e^{-i\frac{\hat{H}}{\hbar}t}\sum_j c_j \left|\phi_j\right\rangle = \sum_j c_j\, e^{-i\frac{\hat{H}}{\hbar}t}\left|\phi_j\right\rangle$$

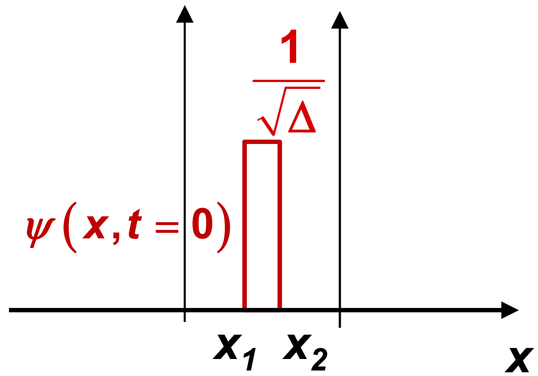
$$= \sum_j c_j\, e^{-i\frac{E_j}{\hbar}t}\left|\phi_j\right\rangle$$

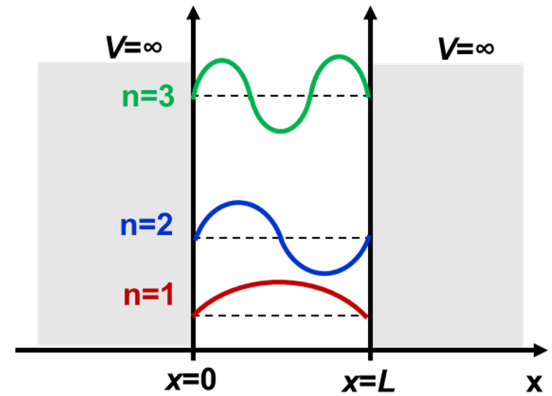
$$i\hbar\frac{\partial}{\partial t}\left|\psi(t)\right\rangle = \hat{H}\left|\psi(t)\right\rangle$$

$$\left|\psi(t)\right\rangle = e^{-i\frac{\hat{H}}{\hbar}t}\left|\psi(t=0)\right\rangle$$

# Time Evolution of Quantum States: Infinite Well Problem IV

**Suppose the initial state is:**




**Write the initial state as:**

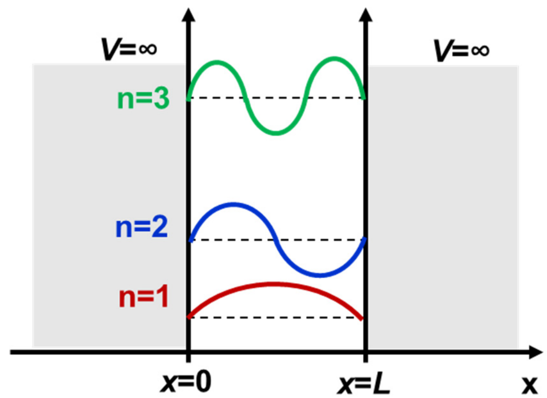
$$\left|\psi(t=0)\right\rangle = \sum_n a_n \left|\phi_n\right\rangle$$

$$a_m = \left\langle\phi_m\middle|\psi(t=0)\right\rangle$$

$$\Rightarrow a_m = \int dx \left\langle\phi_m\middle|x\right\rangle\left\langle x\middle|\psi(t=0)\right\rangle = \int_{-\infty}^{\infty} dx\ \phi_m^*(x)\psi(x,t=0)$$

$$\Rightarrow a_m = \frac{1}{\sqrt{\Delta}}\int_{x_1}^{x_2} dx\ \phi_m^*(x)$$

# Time Evolution of Quantum States: Infinite Well Problem V




Suppose the initial state is:

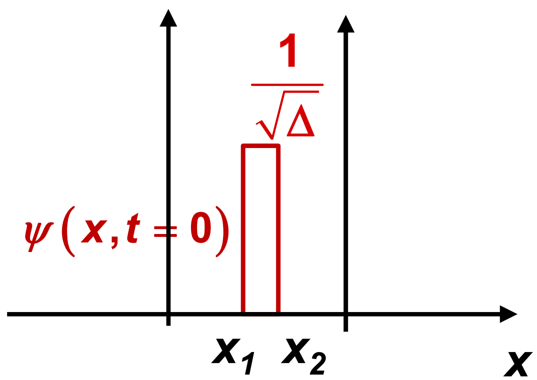
$$\psi(x,t=0) = \begin{cases} \dfrac{1}{\sqrt{\Delta}} & x_1 \le x \le x_2 \\ 0 & \text{otherwise} \end{cases}$$

The state at any later time is:

$$|\psi(t)\rangle = e^{-i\frac{\hat{H}}{\hbar}t}|\psi(t=0)\rangle$$

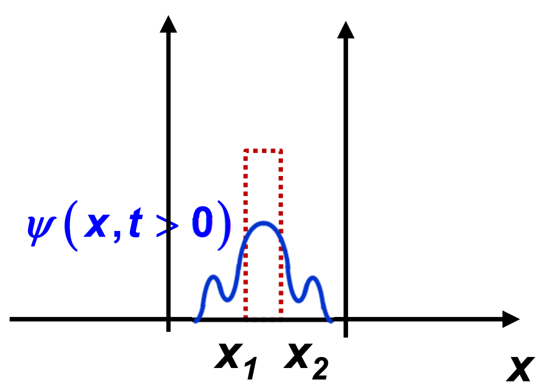
$$= e^{-i\frac{\hat{H}}{\hbar}t}\sum_j c_j|\phi_j\rangle = \sum_j c_j\, e^{-i\frac{\hat{H}}{\hbar}t}|\phi_j\rangle$$

$$= \sum_j c_j\, e^{-i\frac{E_j}{\hbar}t}|\phi_j\rangle$$



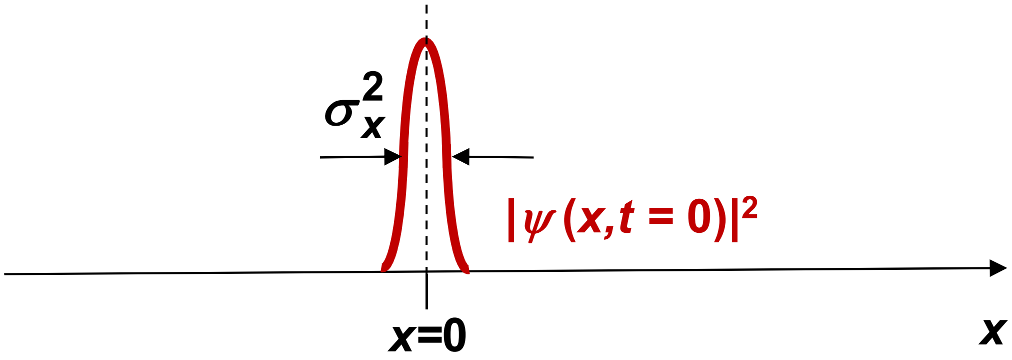

The wavefunction at later time is:

$$\psi(x,t) = \langle x|\psi(t)\rangle = \sum_j c_j\, e^{-i\frac{E_j}{\hbar}t}\phi_j(x)$$



# Time Evolution of Quantum States: Wavepacket Example

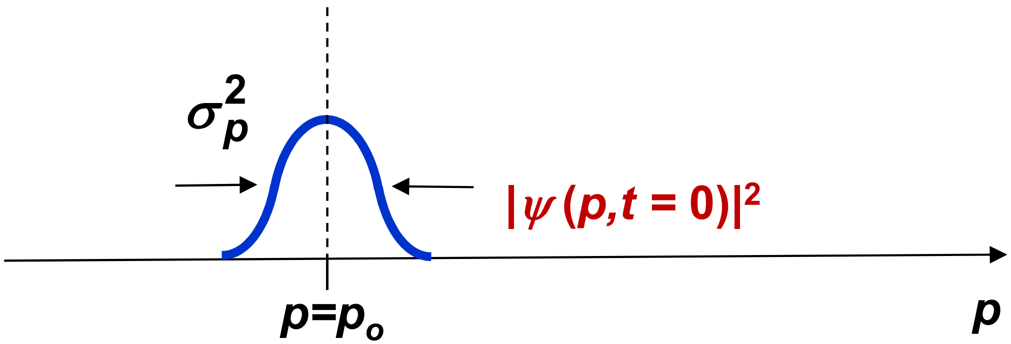
Consider a particle in 1D in free space (potential is zero everywhere)




The wavefunction at time $t = 0$ is:

$$\langle x | \psi(t=0) \rangle = \psi(x, t=0)$$

$$= \left( \frac{1}{2\pi\sigma_x^2} \right)^{1/4} e^{-\frac{x^2}{4\sigma_x^2}} e^{i\frac{p_o}{\hbar}x}$$




Or in momentum basis:

$$\langle p | \psi(t=0) \rangle = \psi(p, t=0)$$

$$= \int_{-\infty}^{\infty} dx \, \psi(x, t=0) \frac{e^{-ipx}}{\sqrt{\hbar}} = \left( \frac{2\pi}{\sigma_p^2} \right)^{1/4} e^{-\frac{(p-p_o)^2}{4\sigma_p^2}} \qquad \left\{ \sigma_p^2 = \frac{\hbar^2}{4\sigma_x^2} \right.$$

Statement of the Problem: Need to find the wavefunction $\psi(x,t)$ for $t > 0$

# Time Evolution of Quantum States: Wavepacket Problem

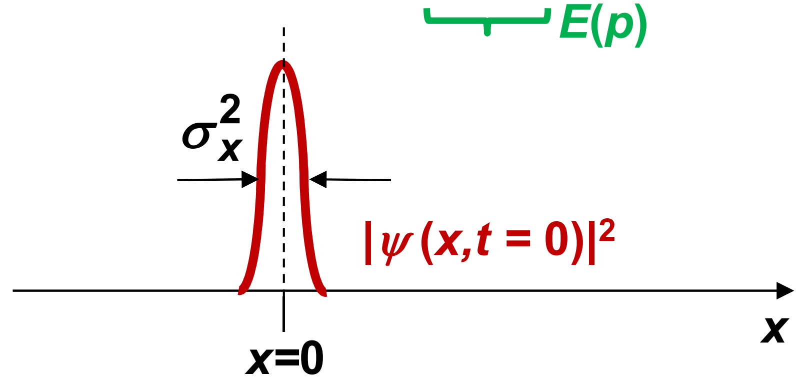
Consider a particle in free-space: $\hat{H} = \dfrac{\hat{p}^2}{2m}$

Momentum eigenstates are also energy eigenstates

$$\begin{cases} \hat{p}|p\rangle = p|p\rangle \\ \hat{H}|p\rangle = \dfrac{\hat{p}^2}{2m}|p\rangle = \underbrace{\dfrac{p^2}{2m}}_{E(p)}|p\rangle \end{cases}$$

Suppose at time $t$=0:

$$\langle x|\psi(t=0)\rangle = \psi(x,t=0) = \text{given}$$




STEP 1: Express the initial state in the energy (momentum) eigenstates:

$$|\psi(t=0)\rangle = \hat{1}|\psi(t=0)\rangle = \left(\int_{-\infty}^{\infty}\frac{dp}{2\pi}|p\rangle\langle p|\right)|\psi(t=0)\rangle = \int_{-\infty}^{\infty}\frac{dp}{2\pi}\psi(p,t=0)\,|p\rangle$$

What is the quantum state at time $t > 0$ ??

## Time Evolution of Quantum States: Wavepacket Problem

**STEP 2: Find the state at later time as follows:**

$$\left|\psi(t)\right\rangle = e^{-i\frac{\hat{H}}{\hbar}t}\left|\psi(t=0)\right\rangle$$

$$= e^{-i\frac{\hat{H}}{\hbar}t}\int_{-\infty}^{\infty}\frac{dp}{2\pi}\psi(p,t=0)\left|p\right\rangle = \int_{-\infty}^{\infty}\frac{dp}{2\pi}\psi(p,t=0)\,e^{-i\frac{\hat{H}}{\hbar}t}\left|p\right\rangle$$

$$= \int_{-\infty}^{\infty}\frac{dp}{2\pi}\psi(p,t=0)\,e^{-i\frac{E(p)}{\hbar}t}\left|p\right\rangle$$

**The wavefunction at later time is then:**

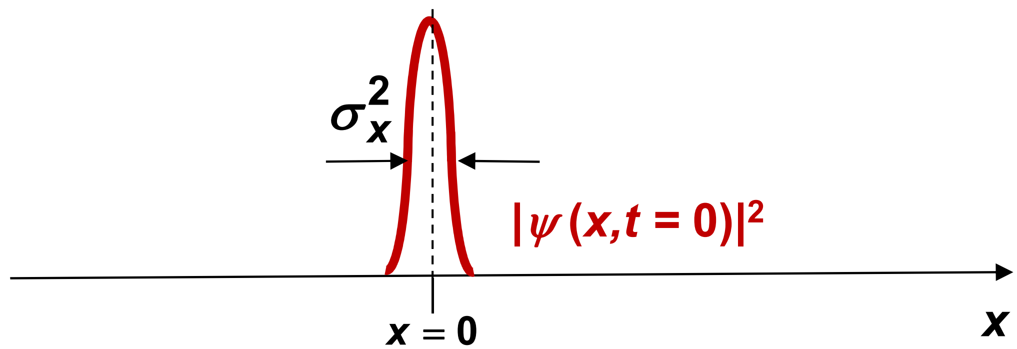
$$\Rightarrow \psi(x,t) = \left\langle x\middle|\psi(t)\right\rangle = \int_{-\infty}^{\infty}\frac{dp}{2\pi}\psi(p,t=0)\,e^{-i\frac{E(p)}{\hbar}t}\left\langle x\middle|p\right\rangle$$

$$= \int_{-\infty}^{\infty}\frac{dp}{2\pi}\,\psi(p,t=0)\,\frac{e^{i\frac{p}{\hbar}x}\,e^{-i\frac{E(p)}{\hbar}t}}{\sqrt{\hbar}}$$

# Time Evolution of Quantum States: Wavepacket Problem

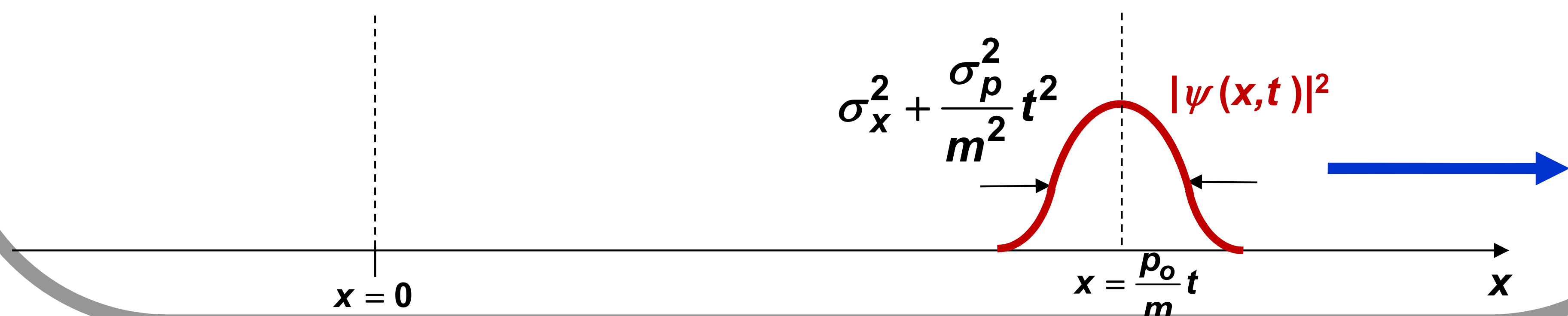
$$\psi(x,t) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \left(\frac{2\pi}{\sigma_p^2}\right)^{1/4} e^{-\frac{(p-p_o)^2}{4\sigma_p^2}} \frac{e^{i\frac{p}{\hbar}x}\, e^{-i\frac{E(p)}{\hbar}t}}{\sqrt{\hbar}} \qquad \left\{ \sigma_p^2 = \frac{\hbar^2}{4\sigma_x^2} \right.$$

$$= \left(\frac{\sigma_x^2}{2\pi}\right)^{1/4} \frac{1}{\sqrt{\sigma_x^2 + i\frac{\hbar}{2m}t}}\, e^{-\frac{\left(x-\frac{p_o}{m}t\right)^2}{4\left[\sigma_x^2 + i\frac{\hbar}{2m}t\right]}}\, e^{ip_o\left(x-\frac{p_o}{2m}t\right)}$$




**Wavepacket group velocity**

$$v_g = \left.\frac{\partial E(p)}{\partial p}\right|_{p=p_o} = \frac{p_o}{m}$$



# The Time Evolution Operator: A Unitary Operator

$$i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

$$|\psi(t)\rangle = e^{-i\frac{\hat{H}}{\hbar}t}|\psi(t=0)\rangle$$

**The operator** $e^{-i\frac{\hat{H}}{\hbar}t}$ **is the time evolution operator:** $\hat{U}(t) = e^{-i\frac{\hat{H}}{\hbar}t}$

$$|\psi(t)\rangle = \hat{U}(t)|\psi(t=0)\rangle$$

**This operator is unitary and preserves the norm of the quantum state:**

$$\begin{aligned}\langle\psi(t)|\psi(t)\rangle &= \langle\psi(t=0)|\hat{U}^\dagger(t)\hat{U}(t)|\psi(t=0)\rangle \\ &= \langle\psi(t=0)|\hat{1}|\psi(t=0)\rangle \\ &= \langle\psi(t=0)|\psi(t=0)\rangle\end{aligned}$$

$$\hat{U}(t) = e^{-i\frac{\hat{H}}{\hbar}t}$$

$$\Rightarrow \hat{U}^\dagger(t) = e^{i\frac{\hat{H}^\dagger}{\hbar}t} = e^{-i\frac{\hat{H}}{\hbar}t}$$

$$\Rightarrow \hat{U}^\dagger(t)\hat{U}(t) = \hat{1}$$

# Energy Measurement Problem

**Suppose we know the Hamiltonian operator for a particle:** $\hat{H}$

**And the quantum state at time *t* is:** $|\psi(t)\rangle$

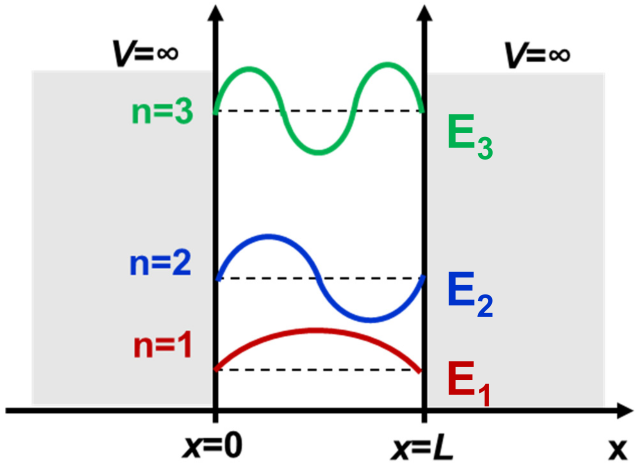
**If we want to find out the a-priori probability of finding the particle at location *x* upon making a measurement, the answer would be:** $|\psi(x,t)|^2$

**Million Dollar Question: Now if particle energy is measured, what are the possible results? And what are the a-priori probabilities of finding each result ??**

**Some comments:**

1) **Consider the infinite potential well problem. A particle inside the well can only have energies $E_1$, $E_2$, $E_3$, ….. i.e. the energy eigenvalues of the Hamiltonian!**

2) **Therefore, the possible result of any energy measurement will always be one of the eigenvalues of the Hamiltonian**



# The Position Measurement Problem Revisited

**Given a state $\left|\psi(t)\right\rangle$, the a-priori probability of finding the particle location to be *x* upon making a measurement is:**

$$\left|\psi(x,t)\right|^2$$

**How did we get the above result ???**

**Max Born's Interpretation (Born Ansatz):**

**We took the state $\left|\psi(t)\right\rangle$ and we took the inner product,**

$$\psi(x) = \left\langle x \middle| \psi(t) \right\rangle$$

**And then we said that the a-priori probability of finding the particle at the location *x* (i.e. at the location corresponding to the ket $\left|x\right\rangle$ ) is:**

$$\left|\psi(x,t)\right|^2 = \left|\left\langle x \middle| \psi(t) \right\rangle\right|^2$$

# The Position Measurement Problem Revisited

## Max Born's Interpretation dissected:

- We are measuring position

- Position is an observable and is represented by the operator $\hat{x}$

- Position operator has a complete set of eigenkets or eigenstates:

$$\hat{x}\,|x'\rangle = x'\,|x'\rangle \quad \longleftrightarrow \quad \int_{-\infty}^{\infty} dx\,|x\rangle\langle x| = \hat{1}$$

- According to the Born interpretation, given a quantum state $|\psi(t)\rangle$ the a-priori probability of measuring the particle position to be *x'* is given by taking the eigenket $|x'\rangle$ corresponding to the eigenvalue *x'* and then computing:

$$|\psi(x',t)|^2 = |\langle x'|\psi(t)\rangle|^2$$

# Back to the Energy Measurement Problem

**Question:** Given a quantum state $\left|\psi(t)\right\rangle$, if particle energy is measured, what are the possible results? And what are the a-priori probabilities of finding each result ??

Suppose we know the Hamiltonian operator for a particle: $\hat{H}$

And its eigenstates and eigenvalues are: $\hat{H}\left|\phi_n\right\rangle = E_n\left|\phi_n\right\rangle \longleftrightarrow \sum_j \left|\phi_j\right\rangle\left\langle\phi_j\right| = \hat{1}$

A possible result of any energy measurement will always be one of the eigenvalues of the Hamiltonian. The a-priori probability of finding an energy eigenvalue $E_n$ can be computed as follows:

We take the state $\left|\psi(t)\right\rangle$ and we compute the inner product:

$$\left\langle\phi_n\middle|\psi(t)\right\rangle$$

Then we take the squared magnitude of this inner product:

$$\left|\left\langle\phi_n\middle|\psi(t)\right\rangle\right|^2$$

The a-priori probability of finding the particle energy to be $E_n$ is then:

$$\left|\left\langle\phi_n\middle|\psi(t)\right\rangle\right|^2$$

# Energy Measurement Problem

**Example:**

**Suppose we know the Hamiltonian operator for a particle:** $\hat{H}$

**And its eigenstates and eigenvalues are:** $\hat{H}\left|\phi_n\right\rangle = E_n\left|\phi_n\right\rangle$

**And suppose we expans the quantum state in terms of the energy eigenstates:**

$$\left|\psi(t)\right\rangle = \sum_j a_j \left|\phi_j\right\rangle$$

**If an energy measurement is made, then the probability of finding the particle with energy $E_n$ is:**

$$\left|\left\langle\phi_n\middle|\psi(t)\right\rangle\right|^2 = \left|a_n\right|^2$$

**If the particle is in an energy eigenstate, e.g. $\left|\psi(t)\right\rangle = \left|\phi_m\right\rangle$, then upon measurement the particle will be found to have the corresponding eigenvalue $E_m$ as the energy with probability one**

**The above arguments go over to the measurement of all observables!**

# Observables and A-Priori Measurement Probabilities

Suppose at time $t$ the quantum state is: $\left|\psi(t)\right\rangle$

Suppose at time $t$ the observable $O$ is measured

The corresponding operator $\hat{O}$ has the following eigenvalues and eigenstates:

$$\hat{O}\left|v_j\right\rangle = \lambda_j\left|v_j\right\rangle \quad \longleftrightarrow \quad \sum_j \left|v_j\right\rangle\left\langle v_j\right| = \hat{1}$$

Then:

1) The result of the measurement can only be one of the eigenvalues $\lambda_j$ of the operator $\hat{O}$

2) If a measurement of the observable $O$ is made, then the a-priori probability of finding the result $\lambda_n$ (i.e. one of the eigenvalues of the operator $\hat{O}$ ) is:

$$\left|\left\langle v_n \middle| \psi(t)\right\rangle\right|^2$$

3) All the a-priori probabilities must add up to unity:

$$\sum_n \left|\left\langle v_n \middle| \psi(t)\right\rangle\right|^2 = 1$$

$$\begin{aligned}
&\sum_n \left|\left\langle v_n \middle| \psi(t)\right\rangle\right|^2 \\
&= \sum_n \left\langle \psi(t) \middle| v_n\right\rangle\left\langle v_n \middle| \psi(t)\right\rangle \\
&= \left\langle \psi(t)\right|\left(\sum_n \left|v_n\right\rangle\left\langle v_n\right|\right)\left|\psi(t)\right\rangle \\
&= \left\langle \psi(t) \middle| \psi(t)\right\rangle = 1
\end{aligned}$$

## Observables and A-Priori Measurement Probabilities: Mean Values

Suppose at time $t$ the quantum state is: $\left|\psi(t)\right\rangle$

Suppose at time $t$ the observable $O$ is measured

The corresponding operator $\hat{O}$ has the following eigenvalues and eigenstates:

$$\hat{O}\left|v_j\right\rangle = \lambda_j \left|v_j\right\rangle$$

**2) If a measurement of the observable *O* is made, then the a-priori probability of finding the result $\lambda_n$ is:**

$$\left|\left\langle v_n \middle| \psi(t)\right\rangle\right|^2$$

Mean value of the observable O (after many measurements) is:

$$\sum_n \left|\left\langle v_n \middle| \psi(t)\right\rangle\right|^2 \lambda_n$$

$$= \sum_n \lambda_n \left\langle \psi(t) \middle| v_n \right\rangle \left\langle v_n \middle| \psi(t) \right\rangle$$

$$= \sum_n \left\langle \psi(t) \middle| \hat{O} \middle| v_n \right\rangle \left\langle v_n \middle| \psi(t) \right\rangle$$

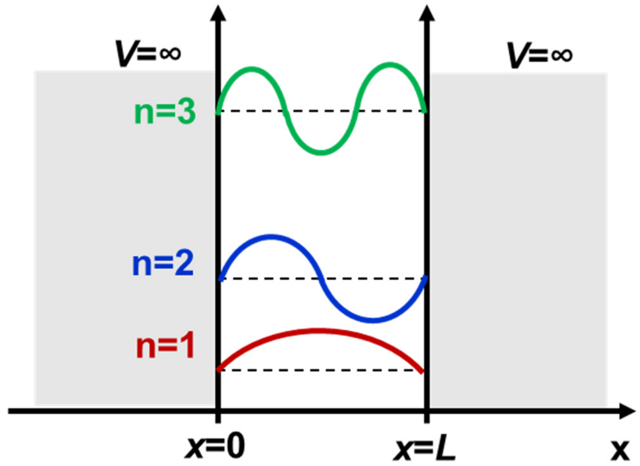
$$= \left\langle \psi(t) \right| \hat{O} \left( \sum_n \left|v_n\right\rangle\left\langle v_n\right| \right) \left|\psi(t)\right\rangle$$

$$= \left\langle \psi(t) \middle| \hat{O} \middle| \psi(t) \right\rangle$$

→ Important !!

# Momentum Measurement Problem

**Consider an electron in an infinite potential well:**

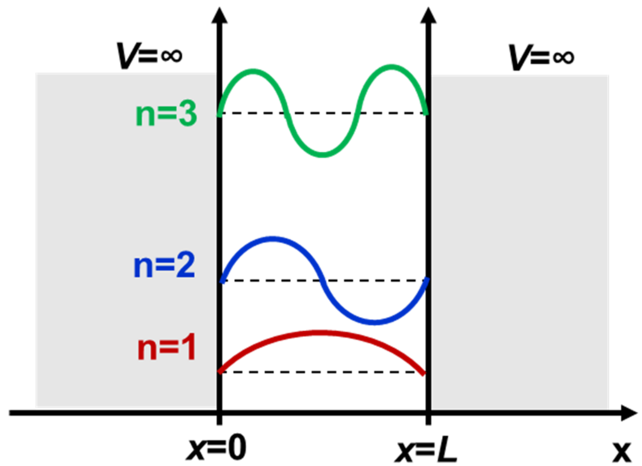



**The quantum state at time *t* is:**

$$\left|\psi(t)\right\rangle = \sum_j c_j \left|\phi_j\right\rangle$$

**If particle <u>momentum</u> is measured, what are the possible results? And what are the a-priori probabilities of finding each result ??**

# Momentum Measurement Problem




$$|\psi(t)\rangle = \sum_j c_j |\phi_j\rangle$$

The momentum operator $\hat{p}$ has the following eigenstates and eigenvalues:

$$\hat{p}|p'\rangle = p'|p'\rangle$$

Following the same rules, the probability of finding the momentum $p$ upon making a momentum measurement is:

$$|\langle p|\psi(t)\rangle|^2$$

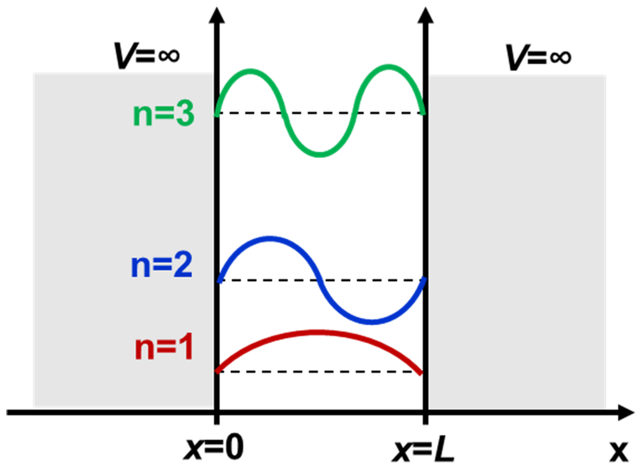
The mean value or the expectation value of the momentum will be:

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi} |\langle p|\psi(t)\rangle|^2 p = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \langle \psi(t)|p\rangle\langle p|\psi(t)\rangle\, p = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \langle \psi(t)|\hat{p}|p\rangle\langle p|\psi(t)\rangle$$

$$= \langle \psi(t)|\hat{p}\left(\int_{-\infty}^{\infty} \frac{dp}{2\pi} |p\rangle\langle p|\right)|\psi(t)\rangle = \langle \psi(t)|\hat{p}\,\hat{1}|\psi(t)\rangle = \langle \psi(t)|\hat{p}|\psi(t)\rangle$$

# Momentum Measurement Problem

$$\left|\psi(t)\right\rangle = \sum_j c_j \left|\phi_j\right\rangle$$




$$\left\langle p \middle| \psi(t)\right\rangle = \sum_j c_j \left\langle p \middle| \phi_j\right\rangle = \sum_j c_j \phi_j(p)$$

$$\Rightarrow \left|\left\langle p \middle| \psi(t)\right\rangle\right|^2 = \left|\sum_j c_j \phi_j(p)\right|^2$$

Probability of finding the momentum *p* upon making a momentum measurement

Suppose: $\left|\psi(t)\right\rangle = \left|\phi_m\right\rangle$

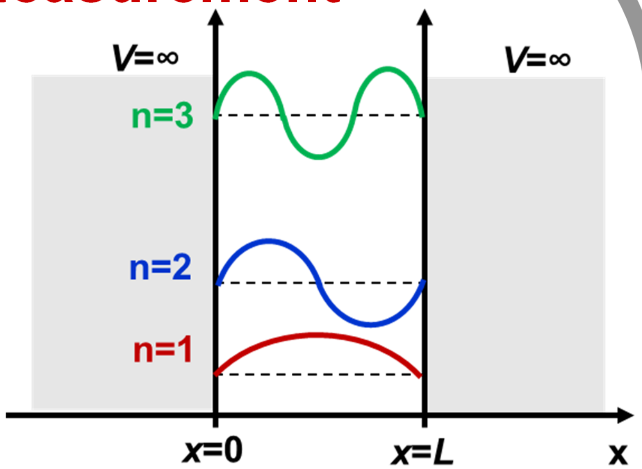
Then: $\left|\left\langle p \middle| \psi(t)\right\rangle\right|^2 = \left|\phi_m(p)\right|^2$

The Fourier transform magnitude squared!!

# Collapse of the Quantum State Upon Measurement

Suppose at time $t = t_1$ the quantum state is:

$$\left|\psi\left(t = t_1\right)\right\rangle = \sum_j c_j \left|\phi_j\right\rangle$$




Suppose at time $t = t_1$ the energy of the particle is measured

Suppose the result of this measurement was: $E_m$

**Question: what is the quantum state immediately after the measurement?**

**The Copenhagen Interpretation:**

**The quantum state represents ALL that is there and is knowable about reality, and therefore the quantum state immediately after the measurement must reflect this knowledge gained (by a conscious observer) from the act of measurement**

Therefore, immediately after the measurement the quantum state must be:

$$\left|\psi\left(t = t_1^+\right)\right\rangle = \left|\phi_m\right\rangle$$

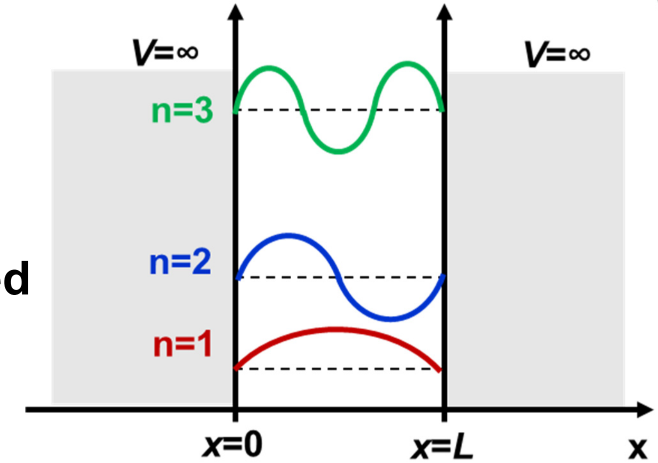
1) The superposition in the quantum state has collapsed!!!

2) The quantum state collapses into the eigenstate of the operator corresponding to the measured eigenvalue

# Collapse of the Quantum State Upon Measurement

Suppose at time $t = t_1$ the quantum state is:

$$\left|\psi(t=t_1)\right\rangle = \sum_j c_j \left|\phi_j\right\rangle$$




Suppose at time $t = t_1$ the energy of the particle is measured

Suppose the result of this measurement was: $E_m$

How to find the quantum state post-measurement?

1) Make a projection operator using the eigenstate corresponding to the eigenvalue measured:

$$\hat{P}_r = \left|\phi_m\right\rangle\left\langle\phi_m\right|$$

2) Apply the projection operator to the quantum state just before the measurement:

$$\hat{P}_r\left|\psi(t=t_1)\right\rangle = \left|\phi_m\right\rangle\left\langle\phi_m\middle|\psi(t=t_1)\right\rangle = \left\langle\phi_m\middle|\psi(t=t_1)\right\rangle\left|\phi_m\right\rangle$$

3) Normalize the resulting state:

$$\frac{\left\langle\phi_m\middle|\psi(t=t_1)\right\rangle\left|\phi_m\right\rangle}{\sqrt{\left|\left\langle\phi_m\middle|\psi(t=t_1)\right\rangle\right|^2}} \propto \left|\phi_m\right\rangle$$

This is the answer!
(up to an irrelevant overall phase factor)

The quantum state collapses into the eigenstate of the operator corresponding to the measured eigenvalue

# Collapse of the Quantum State Upon Measurement: General Treatment

Suppose at time $t = t_1$ the quantum state is: $\left|\psi\left(t = t_1\right)\right\rangle$

Suppose at time $t = t_1$ the observable $O$ is measured

The corresponding operator $\hat{O}$ has the following eigenvalues and eigenstates:

$$\hat{O}\left|v_j\right\rangle = \lambda_j\left|v_j\right\rangle$$

Suppose the result of this measurement was: $\lambda_m$

How to find the quantum state post-measurement?

We also throw in an added complexity: three eigenvectors of $\hat{O}$ have the same eigenvalue $\lambda_m$

$$\hat{O}\left|v_m\right\rangle = \lambda_m\left|v_m\right\rangle$$
$$\hat{O}\left|v_n\right\rangle = \lambda_m\left|v_n\right\rangle$$
$$\hat{O}\left|v_p\right\rangle = \lambda_m\left|v_p\right\rangle$$

Eigenvectors are different but the corresponding eigenvalues of the operator $\hat{O}$ are the same

Example: The eigenvectors $\left|p\right\rangle$ and $\left|-p\right\rangle$ of the Hamiltonian for a free particle have the same energy eigenvalue $E(p)$

# Collapse of the Quantum State Upon Measurement: General Treatment

## How to find the quantum state post-measurement?

1) Make a projection operator using all the eigenstates of $\hat{O}$ that have the eigenvalue that is measured:

$$\hat{P}_r = |v_m\rangle\langle v_m| + |v_n\rangle\langle v_n| + |v_p\rangle\langle v_p|$$

2) Apply this projection operator to the quantum state just before the measurement:

$$\hat{P}_r |\psi(t=t_1)\rangle = \left(|v_m\rangle\langle v_m| + |v_n\rangle\langle v_n| + |v_p\rangle\langle v_p|\right)|\psi(t=t_1)\rangle$$

$$= \langle v_m|\psi(t=t_1)\rangle|v_m\rangle + \langle v_n|\psi(t=t_1)\rangle|v_n\rangle + \langle v_p|\psi(t=t_1)\rangle|v_p\rangle$$

3) Normalize the resulting state:

$$\frac{\langle v_m|\psi(t=t_1)\rangle|v_m\rangle + \langle v_n|\psi(t=t_1)\rangle|v_n\rangle + \langle v_p|\psi(t=t_1)\rangle|v_p\rangle}{\sqrt{\left|\langle v_m|\psi(t=t_1)\rangle\right|^2 + \left|\langle v_n|\psi(t=t_1)\rangle\right|^2 + \left|\langle v_p|\psi(t=t_1)\rangle\right|^2}}$$

**This is the answer!**
(up to an irrelevant overall phase factor)

**The quantum state collapses into the eigen-subspace of the operator corresponding to the measured eigenvalue**