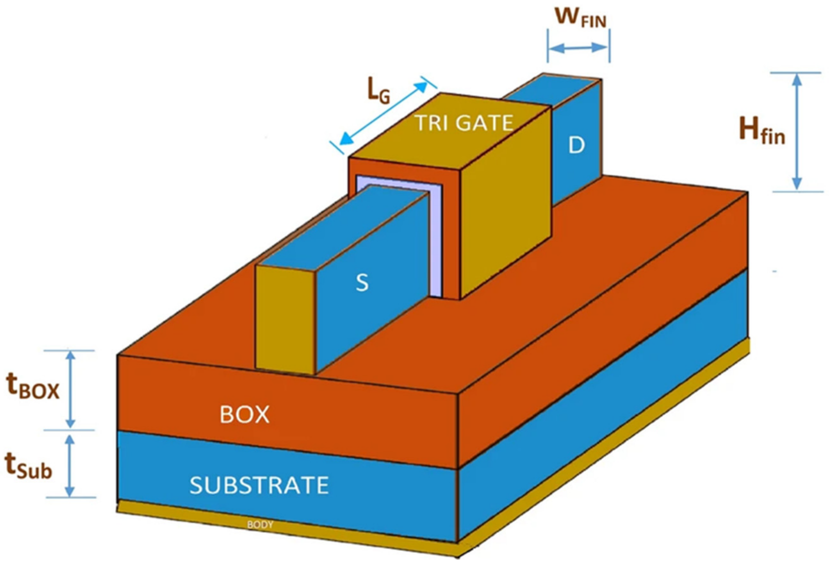
Lecture 6

Schrödinger Equation: The Time-Independent Form

In this lecture you will learn:

- Schrödinger equation – the time-independent form
- Particle in an infinite potential well
- Quantum mechanical tunneling



Schrödinger Equation: What We Learned from the Previous Lecture

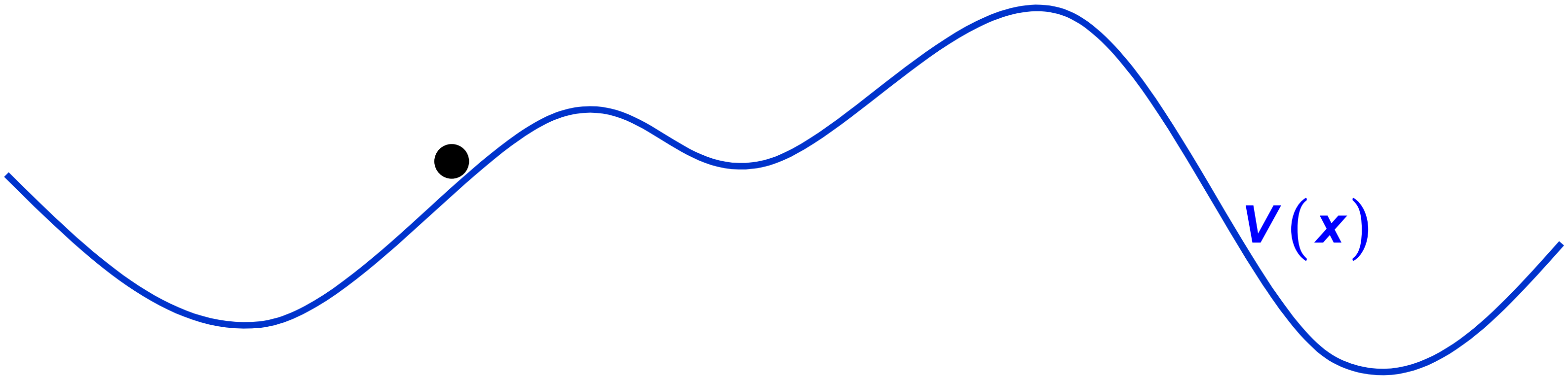
The mean value of the energy is:

$$\langle E\rangle(t) = \int_{-\infty}^{\infty} dx\,\psi^*(x,t)\underbrace{\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x)\right]}_{\text{The total energy operator}}\psi(x,t)$$

The Schrodinger equation is:

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = \underbrace{\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x)\right]}_{\text{The total energy operator}}\psi(x,t)$$

Energy Conservation in Newtonian Physics



Consider a particle of mass *m* with potential energy given by: $V(x)$

The potential is not explicitly time dependent

The total energy *E* of the particle, given by,

$$E = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + V(x)$$

is conserved, i.e.,

$$\frac{dE}{dt} = m\left(\frac{dx}{dt}\right)\left(\frac{d^2x}{dt^2}\right) + \frac{dV(x)}{dx}\left(\frac{dx}{dt}\right) = 0$$

Using Newton's second law:

$$m\left(\frac{d^2x}{dt^2}\right) = -\frac{dV(x)}{dx}$$

Total energy is time-independent

Schrödinger Equation: Time-Independent Form (1D)

Many cases of practical interest involve problems in which the potential is not a function of time:

$$i\hbar\frac{\partial\psi\left(x,t\right)}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\psi\left(x,t\right)}{\partial x^2}+V\left(x\right)\psi\left(x,t\right)$$

In these cases, the TOTAL energy of the particle DOES NOT CHANGE with time and therefore we seek solutions of the form:

$$\psi\left(x,t\right)=\phi\left(x\right)e^{-i\frac{E}{\hbar}t}$$

Energy *E*

Normalization requirement gives:

$$\int_{-\infty}^{\infty}dx\,\psi^*\left(x,t\right)\psi\left(x,t\right)=\int_{-\infty}^{\infty}dx\,\phi^*\left(x\right)\phi\left(x\right)=1$$

Schrödinger Equation: Time-Independent Form (1D)

$$i\hbar\frac{\partial\psi(x,t)}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2}+V(x)\psi(x,t)$$

$$\psi(x,t)=\phi(x)e^{-i\frac{E}{\hbar}t} \longleftarrow$$

Energy *E*

The mean value of the energy is:

$$\langle E\rangle(t)=\int_{-\infty}^{\infty}dx\,\psi^*(x,t)\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x)\right]\psi(x,t)$$

$$=\int_{-\infty}^{\infty}dx\,\psi^*(x,t)\left[i\hbar\frac{\partial}{\partial t}\right]\psi(x,t)$$

$$=\int_{-\infty}^{\infty}dx\,\psi^*(x,t)\left[i\hbar\frac{\partial\psi(x,t)}{\partial t}\right]=\int_{-\infty}^{\infty}dx\,\phi^*(x)e^{i\frac{E}{\hbar}t}\left[E\phi(x)e^{-i\frac{E}{\hbar}t}\right]$$

$$=E\int_{-\infty}^{\infty}dx\,\phi^*(x)\phi(x)=E \longrightarrow$$

Independent of time

The standard deviation in energy can be shown to be zero:

$$\left\langle E^2\right\rangle(t)-\left[\langle E\rangle(t)\right]^2=0$$

So energy of the state is precise!

Schrödinger Equation: Time-Independent Form (1D)

$$i\hbar\frac{\partial\psi(x,t)}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2}+V(x)\psi(x,t)$$

$$\psi(x,t)=\phi(x)e^{-i\frac{E}{\hbar}t}$$

Plug the solution form in the Schrödinger equation to get the <u>time-independent</u> form of the Schrödinger equation:

$$\boxed{-\frac{\hbar^2}{2m}\frac{\partial^2\phi(x)}{\partial x^2}+V(x)\phi(x)=E\phi(x)}$$

We need to solve this equation!

$$\underbrace{\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x)\right]}_{\text{The total energy operator}}\phi(x)=E\phi(x)$$

Schrödinger Equation: Time-Independent Form (3D)

Many cases of practical interest involve problems in which the potential is not a function of time:

$$i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi(\vec{r},t) + V(\vec{r})\psi(\vec{r},t)$$

In these cases, the TOTAL energy of the particle DOES NOT CHANGE with time and therefore we seek solutions of the form:

$$\psi(\vec{r},t) = \phi(\vec{r})e^{-i\frac{E}{\hbar}t}$$

← Energy *E*

Plug the solution form in the Schrödinger equation to get the time-independent form of the Schrödinger equation:

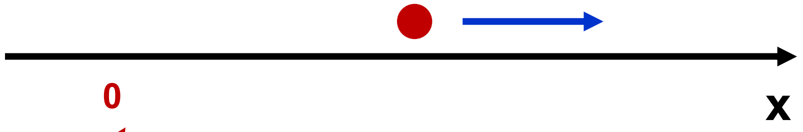
$$\boxed{\left[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r})\right]\phi(\vec{r}) = E\phi(\vec{r})}$$

We need to solve this equation!
The solutions will represent states with fixed precise energies

The Free Particle in 1D

Consider a free particle in 1D

The potential energy *V*(*x*) is 0 everywhere:



$$i\hbar\frac{\partial\psi(x,t)}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2}+\cancelto{0}{V(x)}\psi(x,t)$$

$$\psi(x,t)=\phi(x)e^{-i\frac{E}{\hbar}t}$$

$$\boxed{-\frac{\hbar^2}{2m}\frac{\partial^2\phi(x)}{\partial x^2}=E\phi(x)}$$

Assume: $\phi(x)=Ae^{ikx}$ → **Plane wave solution**

And:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi(x)}{\partial x^2}=E\phi(x)$$

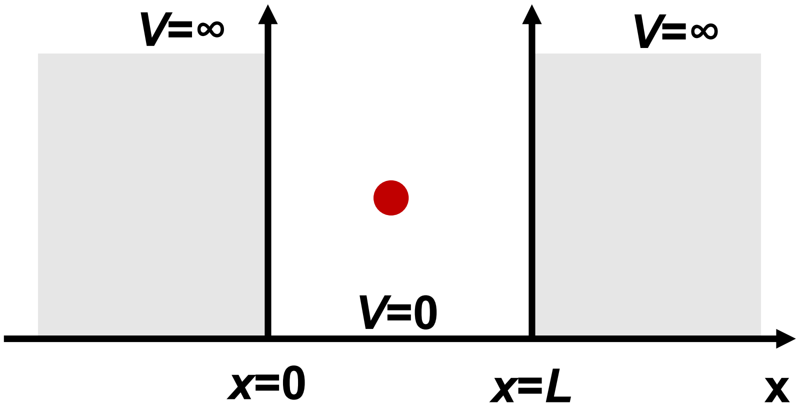
$$\Rightarrow\frac{\hbar^2k^2}{2m}Ae^{ikx}=EAe^{ikx}$$

$$\Rightarrow E=\frac{\hbar^2k^2}{2m}$$ → $$\psi(x,t)=Ae^{ikx}e^{-i\frac{E}{\hbar}t}=Ae^{ikx}e^{-i\frac{\hbar k^2}{2m}t}$$

The Infinite Potential Well Problem in 1D

Consider a particle placed inside a 1D box

Inside the box the potential energy *V*(*x*) is 0
Outside the box the potential energy *V*(*x*) is ∞



The infinite potential at the boundary walls (at *x*=0 and at *x*=*L*) ensure that the particle has no chance of ever being outside the box

We need to figure out the quantum mechanical behavior of the confined particle:

$$\psi(x,t) = \phi(x) e^{-i\frac{E}{\hbar}t}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi(x)}{\partial x^2} + V(x)\phi(x) = E\phi(x)$$

The Time-Independent Schrodinger Equation: An Eigenvalue Equation

$$\psi(x,t)=\phi(x)e^{-i\frac{E}{\hbar}t}$$

$$\underbrace{\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x)\right]}_{\text{Linear energy operator}}\underbrace{\phi(x)}_{\text{Eigenfunction}}=\overset{\substack{\text{Energy Eigenvalue}\\ \downarrow}}{E}\underbrace{\phi(x)}_{}$$

Compared with a matrix eigenvalue equation:

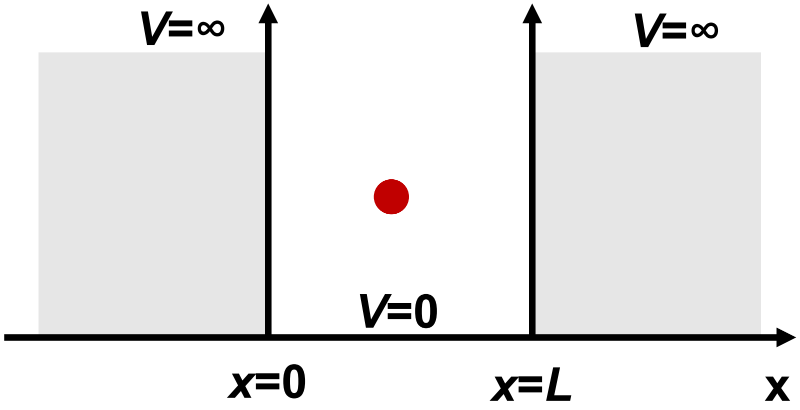
Eigenvalue

$$\bar{\bar{A}}v=\lambda v$$

Linear operator (matrix)

Eigenvector (Column vector)

The Infinite Potential Well Problem in 1D



1) Inside the potential well we can use:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi(x)}{\partial x^2}=E\phi(x)$$

$\Big\{$ $V(x)$=0 inside

2) Outside the potential well we must have: $\phi(x)=0$

Otherwise the average potential energy of the particle will be infinite:

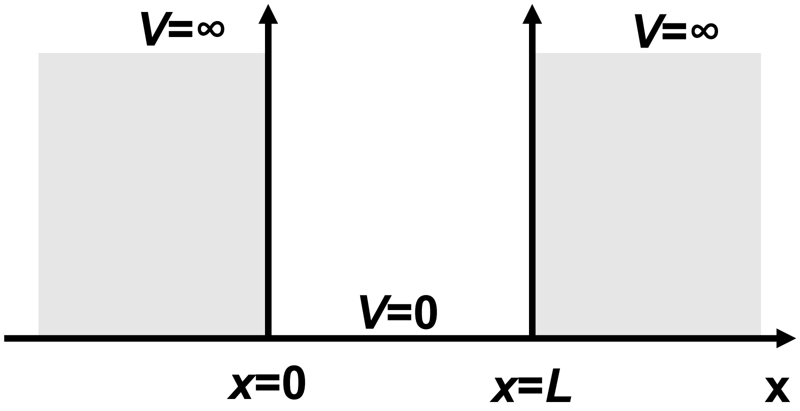
$$\Big\{\ \langle V(x)\rangle=\int_{-\infty}^{\infty}dx\ \ \phi^*(x)\big[V(x)\big]\phi(x)$$

This is a second order differential equation → needs two boundary conditions!

$$\phi(x=0)=\phi(x=L)=0$$

So that the energy of the particle must not become infinite!

The Infinite Potential Well Problem in 1D



Solve: $$-\frac{\hbar^2}{2m}\frac{\partial^2\phi(x)}{\partial x^2}=E\phi(x)\xrightarrow{\text{Subject to BCs}}\phi(x=0)=\phi(x=L)=0$$

Try a solution: $\phi(x)=A\sin(kx)$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi(x)}{\partial x^2}=E\phi(x)\Rightarrow\frac{\hbar^2k^2}{2m}A\sin(kx)=EA\sin(kx)\Rightarrow E=\frac{\hbar^2k^2}{2m}$$

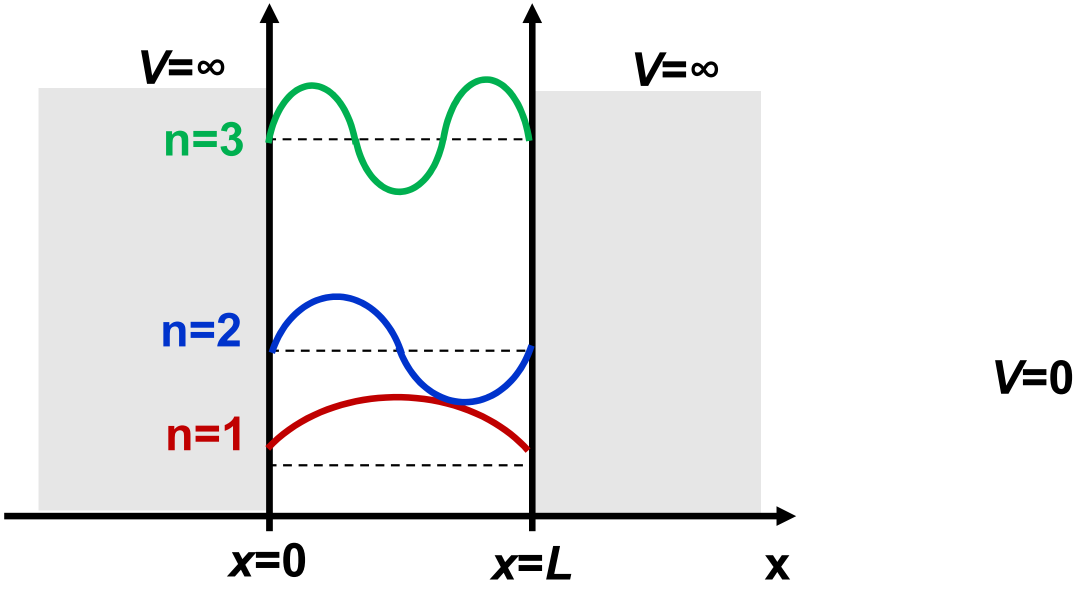
What about the Boundary Conditions? $\phi(x=0)=\phi(x=L)=0$

$\phi(x=0)$ is good

$$\phi(x=L)=0\Rightarrow\sin(kL)=0\Rightarrow k=n\frac{\pi}{L}\qquad\{n=1,2,3....$$

k can only take values that are integral multiple of π/L

The Infinite Potential Well Problem in 1D



$$k = n\frac{\pi}{L} \qquad \left\{ n = 1,2,3.... \right. \qquad E = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2m}\left(n\frac{\pi}{L}\right)^2$$

So the corresponding solutions are:

$$\phi_n(x) = A\sin\left(n\frac{\pi}{L}x\right) \qquad \left\{ n = 1,2,3.... \right.$$

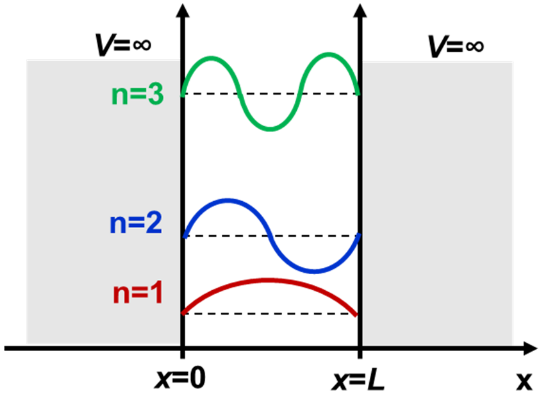
And the corresponding energies are: $E_n = \frac{\hbar^2}{2m}\left(n\frac{\pi}{L}\right)^2 \qquad \left\{ n = 1,2,3.... \right.$

Energy quantization!

The Infinite Potential Well Problem in 1D

$$\phi_n(x) = A\sin\left(n\frac{\pi}{L}x\right) \qquad \left\{n = 1, 2, 3....\right.$$

$$E_n = \frac{\hbar^2}{2m}\left(n\frac{\pi}{L}\right)^2$$



Proper normalization:

We must have:

$$\int_{-\infty}^{\infty} dx\,\left|\phi_n(x)\right|^2 = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} dx\,|A|^2 \sin^2\left(n\frac{\pi}{L}x\right) = 1$$

$$\Rightarrow |A|^2 = \frac{2}{L}$$

$$\Rightarrow |A| = \sqrt{\frac{2}{L}}\,e^{i\phi}$$

Choose the phase ϕ to be 0:

$$\phi_n(x) = \sqrt{\frac{2}{L}}\sin\left(n\frac{\pi}{L}x\right)$$

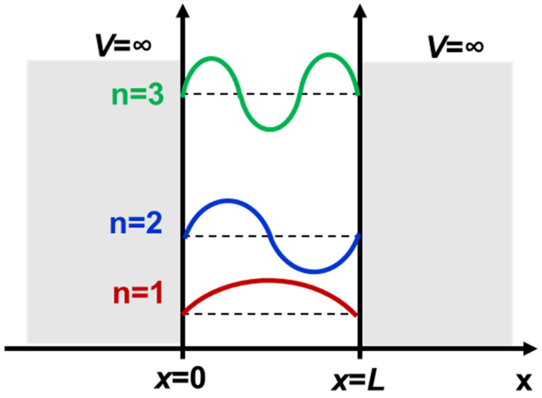
$$E_n = \frac{\hbar^2}{2m}\left(n\frac{\pi}{L}\right)^2$$

$$\left\{n = 1, 2, 3....\right.$$

The Infinite Potential Well Problem in 1D

$$\phi_n(x) = A\sin\left(n\frac{\pi}{L}x\right)$$

$$E_n = \frac{\hbar^2}{2m}\left(n\frac{\pi}{L}\right)^2$$



Orthogonality of the solutions:

Different solutions are orthogonal

$$\int_{-\infty}^{\infty} dx\, \phi_m^*(x)\phi_n(x) = \delta_{nm}$$

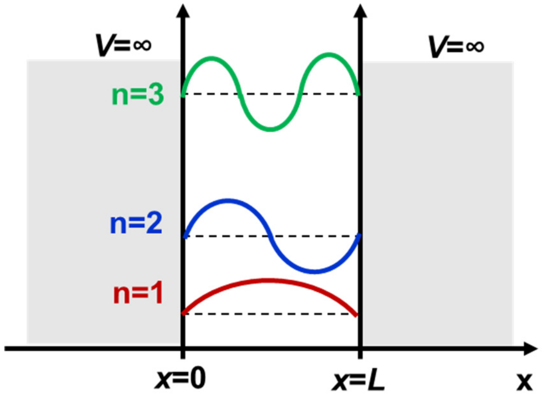
As you will see, this is a very general property of the solutions of the time-independent Schrödinger equation

The Infinite Potential Well Problem in 1D

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(n\frac{\pi}{L}x\right)$$

$$E_n = \frac{\hbar^2}{2m}\left(n\frac{\pi}{L}\right)^2$$

$$\{n = 1,2,3....$$



Since we had assumed the actual time-dependent wavefunction is:

$$\psi(x,t) = \phi(x)e^{-i\frac{E}{\hbar}t}$$

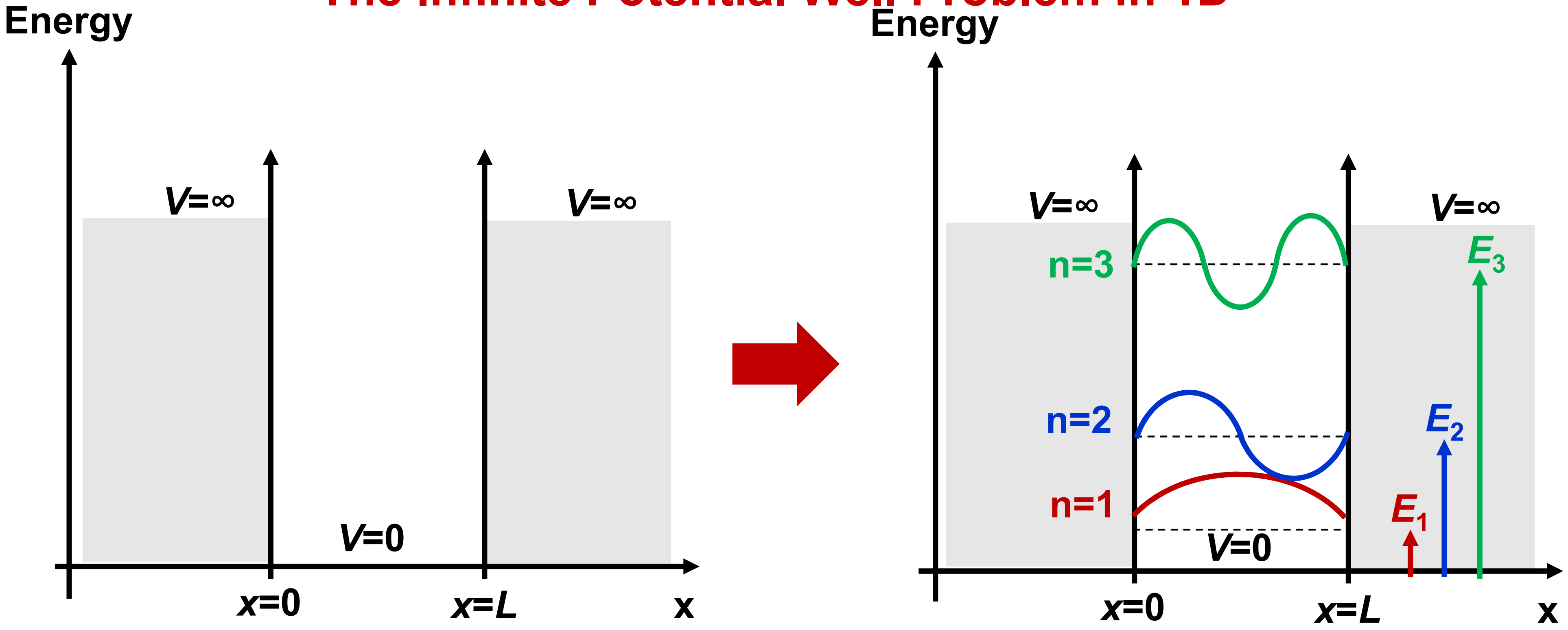
Solutions of the time-dependent Schrödinger equation for the infinite well would look like:

If: $\psi(x,t=0) = \phi_n(x)$ $\{n = 1,2,3....$

$$\Rightarrow \psi(x,t) = \phi_n(x)e^{-i\frac{E_n}{\hbar}t}$$

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t)$$

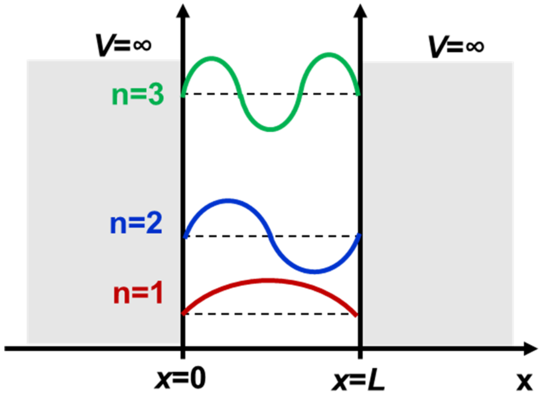
The Infinite Potential Well Problem in 1D



Superposition in Quantum Mechanics

Schrödinger equation is a linear differential equation

This means that a <u>superposition</u> (i.e a sum) of functions that satisfy the equation will also satisfy the equation



For the infinite potential well, since we have found that the following functions satisfy the Schrodinger equation:

$$\phi_n(x)e^{-i\frac{E_n}{\hbar}t} \qquad \{n = 1,2,3....$$

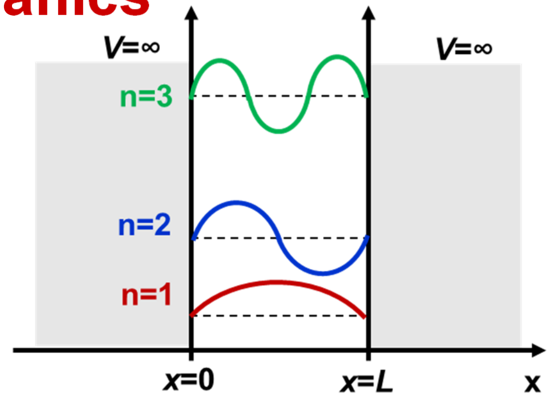
Therefore, a superposition (i.e. a sum) of these functions, weighted by arbitrary complex coefficients, will also satisfy the time-dependent Schrodinger equation:

$$\psi(x,t) = \sum_n \underset{\uparrow}{a_n}\, \phi_n(x)e^{-i\frac{E_n}{\hbar}t}$$

Complex coefficients

$$\left\{ i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t) \right.$$

Superposition in Quantum Mechanics



$$\psi(x,t)=\sum_n a_n\,\phi_n(x)\,e^{-i\frac{E_n}{\hbar}t}$$

Complex coefficients

- **This solution does not have a fixed precise energy.**
- **It is a <u>superposition</u> of states each of which has a fixed precise energy.**

$$\left\{ i\hbar\frac{\partial\psi(x,t)}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2}+V(x)\psi(x,t)\right.$$

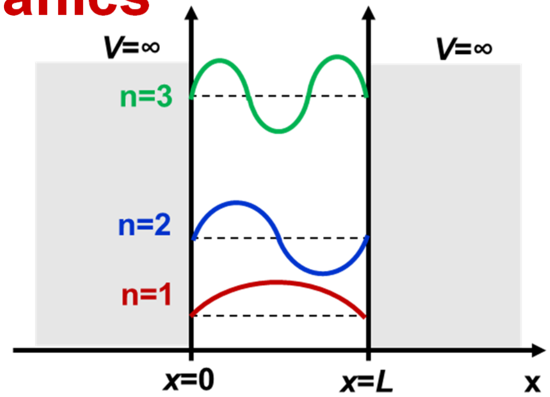
Normalization:

$$\int_{-\infty}^{\infty}dx\,\psi^*(x,t)\psi(x,t)=1$$

$$\Rightarrow\sum_n\sum_m a_n^*a_m\int_{-\infty}^{\infty}dx\,\phi_n^*(x)e^{i\frac{E_n}{\hbar}t}\left[\phi_m(x)e^{-i\frac{E_m}{\hbar}t}\right]=\sum_n\sum_m a_n^*a_m e^{i\frac{E_n}{\hbar}t}e^{-i\frac{E_m}{\hbar}t}\delta_{nm}=1$$

$$\Rightarrow\sum_n\left|a_n\right|^2=1$$

Superposition in Quantum Mechanics



$$\psi(x,t) = \sum_n a_n \, \phi_n(x) \, e^{-i\frac{E_n}{\hbar}t}$$

Complex coefficients

- This solution does not have a fixed precise energy.
- It is a <u>superposition</u> of states each of which has a fixed precise energy.

$$\left\{ i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t) \right.$$

Mean energy:

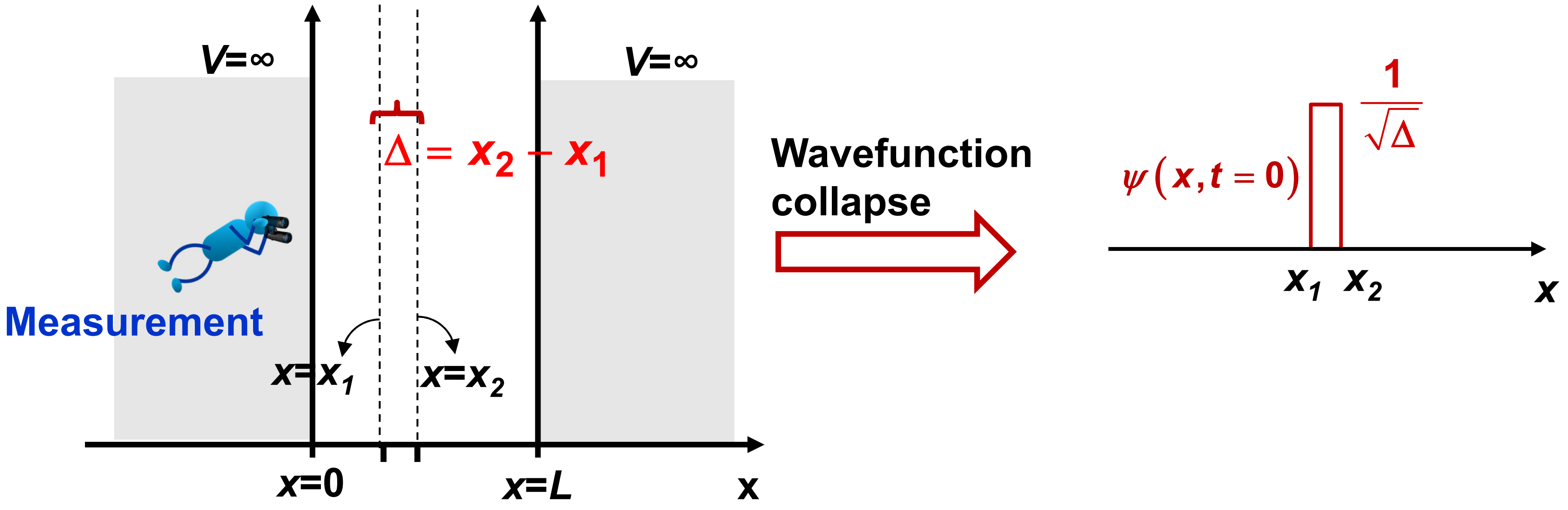
$$\langle E\rangle(t) = \int_{-\infty}^{\infty} dx \, \psi^*(x,t)\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\psi(x,t)$$

$$= \int_{-\infty}^{\infty} dx \, \psi^*(x,t)\left[i\hbar\frac{\partial}{\partial t}\right]\psi(x,t) = \sum_m \sum_n a_n^* a_m \int_{-\infty}^{\infty} dx \, \phi_n^*(x) e^{i\frac{E_n}{\hbar}t}\left[E_m\phi_m(x) e^{-i\frac{E_m}{\hbar}t}\right]$$

$$= \sum_n |a_n|^2 E_n$$

The Infinite Potential Well Problem in 1D: A Post Measurement Time Evolution Problem

Suppose a particle is known to be in a infinite potential well box



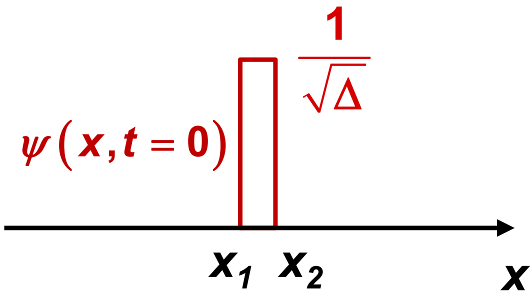
A time t=0, a measurement is made to locate the particle

The measurement is not very precise and post-measurement the particle is known to be somewhere inside the dashed region of thickness Δ

So immediately after the measurement its wavefunction can be taken to be roughly:

$$\psi\left(x,t=0^{+}\right)=\begin{cases}\dfrac{1}{\sqrt{\Delta}} & x_1 \le x \le x_2 \\ 0 & \text{otherwise}\end{cases} \quad \Longrightarrow \quad \int_{-\infty}^{\infty} dx\,\left|\psi\left(x,t=0\right)\right|^2 = 1$$

The Infinite Potential Well Problem in 1D: A Post Measurement Time Evolution Problem



Question: What is the particle wavefunction for time $t>0$??

We try a superposition solution to match the initial condition:

$$\psi(x,t) = \sum_n a_n \phi_n(x) e^{-i\frac{E_n}{\hbar}t}$$

We need to find this coefficient only!

This implies:

$$\psi(x,t=0) = \sum_n a_n \phi_n(x)$$

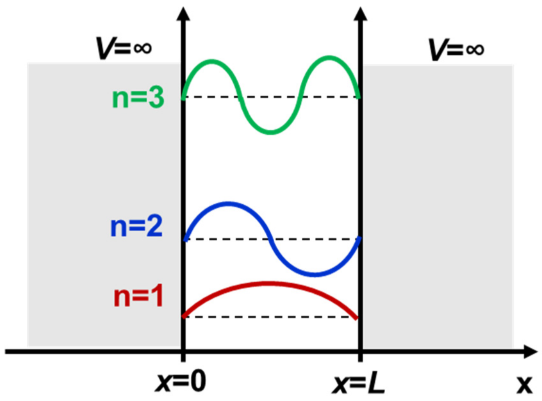
Multiply both sides by $\phi_m^*(x)$ and integrate:

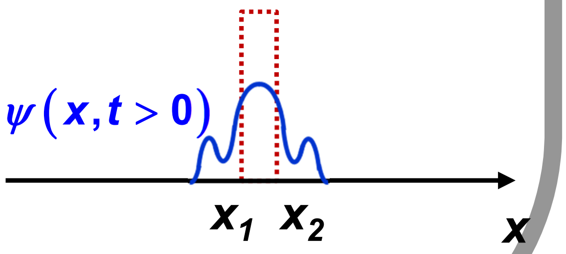
$$\int_{-\infty}^{\infty} dx\, \phi_m^*(x)\psi(x,t=0) = \sum_n a_n \int_{-\infty}^{\infty} dx\, \phi_m^*(x)\phi_n(x)$$

$$\Rightarrow \int_{-\infty}^{\infty} dx\, \phi_m^*(x)\psi(x,t=0) = \sum_n a_n \delta_{mn}$$

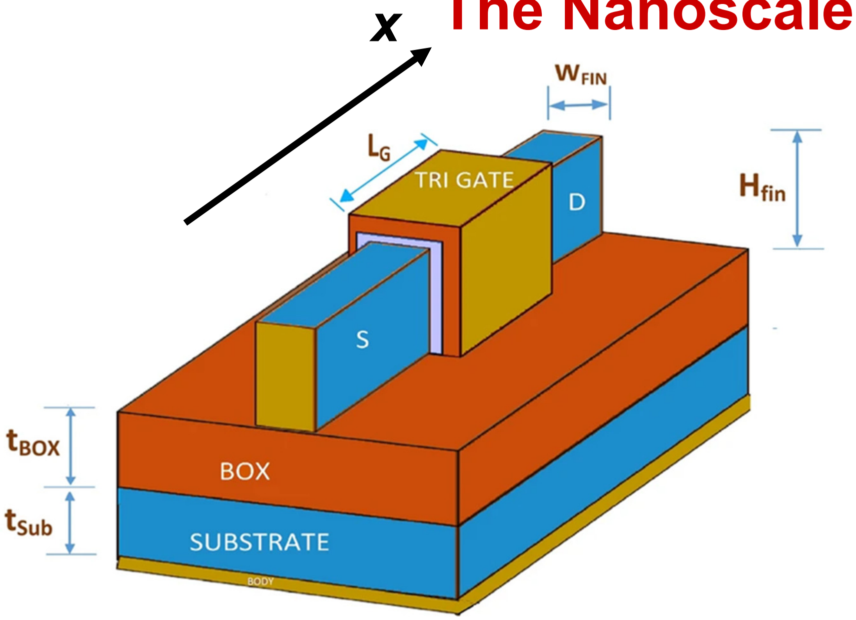
$$\Rightarrow \int_{-\infty}^{\infty} dx\, \phi_m^*(x)\psi(x,t=0) = a_m$$

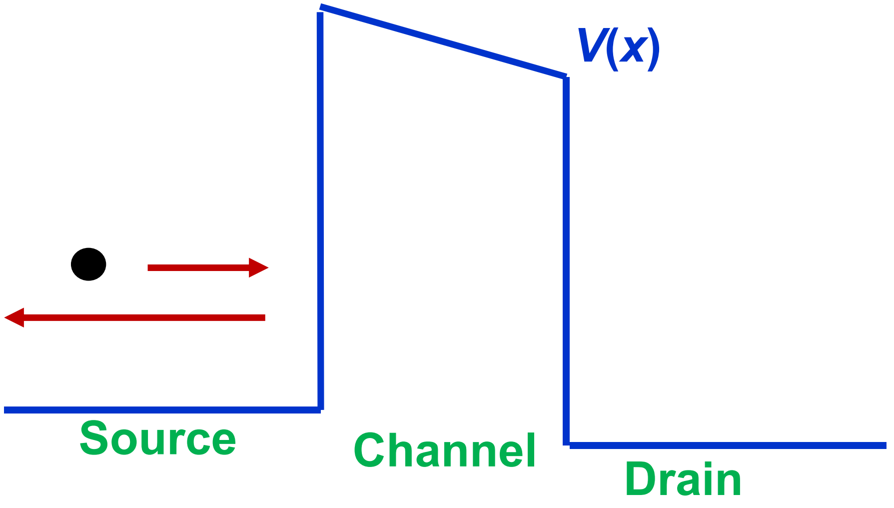
$$\Rightarrow a_m = \frac{1}{\sqrt{\Delta}} \int_{x_1}^{x_2} dx\, \phi_m^*(x)$$





The Nanoscale FET: Classical Picture

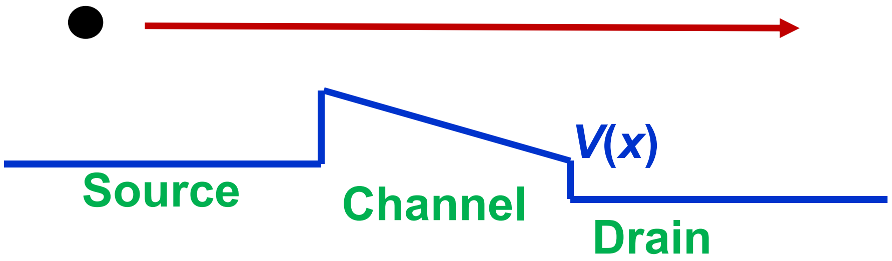




Potential energy (transistor off)

<u>No source-drain electron current</u>

In a FinFET, the potential energy profile seen by the electron can be changed by the application of a gate voltage

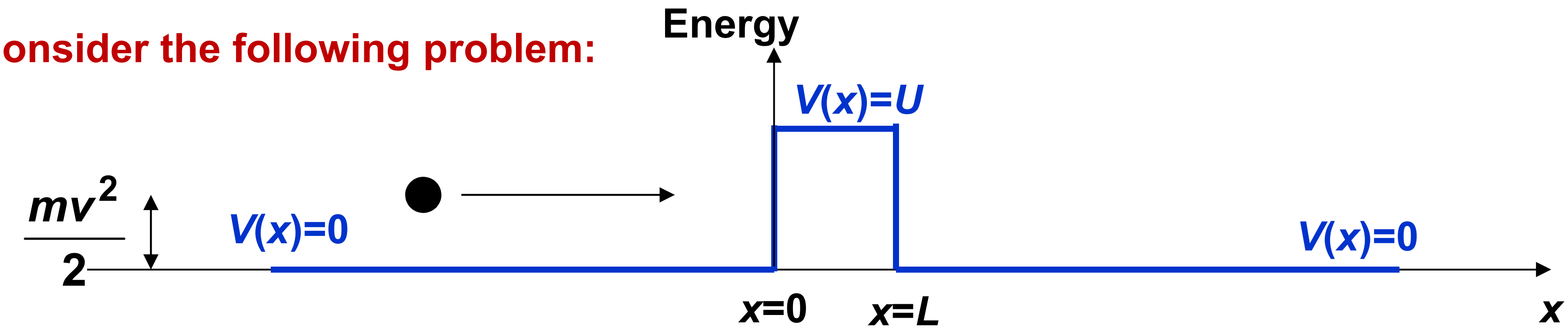


Potential energy (transistor on)

<u>Source-drain electron current</u>

Barrier Tunneling in Quantum Physics

Consider the following problem:



Classical physics:

A particle with KE equal to $mv^2/2$ is coming from the left towards a region where the potential is U over a distance L

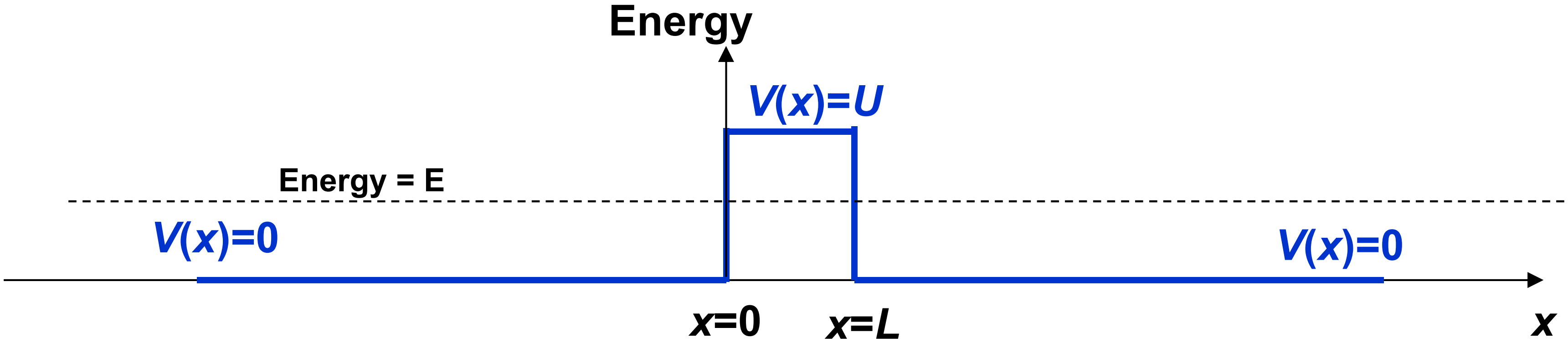
The KE of the particle is less than U: $E = \frac{mv^2}{2} < U$

The particle doesn't have enough KE to go through the potential hill, and will therefore bounce back. The potential hill is also called a potential barrier.

Quantum physics:

There is a chance that the particle will go through, or tunnel through, the potential barrier and there is a chance that the particle with bounce back!

Barrier Tunneling in Quantum Physics



We seek solutions of the form:

$$\psi(x,t)=\phi(x)e^{-i\frac{E}{\hbar}t} \Longrightarrow -\frac{\hbar^2}{2m}\frac{\partial^2\phi(x)}{\partial x^2}+V(x)\phi(x)=E\phi(x)$$

Region I:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi(x)}{\partial x^2}=E\phi(x)$$

Region II:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi(x)}{\partial x^2}+U\phi(x)=E\phi(x)$$

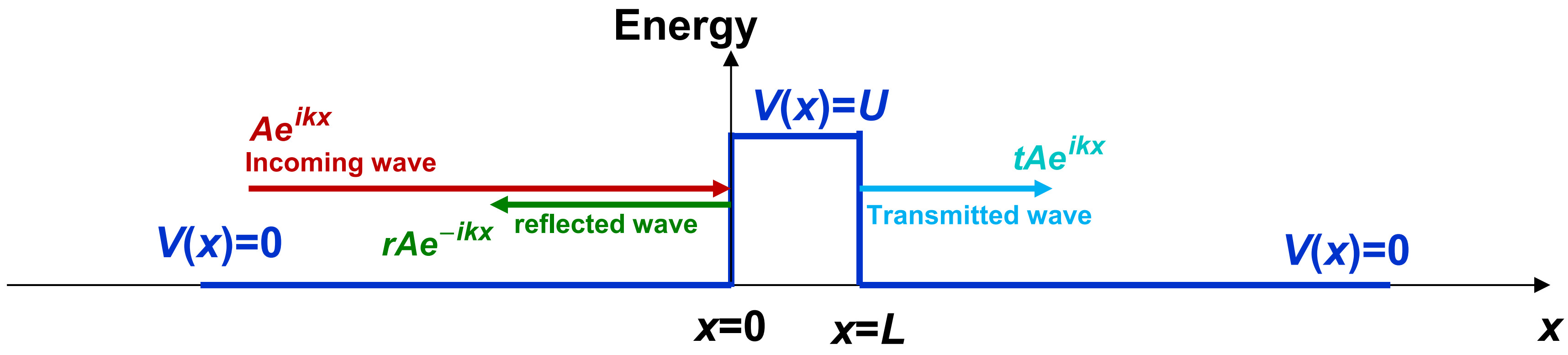
Region III:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi(x)}{\partial x^2}=E\phi(x)$$

All same *E*

$E < U$

Barrier Tunneling in Quantum Physics: Solutions



Region I:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi(x)}{\partial x^2}=E\phi(x)$$

$$\phi(x)=Ae^{ikx}+rAe^{-ikx} \longrightarrow$$

Reflection amplitude

Superposition of two waves

Each wave has energy: $E=\frac{\hbar^2k^2}{2m}<U$

Incoming wave has momentum: $+\hbar k$

Reflected wave has momentum: $-\hbar k$

Region III:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi(x)}{\partial x^2}=E\phi(x)$$

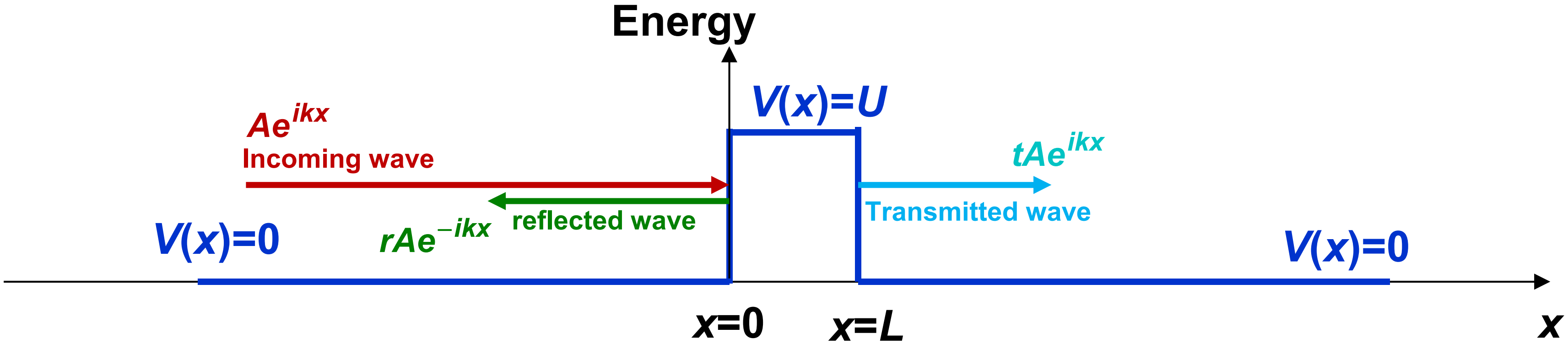
$$\phi(x)=tAe^{ikx} \longrightarrow$$

Transmission amplitude

Wave has energy: $E=\frac{\hbar^2k^2}{2m}<U$

Transmitted wave has momentum: $+\hbar k$

Barrier Tunneling in Quantum Physics: Classically Forbidden Region



Region II (this is a classically forbidden region): $$-\frac{\hbar^2}{2m}\frac{\partial^2\phi(x)}{\partial x^2}+U\phi(x)=E\phi(x)$$

If one assumes a solution which is a plane wave: $\phi(x)\propto e^{\pm iqx}$

Then one gets: $$\frac{\hbar^2q^2}{2m}+U=E$$

But we also know that the energy $E=\dfrac{\hbar^2k^2}{2m}<U$

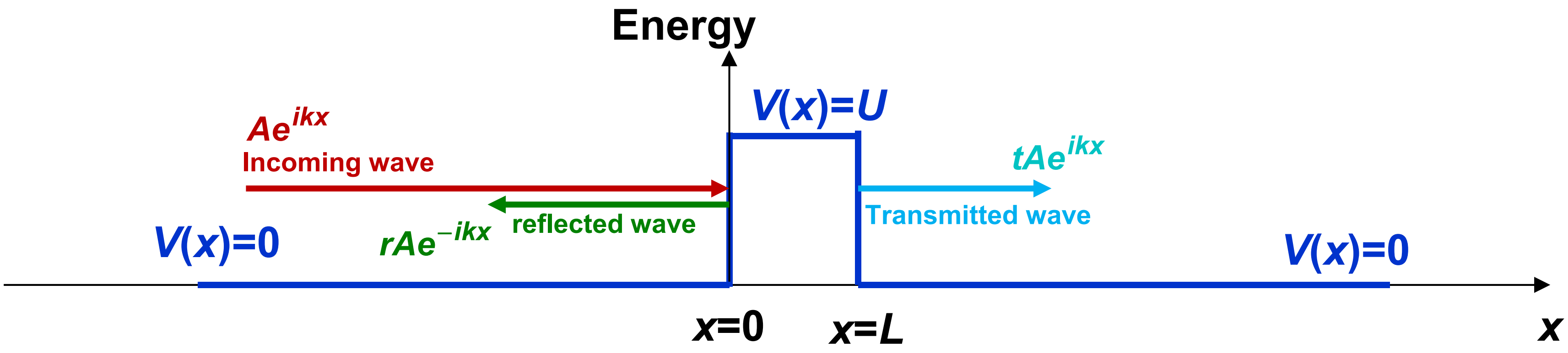
Therefore: $$\frac{\hbar^2q^2}{2m}=-(U-E) \longrightarrow$$ ***q* has to be imaginary!!**

$$q=\pm i\sqrt{\frac{2m}{\hbar^2}(U-E)}=\pm i\gamma$$

So the solution form is: $\phi(x)\propto Be^{-\gamma x}+Ce^{+\gamma x}$

Decaying and growing exponentials!

Barrier Tunneling in Quantum Physics: Solutions



Region I: $\phi(x) = Ae^{ikx} + rAe^{-ikx}$

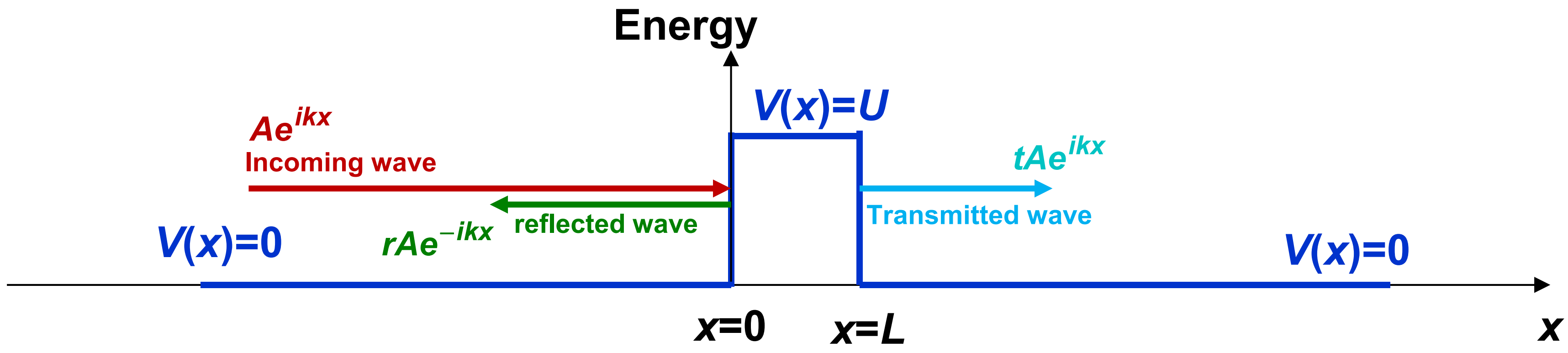
Region II (this is a classically forbidden region): $\phi(x) \propto Be^{-\gamma x} + Ce^{+\gamma x}$

Region III: $\phi(x) = tAe^{ikx}$

Now we need to stitch together these solutions at the boundaries

We have 4 unknown coefficients, which means we need four boundary conditions!

Barrier Tunneling in Quantum Physics: Boundary Conditions



Boundary conditions:

The wavefunction $\phi(x)$ and its derivative $\dfrac{\partial \phi(x)}{\partial x}$ both must be continuous at all boundaries

Ok – but why??

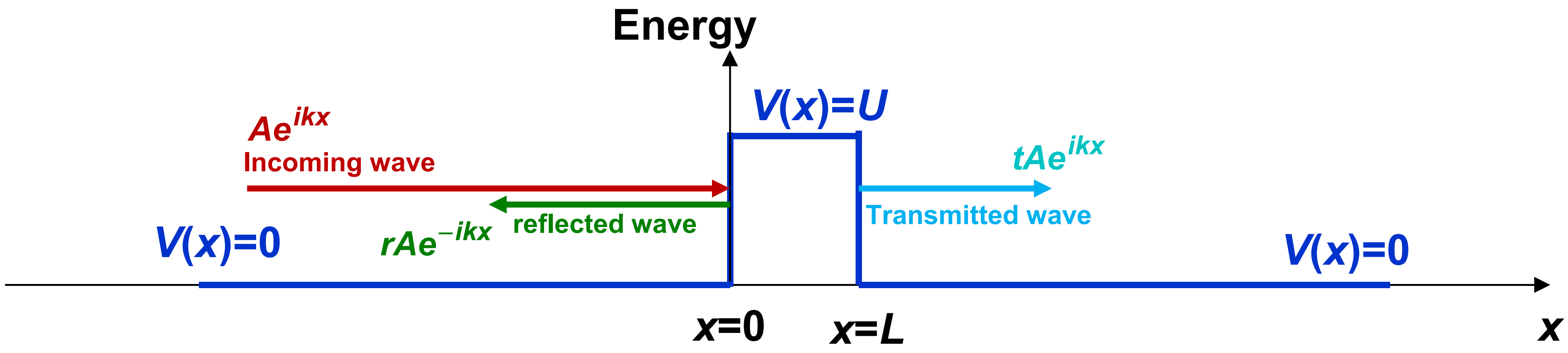
Recall that:

$$\langle p \rangle = \int_{-\infty}^{\infty} dx\ \ \phi^*(x)\left[\frac{\hbar}{i}\frac{\partial}{\partial x}\right]\phi(x)$$

$$\langle E \rangle = \int_{-\infty}^{\infty} dx\ \ \phi^*(x)\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x)\right]\phi(x)$$

$$\left.\begin{array}{l}\text{A discontinuity in } \phi(x) \text{ somewhere would imply an infinite momentum} \\ \text{A discontinuity in } \dfrac{\partial \phi(x)}{\partial x} \text{ somewhere would imply an infinite energy}\end{array}\right\} \text{Both are unphysical}$$

Barrier Tunneling in Quantum Physics: Solutions



Region I: $\phi(x) = Ae^{ikx} + rAe^{-ikx}$

Region II (this is a classically forbidden region): $\phi(x) \propto Be^{-\gamma x} + Ce^{+\gamma x}$

Region III: $\phi(x) = tAe^{ikx}$

Apply all the boundary conditions:

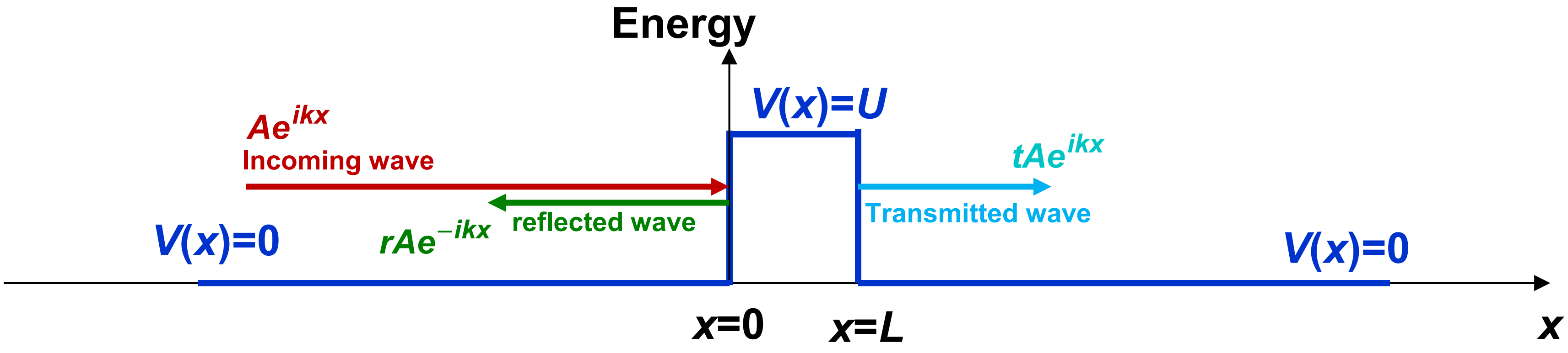
$$A(1+r) = B + C$$

$$ikA(1-r) = -\gamma(B-C)$$

$$Be^{-\gamma L} + Ce^{+\gamma L} = tAe^{ikL}$$

$$-\gamma\left(Be^{-\gamma L} - Ce^{+\gamma L}\right) = iktAe^{ikL}$$

Barrier Tunneling in Quantum Physics: Solutions



Region I: $\phi(x) = Ae^{ikx} + rAe^{-ikx}$

Region II (this is a classically forbidden region): $\phi(x) \propto Be^{-\gamma x} + Ce^{+\gamma x}$

Region III: $\phi(x) = tAe^{ikx}$

We get:

$$t = \frac{4ik\gamma e^{-ikL}}{(\gamma + ik)^2 e^{-\gamma L} - (\gamma - ik)^2 e^{+\gamma L}}$$

$$r = \frac{2\left(\gamma^2 - (ik)^2\right)\sinh(\gamma L)}{(\gamma + ik)^2 e^{-\gamma L} - (\gamma - ik)^2 e^{+\gamma L}}$$

As $L \to 0$:

$$t \to 1$$

$$r \to 0$$

As expected!

As $L >> 1/\gamma$:

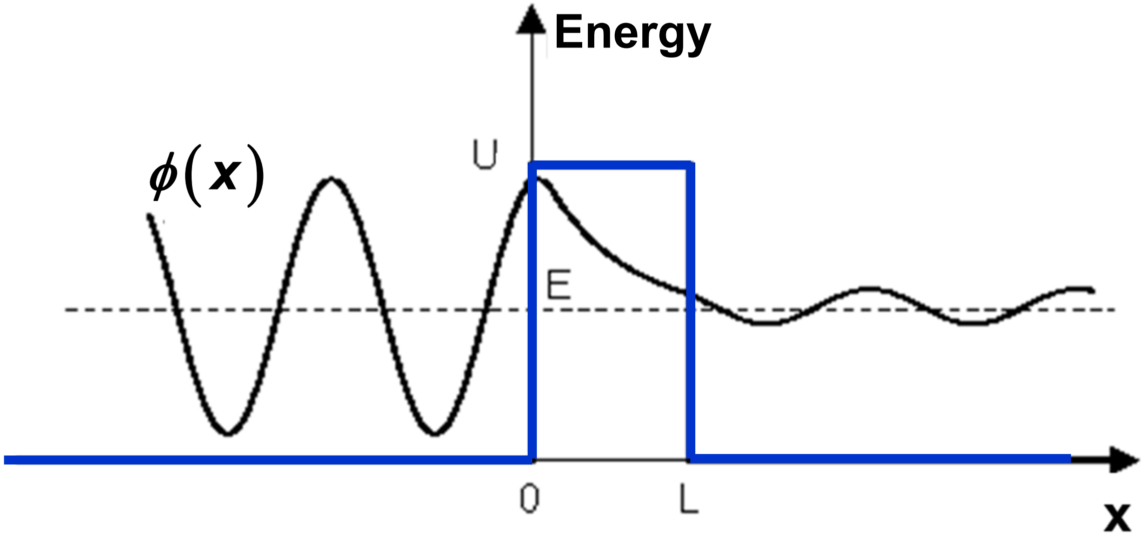
$$t \to -\frac{4ik\gamma e^{-ikL}}{(\gamma - ik)^2} e^{-\gamma L}$$

Exponentially small, but non-zero transmission amplitude

$$r \to -\frac{\gamma + ik}{\gamma - ik}$$

Note that $|r| \to 1$

Barrier Tunneling in Quantum Physics: Solutions



As $L >> 1/\gamma$:

$$t \rightarrow -\frac{4ik\gamma e^{-ikL}}{(\gamma - ik)^2} e^{-\gamma L}$$

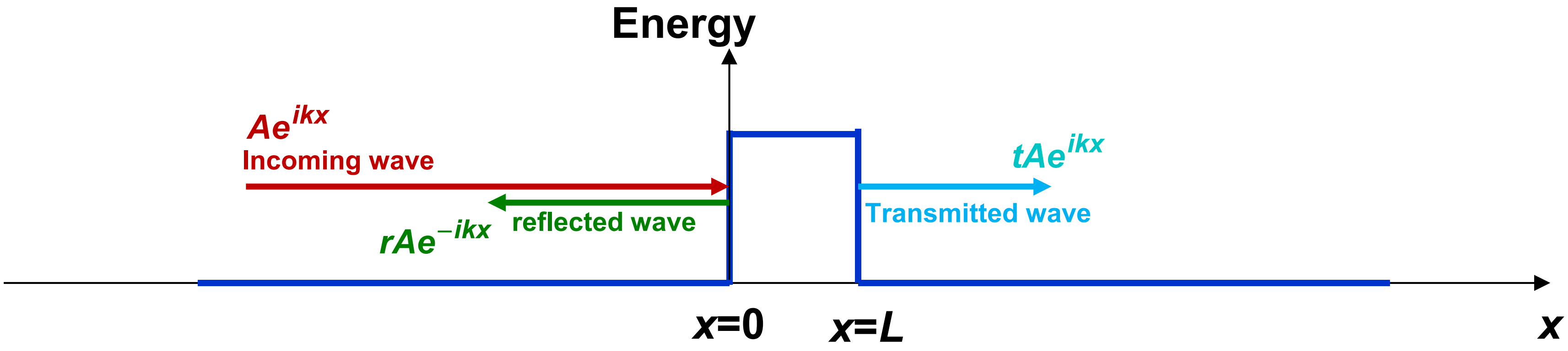
Exponentially small, but non-zero transmission amplitude

$$r \rightarrow -\frac{\gamma + ik}{\gamma - ik}$$

Note that $|r| \rightarrow 1$

$$t \sim e^{-\gamma L} = e^{-\sqrt{\frac{2m}{\hbar^2}(U-E)}\,L}$$

Barrier Tunneling in Quantum Physics: Probability Current



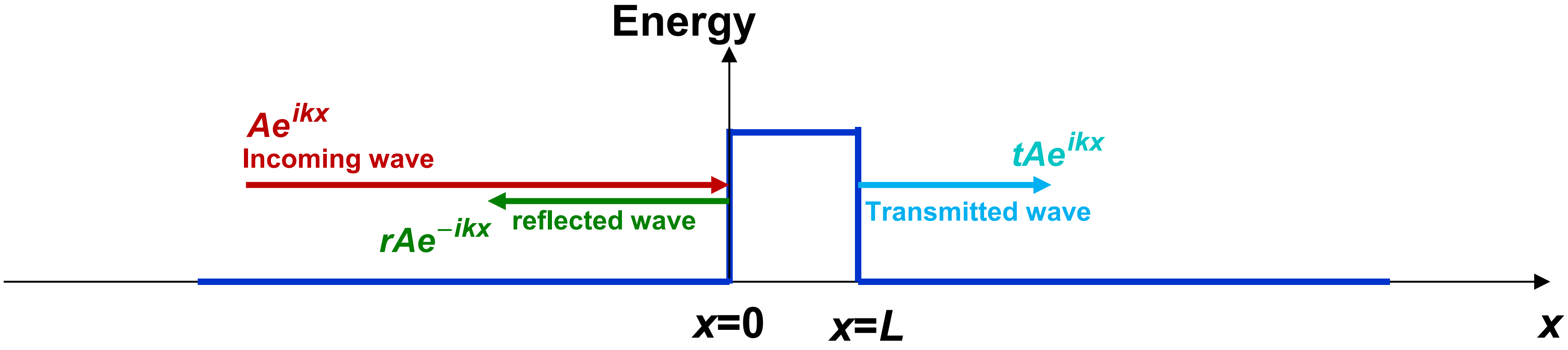
Region I: $\phi(x) = Ae^{ikx} + rAe^{-ikx}$

$$I(x,t)=\psi^*(x,t)\frac{\hbar}{2im}\frac{\partial\psi(\vec{r},t)}{\partial x}-\psi(x,t)\frac{\hbar}{2im}\frac{\partial\psi^*(\vec{r},t)}{\partial x}=\underbrace{|A|^2\frac{\hbar k}{m}}_{\text{Incoming prob. current}}-\underbrace{|A|^2\frac{\hbar k}{m}|r|^2}_{\text{Reflected prob. current}}$$

Region II: $\phi(x) = tAe^{-ikx}$

$$I(x,t)=\psi^*(x,t)\frac{\hbar}{2im}\frac{\partial\psi(\vec{r},t)}{\partial x}-\psi(x,t)\frac{\hbar}{2im}\frac{\partial\psi^*(\vec{r},t)}{\partial x}=\underbrace{|A|^2\frac{\hbar k}{m}|t|^2}_{\text{Transmitted prob. current}}$$

Barrier Tunneling in Quantum Physics: Probability Current



Conservation of probability: **Incoming probability current (coming towards the potential barrier) must equal the outgoing probability current (going away from the potential barrier)**

This implies:

Incoming probability current = reflected probability current
+ transmitted probability current

$$|A|^2 \frac{\hbar k}{m} = |A|^2 \frac{\hbar k}{m} |r|^2 + |A|^2 \frac{\hbar k}{m} |t|^2$$

$$\Rightarrow 1 = |r|^2 + |t|^2$$

Barrier Tunneling in Quantum Physics: Probability Current