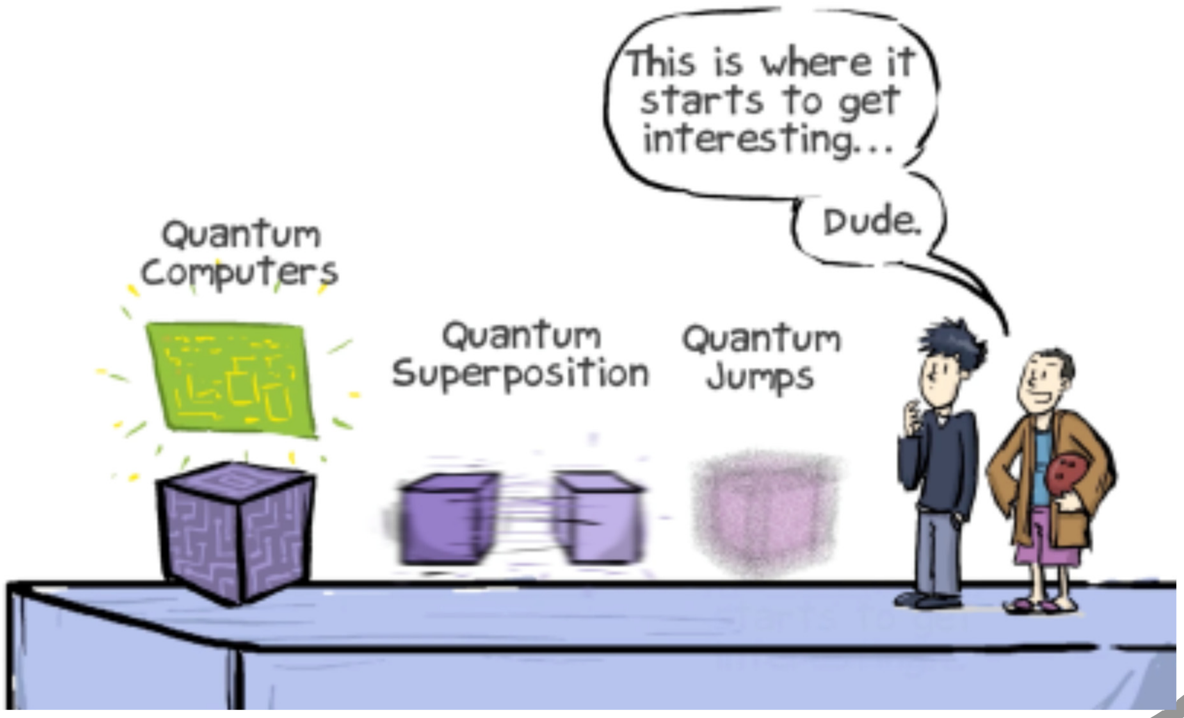
# Lecture 5

# Mean or Expectation Values of Observables in Quantum Mechanics

**In this lecture you will learn:**

• **How to obtain mean values of various physical quantities (observables) from wavefunctions**



# The Schrödinger Equation: Recap

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x)\psi(x,t)$$

When the potential is zero everywhere in space:

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x,t)}{\partial x^2}$$

The solution is a plane wave:

$$\psi(x,t) = A\, e^{ikx} e^{-i\frac{E}{\hbar}t}$$

Energy of the particle: $E(k) = \dfrac{\hbar^2 k^2}{2m}$

Imagine the particle is in a very large box of length *L*:

$$\int_{-\infty}^{\infty} dx\, |\psi(x,t)|^2 = \int_{0}^{L} dx\, |\psi(x,t)|^2 = |A|^2 L = 1 \quad \Rightarrow \quad |A| = \frac{1}{\sqrt{L}}$$

Quantum wavefunction is defined only up to an overall phase factor:

$$A = \frac{e^{i\phi}}{\sqrt{L}}$$

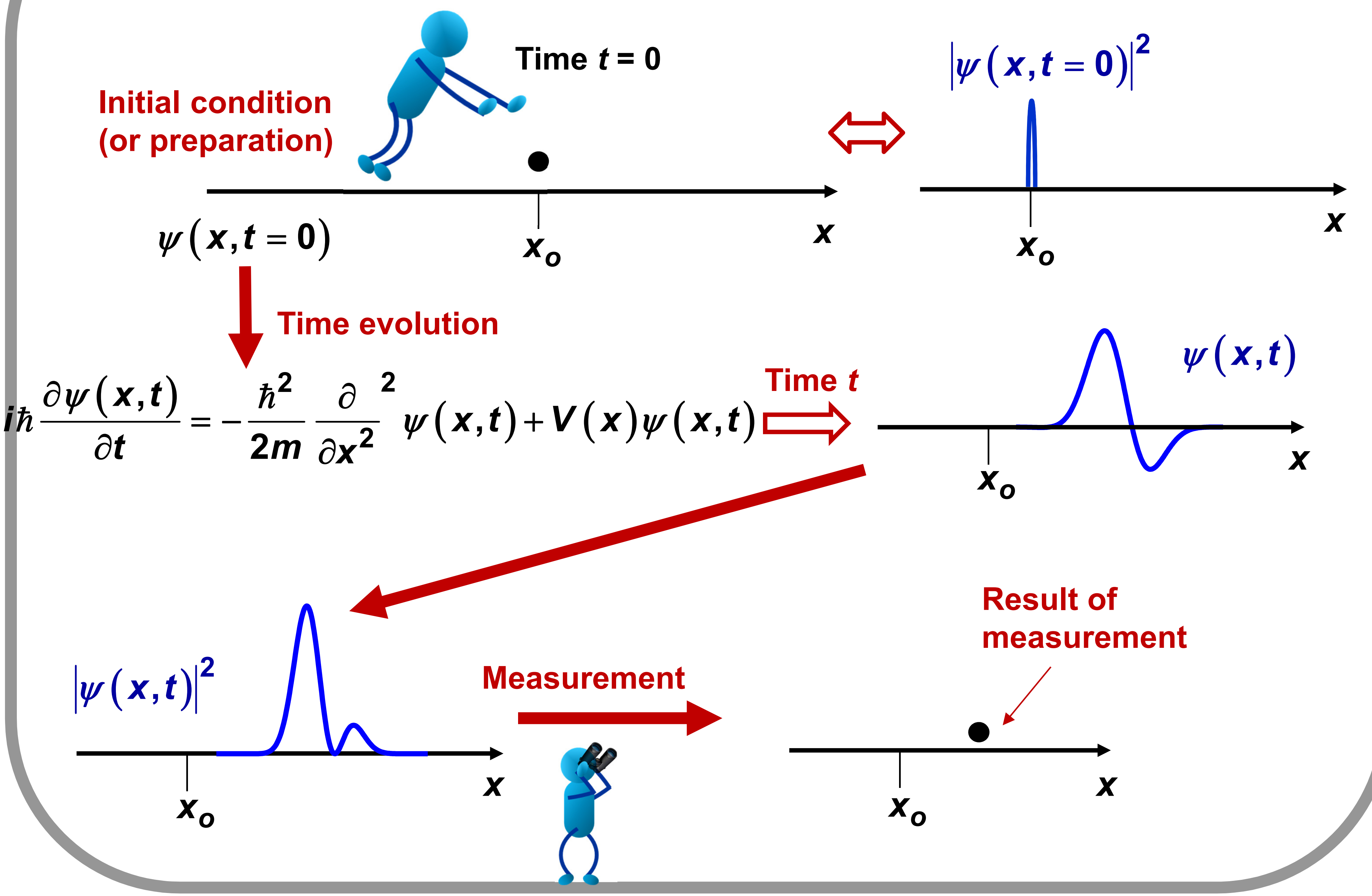
# Observables in Quantum Physics

An observable in quantum physics is any physical quantity that can be measured

Examples of observables associated with a particle include its position, its momentum, its energy, its angular momentum, its spin, etc

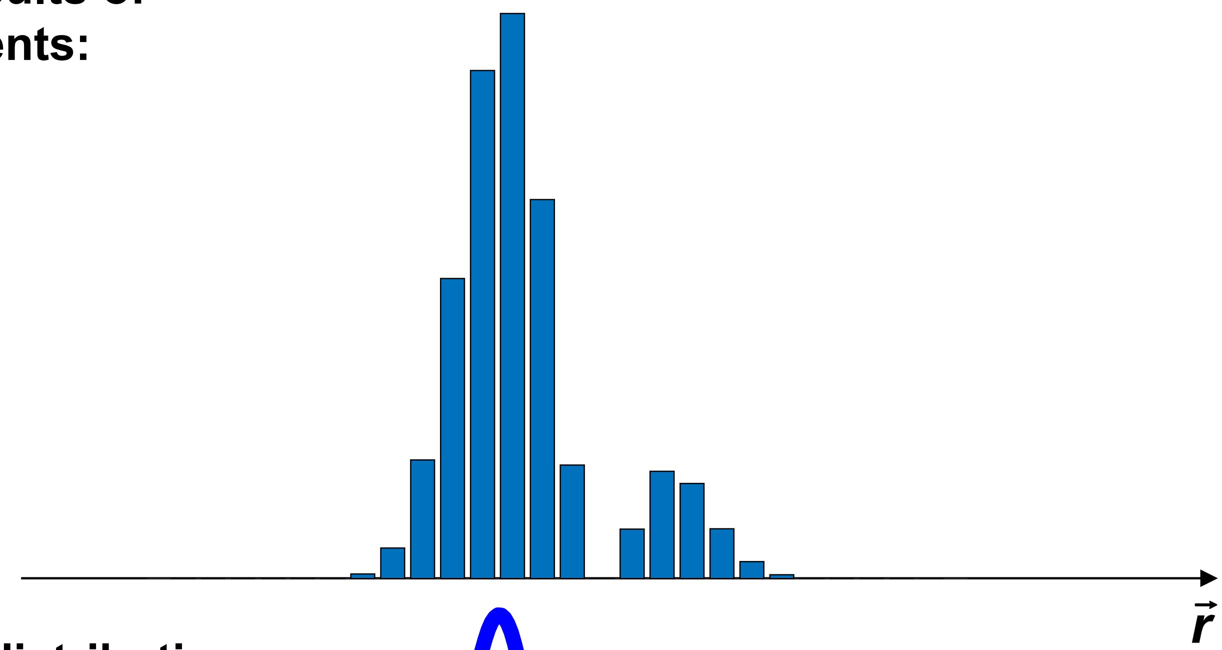
Question: How do we obtain the mean values of various physical quantities (observables) from the wavefunction?

# The Statistical Interpretation of Quantum Probabilities

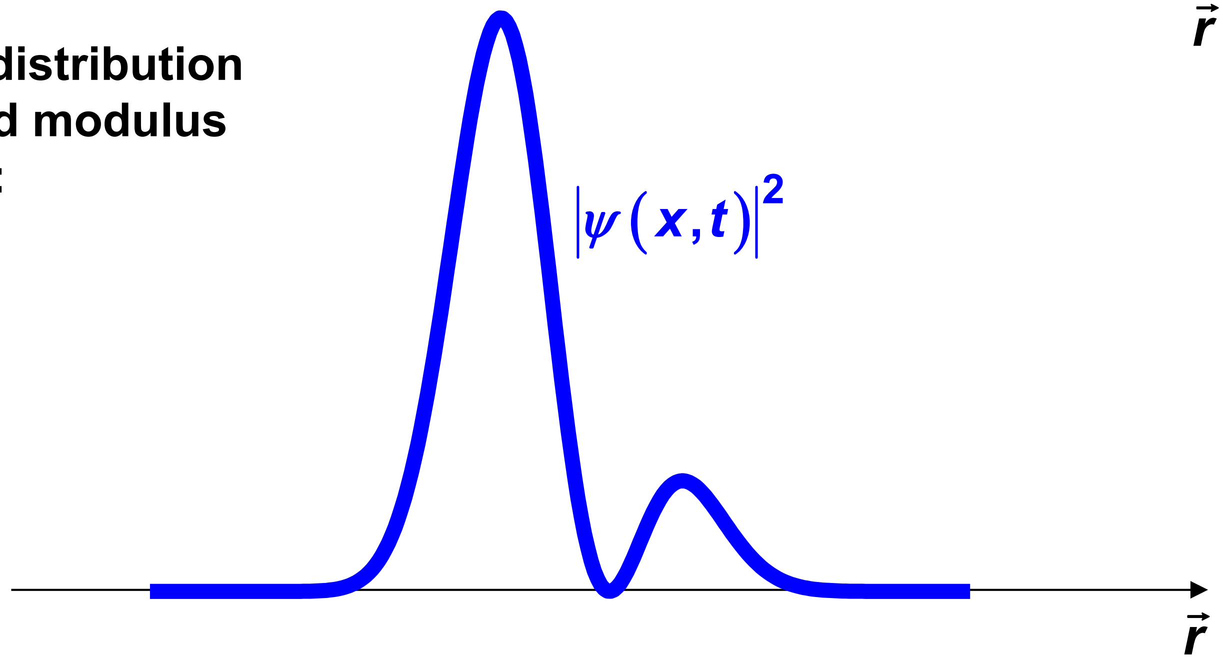


# The Statistical Interpretation of Quantum Probabilities

**Histogram of the results of position measurements:**



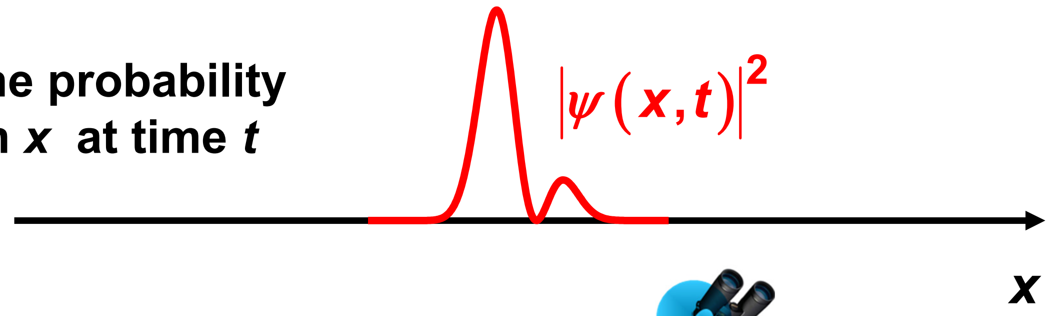

**A-priori probability distribution given by the squared modulus of the wavefunction:**



# Mean Value of Position or Position Expectation Value

Since the a-priori probability (or rather the probability density) of finding the particle at location $x$ at time $t$ is:

$$|\psi(x,t)|^2$$





The average position of the particle at time $t$ is then:

$$\langle x\rangle(t) = \int_{-\infty}^{\infty} dx \;\; x|\psi(x,t)|^2 = \int_{-\infty}^{\infty} dx \;\; \psi^*(x,t)\underbrace{[x]}_{\text{Position operator}}\psi(x,t)$$

This is also called the expectation value of the position, or the mean value of the position, or the average value of the position

**Meaning:** If the experiment is repeated many times under identical conditions, and each time a measurement is made at time $t$ to locate the particle, then the mean location is determined experimentally by taking the average of the measurement results obtained in all these experiments and this experimental mean would equal the expectation value or the mean value given above by the quantum wavefunction

# Mean Value of Momentum or Momentum Expectation Value

We know that in classical physics:

$$p(t) = mv(t) = m\frac{dx(t)}{dt}$$

So we expect the momentum expectation value in quantum mechanics to go as:

$$\langle p\rangle(t) = m\frac{d}{dt}\langle x\rangle(t)$$

$$\begin{cases} i\hbar\dfrac{\partial\psi(x,t)}{\partial t} = -\dfrac{\hbar^2}{2m}\dfrac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t) \\[2ex] -i\hbar\dfrac{\partial\psi^*(x,t)}{\partial t} = -\dfrac{\hbar^2}{2m}\dfrac{\partial^2\psi^*(x,t)}{\partial x^2} + V(x)\psi^*(x,t) \end{cases}$$

We start from:

$$\langle x\rangle(t) = \int_{-\infty}^{\infty} dx\ \ \psi^*(x,t)[x]\psi(x,t)$$

$$\Rightarrow m\frac{d}{dt}\langle x\rangle(t) = \int_{-\infty}^{\infty} dx\ \ m\frac{\partial\psi^*(x,t)}{\partial t}[x]\psi(x,t) + \int_{-\infty}^{\infty} dx\ \ \psi^*(x,t)[x]m\frac{\partial\psi(x,t)}{\partial t}$$

use

use

# Mean Value of Momentum or Momentum Expectation Value

$$\langle x\rangle(t)=\int_{-\infty}^{\infty}dx\;\;\psi^*(x,t)[x]\psi(x,t)$$

$$\Rightarrow m\frac{d}{dt}\langle x\rangle(t)=\int_{-\infty}^{\infty}dx\;\;m\frac{\partial\psi^*(x,t)}{\partial t}[x]\psi(x,t)+\int_{-\infty}^{\infty}dx\;\;\psi^*(x,t)[x]m\frac{\partial\psi(x,t)}{\partial t}$$

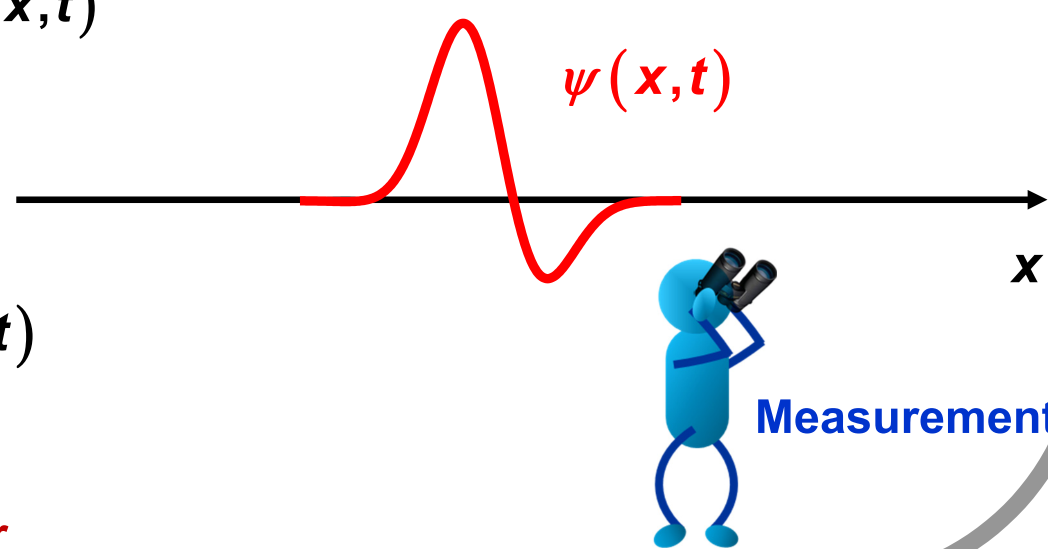
$$=\int_{-\infty}^{\infty}dx\;\;\left(\frac{\hbar}{2i}\right)\frac{\partial^2\psi^*(x,t)}{\partial x^2}[x]\psi(x,t)+\int_{-\infty}^{\infty}dx\;\;\psi^*(x,t)[x]\left(-\frac{\hbar}{2i}\right)\frac{\partial^2\psi(x,t)}{\partial x^2}$$

Integrate by parts both terms

$$=\int_{-\infty}^{\infty}dx\;\;\left(-\frac{\hbar}{2i}\right)\frac{\partial\psi^*(x,t)}{\partial x}\psi(x,t)+\int_{-\infty}^{\infty}dx\;\;\psi^*(x,t)\left(\frac{\hbar}{2i}\right)\frac{\partial\psi(x,t)}{\partial x}$$

Integrate by parts first term

$$=\int_{-\infty}^{\infty}dx\;\;\psi^*(x,t)\left[\frac{\hbar}{i}\frac{\partial}{\partial x}\right]\psi(x,t)$$




**The final answer becomes:**

$$\langle p\rangle(t)=\int_{-\infty}^{\infty}dx\;\;\psi^*(x,t)\underbrace{\left[\frac{\hbar}{i}\frac{\partial}{\partial x}\right]}_{\text{Momentum operator}}\psi(x,t)$$

Momentum operator

# Mean Value of Kinetic Energy (KE) or KE Expectation Value

We know that in classical physics:

$$\text{KE} = \frac{p^2(t)}{2m}$$

So we expect the KE expectation value in quantum mechanics to go as:

$$\langle \text{KE} \rangle (t) = \frac{\langle p^2 \rangle (t)}{2m}$$

Since:

$$\langle p \rangle (t) = \int_{-\infty}^{\infty} dx \;\; \psi^*(x,t) \left[ \frac{\hbar}{i} \frac{\partial}{\partial x} \right] \psi(x,t)$$

We conclude that:

$$\langle \text{KE} \rangle (t) = \frac{\langle p^2 \rangle (t)}{2m} = \frac{1}{2m} \int_{-\infty}^{\infty} dx \;\; \psi^*(x,t) \left[ \frac{\hbar}{i} \frac{\partial}{\partial x} \frac{\hbar}{i} \frac{\partial}{\partial x} \right] \psi(x,t)$$

$$= \int_{-\infty}^{\infty} dx \;\; \psi^*(x,t) \underbrace{\left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right]}_{\text{KE operator}} \psi(x,t)$$

# Mean Value of Potential Energy (PE) or PE Expectation Value

**Since the a-priori probability of finding the particle at location *x* at time *t* is:**

$$\left|\psi(x,t)\right|^2$$

**And the average position of the particle at time *t* is:**

$$\langle x \rangle(t) = \int_{-\infty}^{\infty} dx \;\; \psi^*(x,t)\left[x\right]\psi(x,t)$$

**We expect:**

$$\langle V(x) \rangle(t) = \int_{-\infty}^{\infty} dx \;\; \psi^*(x,t)\underbrace{\left[V(x)\right]}_{\text{PE operator}}\psi(x,t)$$

**PE operator**

# Mean Value of Energy or Energy Expectation Value

**The total energy of a particle is the sum of its kinetic and potential energies**

**So we expect that in quantum mechanics the mean value of the total energy will be:**

$$\langle E\rangle(t) = \langle \mathrm{KE}\rangle(t) + \langle \mathrm{PE}\rangle(t) = \frac{\langle p^2\rangle(t)}{2m} + \langle V(x)\rangle(t)$$

$$= \int_{-\infty}^{\infty} dx \;\; \psi^*(x,t)\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\psi(x,t)$$

Total energy operator

**Another way of writing the Schrodinger equation:**

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t)$$

$$\Rightarrow i\hbar\frac{\partial\psi(x,t)}{\partial t} = \begin{bmatrix}\text{Energy}\\ \text{operator}\end{bmatrix}\psi(x,t)$$

# Mean Values or Expectation Values of Observables in 1D

**Position:**

$$\langle x\rangle(t)=\int_{-\infty}^{\infty}dx\;\;\psi^*(x,t)\left[x\right]\psi(x,t)$$

**Potential Energy:**

$$\langle \mathrm{PE}\rangle(t)=\langle V(x)\rangle(t)=\int_{-\infty}^{\infty}dx\;\;\psi^*(x,t)\left[V(x)\right]\psi(x,t)$$

**Momentum:**

$$\langle p\rangle(t)=\int_{-\infty}^{\infty}dx\;\;\psi^*(x,t)\left[\frac{\hbar}{i}\frac{\partial}{\partial x}\right]\psi(x,t)$$

**Kinetic Energy:**

$$\langle \mathrm{KE}\rangle(t)=\int_{-\infty}^{\infty}dx\;\;\psi^*(x,t)\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right]\psi(x,t)$$

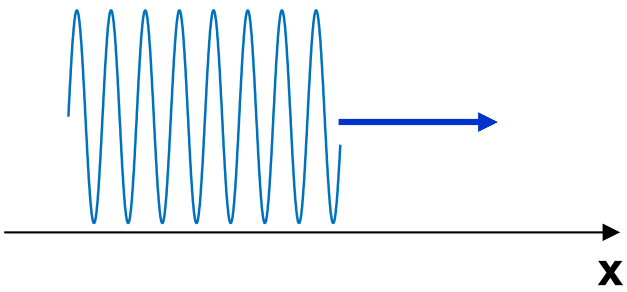
**Total Energy:**

$$\langle E\rangle(t)=\int_{-\infty}^{\infty}dx\;\;\psi^*(x,t)\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x)\right]\psi(x,t)$$

# Mean Values or Expectation Value in 1D: Example

Consider a particle in 1D in free space (potential is zero everywhere):

$$\psi(x,t) = \frac{1}{\sqrt{L}} e^{ikx} e^{-i\frac{E(k)}{\hbar}t}$$




Mean value of momentum:

$$\langle p \rangle(t) = \int_{-\infty}^{\infty} dx \;\; \psi^*(x,t)\left[\frac{\hbar}{i}\frac{\partial}{\partial x}\right]\psi(x,t)$$

$$= \int_{-\infty}^{\infty} dx \;\; \frac{1}{\sqrt{L}} e^{-ikx} e^{+i\frac{E(k)}{\hbar}t}\left[\frac{\hbar}{i}\frac{\partial}{\partial x}\right]\frac{1}{\sqrt{L}} e^{ikx} e^{-i\frac{E(k)}{\hbar}t}$$

$$= \int_{-\infty}^{\infty} dx \;\; \frac{1}{\sqrt{L}} e^{-ikx} e^{+i\frac{E(k)}{\hbar}t}\left[\hbar k\right]\frac{1}{\sqrt{L}} e^{ikx} e^{-i\frac{E(k)}{\hbar}t}$$

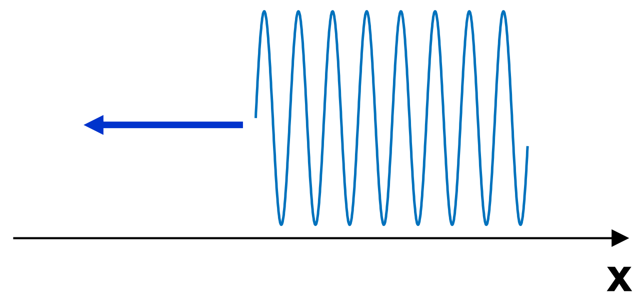
$$= \hbar k \int_{-\infty}^{\infty} dx \;\; \frac{1}{\sqrt{L}} e^{-ikx} e^{+i\frac{E(k)}{\hbar}t}\,\frac{1}{\sqrt{L}} e^{ikx} e^{-i\frac{E(k)}{\hbar}t}$$

$$= \hbar k$$

# Mean Values or Expectation Value in 1D: Example

**Consider a particle in 1D in free space (potential is zero everywhere):**

$$\psi(x,t) = \frac{1}{\sqrt{L}} e^{-ikx} e^{-i\frac{E(-k)}{\hbar}t} = \frac{1}{\sqrt{L}} e^{-ikx} e^{-i\frac{E(k)}{\hbar}t}$$




**Mean value of momentum:**

$$\langle p \rangle(t) = \int_{-\infty}^{\infty} dx \;\; \psi^*(x,t)\left[\frac{\hbar}{i}\frac{\partial}{\partial x}\right]\psi(x,t)$$

$$= \int_{-\infty}^{\infty} dx \;\; \frac{1}{\sqrt{L}} e^{+ikx} e^{+i\frac{E(k)}{\hbar}t} \left[\frac{\hbar}{i}\frac{\partial}{\partial x}\right] \frac{1}{\sqrt{L}} e^{-ikx} e^{-i\frac{E(k)}{\hbar}t}$$

$$= \int_{-\infty}^{\infty} dx \;\; \frac{1}{\sqrt{L}} e^{+ikx} e^{+i\frac{E(k)}{\hbar}t} \left[-\hbar k\right] \frac{1}{\sqrt{L}} e^{-ikx} e^{-i\frac{E(k)}{\hbar}t}$$

$$= -\hbar k \int_{-\infty}^{\infty} dx \;\; \frac{1}{\sqrt{L}} e^{+ikx} e^{+i\frac{E(k)}{\hbar}t} \frac{1}{\sqrt{L}} e^{-ikx} e^{+i\frac{E(k)}{\hbar}t}$$

$$= -\hbar k$$

# Mean Values or Expectation Value in 1D: Example

Consider a particle in 1D in free space (potential is zero everywhere):

$$\psi(x,t) = \frac{1}{\sqrt{2L}} e^{ikx} e^{-i\frac{E(k)}{\hbar}t} + \frac{1}{\sqrt{2L}} e^{-ikx} e^{-i\frac{E(-k)}{\hbar}t}$$

$$= \sqrt{\frac{2}{L}} \cos(kL) e^{-i\frac{E(k)}{\hbar}t}$$

The state is an equal-weight superposition of two plane waves

Mean value of momentum:

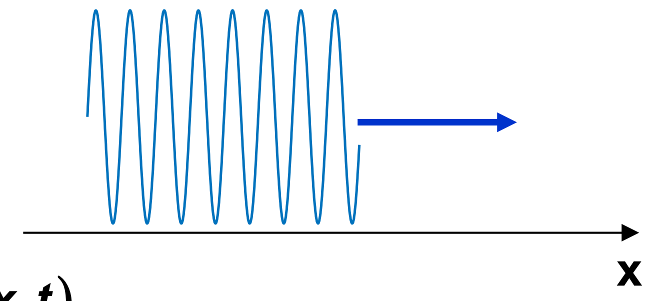
$$\langle p \rangle(t) = \int_{-\infty}^{\infty} dx \;\; \psi^*(x,t) \left[ \frac{\hbar}{i} \frac{\partial}{\partial x} \right] \psi(x,t)$$

$$= 0$$

# Mean Values or Expectation Value in 1D: Example

**Consider a particle in 1D in free space (potential is zero everywhere):**

$$\psi(x,t) = \frac{1}{\sqrt{L}} e^{ikx} e^{-i\frac{E(k)}{\hbar}t}$$




**Mean energy:**

$$\langle E\rangle(t) = \int_{-\infty}^{\infty} dx \;\; \psi^*(x,t)\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\psi(x,t) \quad (V(x) \rightarrow 0)$$

$$= \int_{-\infty}^{\infty} dx \;\; \psi^*(x,t)\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right]\psi(x,t)$$

$$= \int_{-\infty}^{\infty} dx \;\; \frac{1}{\sqrt{L}} e^{-ikx} e^{+i\frac{E(k)}{\hbar}t}\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right]\frac{1}{\sqrt{L}} e^{ikx} e^{-i\frac{E(k)}{\hbar}t}$$

$$= \int_{-\infty}^{\infty} dx \;\; \frac{1}{\sqrt{L}} e^{-ikx} e^{+i\frac{E(k)}{\hbar}t}\left[\frac{\hbar^2 k^2}{2m}\right]\frac{1}{\sqrt{L}} e^{ikx} e^{-i\frac{E(k)}{\hbar}t}$$

$$= \frac{\hbar^2 k^2}{2m}\int_{-\infty}^{\infty} dx \;\; \frac{1}{\sqrt{L}} e^{-ikx} e^{+i\frac{E(k)}{\hbar}t} \frac{1}{\sqrt{L}} e^{ikx} e^{-i\frac{E(k)}{\hbar}t}$$

$$= \frac{\hbar^2 k^2}{2m}$$

# Fourier Transformed Wavefunctions

Consider a particle in 1D with wavefunction: $\psi(x,t)$

The Fourier transform of the wavefunction is:

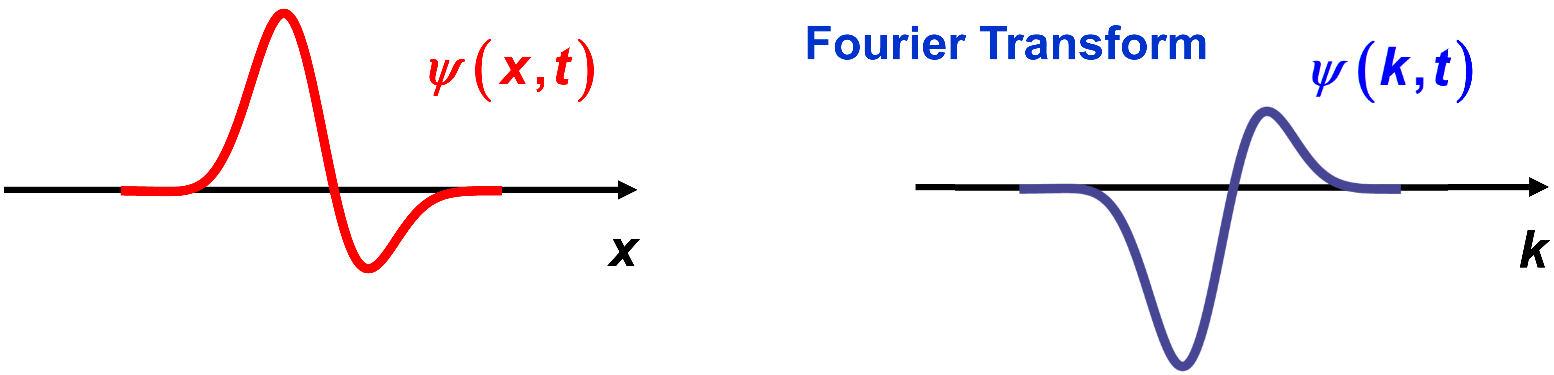
$$\psi(k,t) = \int_{-\infty}^{\infty} dx\, \psi(x,t)\, e^{-ikx}$$

$$\psi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \psi(k,t)\, e^{ikx}$$

Note: unless the particle is free do not assume:

$$\psi(k,t) = \psi(k)\, e^{-\frac{i}{\hbar}E(k)t}$$

$$E(k) = \frac{\hbar^2 k^2}{2m}$$




And:

$$\int_{-\infty}^{\infty} dx\, \left|\psi(x,t)\right|^2 = 1 \quad \Rightarrow \quad \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left|\psi(k,t)\right|^2 = 1$$

# Fourier Transformed Wavefunctions

$$\psi(k,t) = \int_{-\infty}^{\infty} dx\, \psi(x,t)\, e^{-ikx}$$

**Momentum expectation value can also be expressed as:**

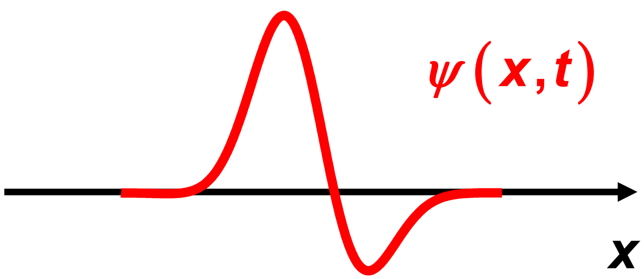
$$\langle p\rangle(t) = \int_{-\infty}^{\infty} dx\ \ \psi^*(x,t)\left[\frac{\hbar}{i}\frac{\partial}{\partial x}\right]\psi(x,t)$$

$$= \int_{-\infty}^{\infty}\frac{dk}{2\pi}\int_{-\infty}^{\infty}\frac{dk'}{2\pi}\int_{-\infty}^{\infty} dx\ \ \psi^*(k,t)e^{-ikx}\left[\hbar k'\right]\psi(k',t)e^{ik'x}$$

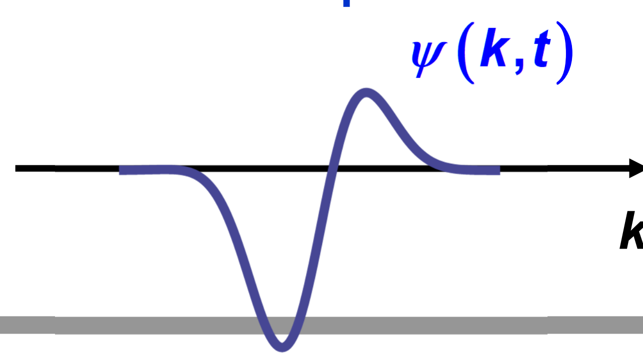
$$= \int_{-\infty}^{\infty}\frac{dk}{2\pi}\int_{-\infty}^{\infty}\frac{dk'}{2\pi}\psi^*(k,t)\left[\hbar k'\right]\psi(k',t)\int_{-\infty}^{\infty} dx\ \ e^{-ikx}e^{ik'x}$$

$$= \int_{-\infty}^{\infty}\frac{dk}{2\pi}\int_{-\infty}^{\infty}\frac{dk'}{2\pi}\hbar k'\,\psi^*(k,t)\psi(k',t)(2\pi)\delta(k-k')$$

$$= \int_{-\infty}^{\infty}\frac{dk}{2\pi}\hbar k\left|\psi(k,t)\right|^2$$




**Momentum space wavefunction**



# Gaussian Wavepacket Example

**Consider a particle in 1D in free space (potential is zero everywhere):**

$$\psi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \psi(k,t)\, e^{ikx}$$

**"Wavefunction" in Fourier space**

**Normalization:**

$$\int_{-\infty}^{\infty} dx\, |\psi(x,t)|^2 = 1 \quad \Rightarrow \quad \int_{-\infty}^{\infty} \frac{dk}{2\pi} |\psi(k,t)|^2 = 1$$

**Free-space Shrodinger equation (zero potential):**

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x,t)$$

**Substitute the Fourier expression:**

$$\psi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \psi(k,t)\, e^{ikx}$$

**To get:**

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi}\, i\hbar \frac{\partial}{\partial t} \psi(k,t)\, e^{ikx} = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\hbar^2 k^2}{2m} \psi(k,t)\, e^{ikx}$$

# Gaussian Wavepacket Example

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi}\, i\hbar \frac{\partial}{\partial t}\psi(k,t)\, e^{ikx} = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\hbar^2 k^2}{2m}\psi(k,t)\, e^{ikx}$$

$E(k)$

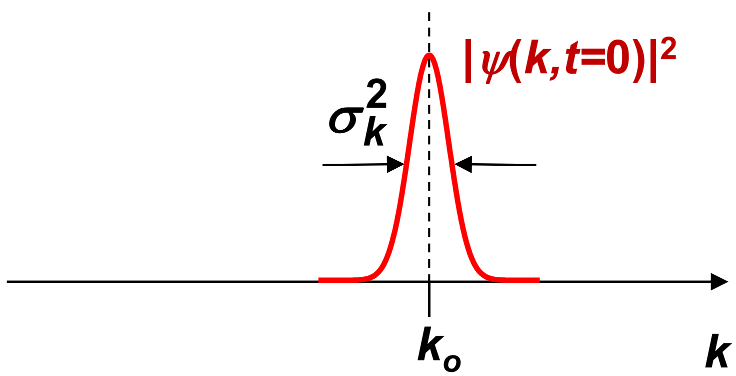
$$\Rightarrow i\hbar \frac{\partial}{\partial t}\psi(k,t) = E(k)\psi(k,t)$$

$$\Rightarrow \psi(k,t) = \phi(k)\, e^{-\frac{i}{\hbar}E(k)t}$$

**This implies:**

$$\psi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\psi(k,t)\, e^{ikx} = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\phi(k)\, e^{ikx} e^{-\frac{i}{\hbar}E(k)t}$$

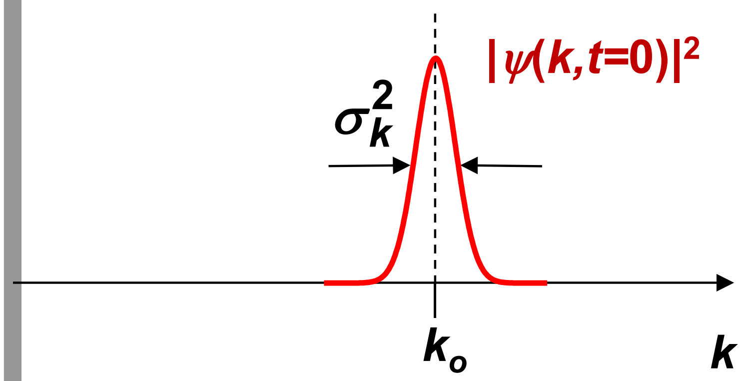
**Consider a Gaussian packet in wavevector space or Fourier space:**




$$\psi(k,t=0) = \phi(k) = \left(\frac{2\pi}{\sigma_k^2}\right)^{1/4} e^{-\frac{(k-k_o)^2}{4\sigma_k^2}}$$

# Gaussian Wavepacket Example

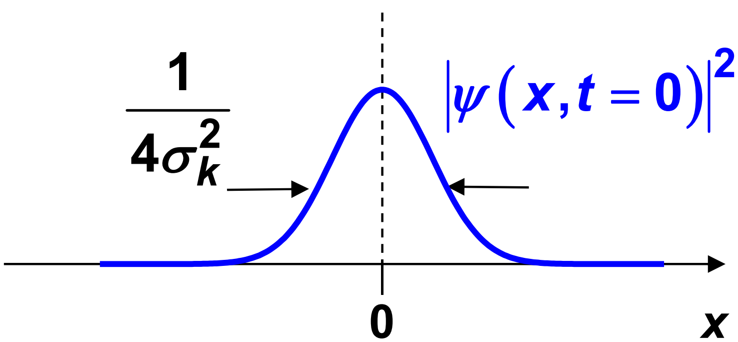
**In wavevector or Fourier space:**




$$\psi(k,t) = \phi(k) e^{-i\frac{E(k)}{\hbar}t} = \left(\frac{2\pi}{\sigma_k^2}\right)^{1/4} e^{-\frac{(k-k_o)^2}{4\sigma_k^2}} e^{-i\frac{E(k)}{\hbar}t}$$

$$\psi(k,t=0) = \phi(k) = \left(\frac{2\pi}{\sigma_k^2}\right)^{1/4} e^{-\frac{(k-k_o)^2}{4\sigma_k^2}}$$

**In real space:**




$$\psi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left[\left(\frac{2\pi}{\sigma_k^2}\right)^{1/4} e^{-\frac{(k-k_o)^2}{4\sigma_k^2}}\right] e^{ikx} e^{-i\frac{E(k)}{\hbar}t}$$

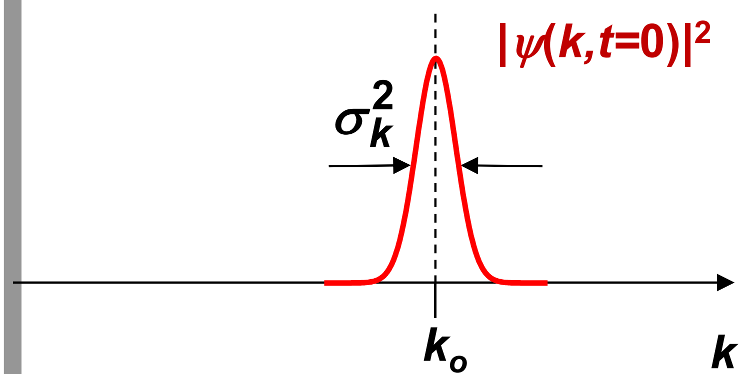
$$\Rightarrow \psi(x,t=0) = \left(\frac{4\sigma_k^2}{2\pi}\right)^{1/4} e^{-\frac{x^2}{4\left(1/4\sigma_k^2\right)}} e^{ik_o x}$$

Nobel prize winning statement

A fat wavepacket in real space requires a small number of superposed plane waves to make it
A slim wavepacket in real space requires a large number of superposed plane waves to make it

# Gaussian Wavepacket Example

**In Fourier space:**




$$\psi(k,t) = \left(\frac{2\pi}{\sigma_k^2}\right)^{1/4} e^{-\frac{(k-k_o)^2}{4\sigma_k^2}} e^{-i\frac{E(k)}{\hbar}t}$$

$$\psi(k,t=0) = \left(\frac{2\pi}{\sigma_k^2}\right)^{1/4} e^{-\frac{(k-k_o)^2}{4\sigma_k^2}}$$

**Width of the packet in Fourier space:** $\sigma_k^2 = \int_{-\infty}^{\infty} \frac{dk}{2\pi}(k-k_o)^2 \left|\psi(k,t=0)\right|^2$

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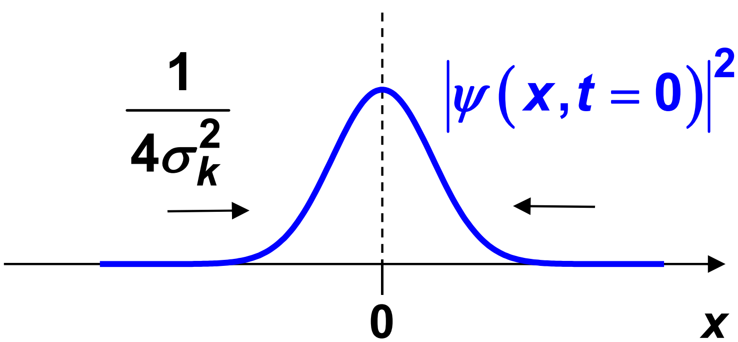
**In real space:**




$$\psi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\psi(k,t)\, e^{ikx}$$

$$\psi(x,t=0) = \left(\frac{4\sigma_k^2}{2\pi}\right)^{1/4} e^{-\frac{x^2}{4\left(1/4\sigma_k^2\right)}} e^{ik_o x}$$

**Width of the packet in real space:** $\sigma_x^2 = \int_{-\infty}^{\infty} dx \;\; x^2 \left|\psi(x,t=0)\right|^2 = \frac{1}{4\sigma_k^2}$

$$\sigma_x^2 \sigma_k^2 = \frac{1}{4}$$

# Gaussian Wavepacket Example

**Now momentum $p$ equals $\hbar k$**

**Therefore, the standard deviation of momentum at time $t$=0 is:**

$$\sigma_p^2 = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left(\hbar k - \hbar k_o\right)^2 \left|\psi\left(k, t=0\right)\right|^2 = \hbar^2 \sigma_k^2$$

**The standard deviation of position was:**

$$\sigma_x^2 = \int_{-\infty}^{\infty} dx \;\; x^2 \left|\psi\left(x, t=0\right)\right|^2 = \frac{1}{4\sigma_k^2}$$

**This implies:**

$$\sigma_x^2 \sigma_p^2 = \frac{\hbar^2}{4}$$

**or**

$$\sigma_x \sigma_p = \frac{\hbar}{2}$$

**A fat wavepacket in real space requires a small number of superposed plane waves to make it**
**A slim wavepacket in real space requires a large number of superposed plane waves to make it**

# Mean Values or Expectation Values in 3D

**Position:**

$$\langle \vec{r} \rangle (t) = \int d^3\vec{r} \;\; \psi^*(\vec{r},t)\left[\vec{r}\right]\psi(\vec{r},t)$$

**Potential Energy:**

$$\langle \mathrm{PE} \rangle (t) = \langle V(\vec{r}) \rangle (t) = \int d^3\vec{r} \;\; \psi^*(\vec{r},t)\left[V(\vec{r})\right]\psi(\vec{r},t)$$

**Momentum:**

$$\langle \vec{p} \rangle (t) = \int d^3\vec{r} \;\; \psi^*(\vec{r},t)\left[\frac{\hbar}{i}\nabla\right]\psi(\vec{r},t)$$

**Kinetic Energy:**

$$\langle \mathrm{KE} \rangle (t) = \left\langle \frac{p^2}{2m} \right\rangle (t) = \int d^3\vec{r} \;\; \psi^*(\vec{r},t)\left[-\frac{\hbar^2}{2m}\nabla^2\right]\psi(\vec{r},t)$$

**Total Energy:**

$$\langle E \rangle (t) = \int d^3\vec{r} \;\; \psi^*(\vec{r},t)\left[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r})\right]\psi(\vec{r},t)$$