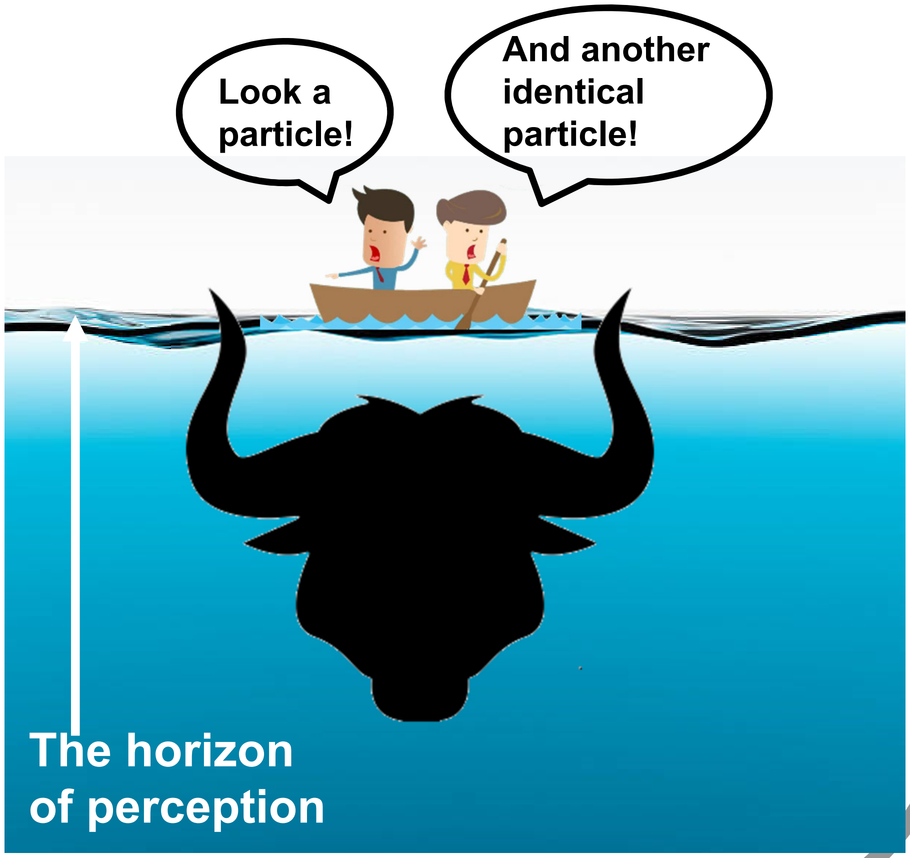
Lecture 25

Many Particle States and Wavefunctions, Identical Particles, Spin-Statistics Theorem, and Pauli's Exclusion Principle

In this lecture you will learn:

- Writing quantum states of many particle systems
- Spin-statistics theorem
- Fermions and Pauli's exclusion principle



Two Distinguishable Spinless Particles

Consider two distinguishable particles:

$\vec{r}_B$ Particle B

Particle A $\vec{r}_A$

Let the quantum state of the system be

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|\phi\rangle_A \otimes |\chi\rangle_B + |\omega\rangle_A \otimes |\phi\rangle_B\right]$$

The wavefunction is obtained by projecting the quantum state onto the position basis of the joint Hilbert space:

$$|\vec{r}_A,\vec{r}_B\rangle = |\vec{r}_A\rangle_A \otimes |\vec{r}_B\rangle_B \quad \xrightarrow{\text{completeness}} \quad \int d^3\vec{r}_B \int d^3\vec{r}_A\, |\vec{r}_A,\vec{r}_B\rangle\langle\vec{r}_A,\vec{r}_B| = \hat{1}$$

This implies that the wavefunction of the system of particles is:

$$\langle\vec{r}_A,\vec{r}_B|\psi\rangle = \psi(\vec{r}_A,\vec{r}_B) = \frac{1}{\sqrt{2}}\left[\phi(\vec{r}_A)\chi(\vec{r}_B) + \omega(\vec{r}_A)\phi(\vec{r}_B)\right]$$

Two Distinguishable Spinless Particles

The wavefunction of the system of particles is:

$$\langle \vec{r}_A, \vec{r}_B | \psi \rangle = \psi(\vec{r}_A, \vec{r}_B) = \frac{1}{\sqrt{2}} \left[\phi(\vec{r}_A) \chi(\vec{r}_B) + \omega(\vec{r}_A) \phi(\vec{r}_B) \right]$$

Probability $P(\vec{r}_A, \vec{r}_B)$ of finding particle A at $\vec{r}_A$ <u>and</u> particle B at $\vec{r}_B$ is:

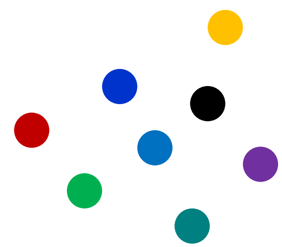
$$P(\vec{r}_A, \vec{r}_B) = \left| \psi(\vec{r}_A, \vec{r}_B) \right|^2$$

Probability $P(\vec{r}_A)$ of finding particle A at $\vec{r}_A$ (irrespective of where particle B might be):

$$P(\vec{r}_A) == \int d^3\vec{r}_B \, P(\vec{r}_A, \vec{r}_B) = \int d^3\vec{r}_B \, \left| \psi(\vec{r}_A, \vec{r}_B) \right|^2$$

Many Particle Systems: Distinguishable Spinless Particles

Consider a system of N distinguishable spinless particles:



Coordinates: $\vec{r}_A, \vec{r}_B, \vec{r}_C, \ldots\ldots\ldots$

Let the quantum state of the system be: $|\psi\rangle$

The wavefunction is obtained by projecting the quantum state onto the position basis of the joint Hilbert space:

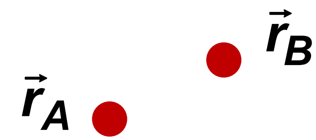
$$|\vec{r}_A, \vec{r}_B, \vec{r}_C, \ldots\ldots\rangle = |\vec{r}_A\rangle_A \otimes |\vec{r}_B\rangle_B \otimes |\vec{r}_C\rangle_C \otimes \ldots\ldots$$

This implies that the wavefunction of the system of particles is:

$$\langle \vec{r}_A, \vec{r}_B, \vec{r}_C, \ldots\ldots | \psi \rangle = \psi(\vec{r}_A, \vec{r}_B, \vec{r}_C, \ldots\ldots)$$

Two Indistinguishable (Identical) Particles

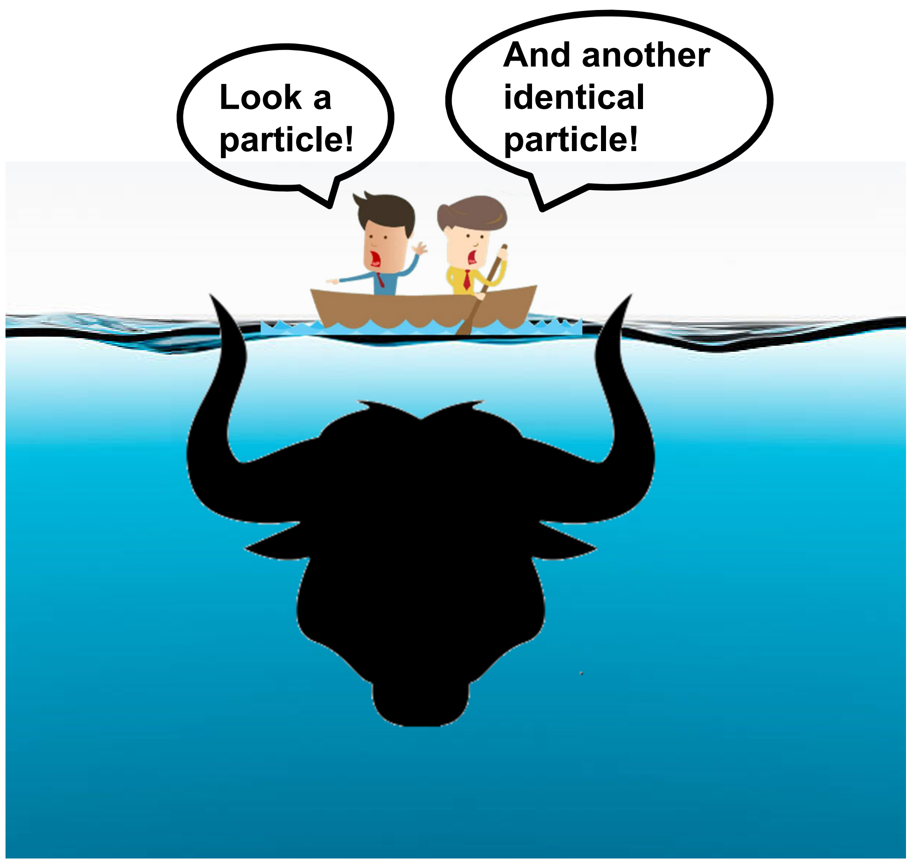
Consider now two indistinguishable (identical) particles:



Let the quantum state of the system be $|\psi\rangle$

Fundamental question: are these really two separate particles or just different manifestations of a single reality?

The horizon of perception



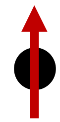
Need to be careful how we think about indistinguishable particles!

They could all be parts of the same one big monster!

Need information about the particles' spins too!

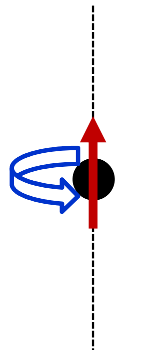
Particles with Spin: What's so Odd About Them?

Consider a particle with spin "*s*" and spin angular momentum operator $\hat{\vec{S}}$:



Let its full quantum state be: $|\psi\rangle$ $\longrightarrow$ $\left\{ \begin{array}{l} \left(\hat{\vec{S}}.\,\hat{n}\right)|\psi\rangle = \lambda|\psi\rangle \\ \lambda_{max} = s\hbar \end{array} \right.$

If we rotate the particle wrt to the spin axis, by angle ϕ, the quantum state becomes:

$|\psi\rangle \longrightarrow$ ϕ  $\longrightarrow e^{-is\phi}|\psi\rangle$

The proof of this is not in this course

If we rotate by 4π, the state must come back to the original state (Why 4π? Why not 2π?):

$$e^{-is4\pi}|\psi\rangle = |\psi\rangle$$

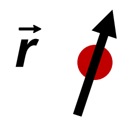
$$\Rightarrow e^{-is4\pi} = 1$$

$$\Rightarrow s = \frac{n}{2} \qquad \{\; n = 0,1,2,3,\ldots\ldots$$

Spin "s" can only be integer, or half-integer !!

Particles with Spin 1/2: Hilbert Space

Consider the quantum state of an electron – a particle with spin 1/2:



The Hilbert space is spanned by states of the form:

$$|\vec{r}\rangle \otimes |z\uparrow\rangle \qquad |\vec{r}\rangle \otimes |z\downarrow\rangle$$

Or, with some abuse of notation:

$$|\vec{r}\rangle |z\uparrow\rangle \qquad |\vec{r}\rangle |z\downarrow\rangle$$

Or, just:

$$|\vec{r}, z\uparrow\rangle \qquad |\vec{r}, z\downarrow\rangle$$

Completeness relation:

$$\int d^3\vec{r}\left[\,|\vec{r}, z\uparrow\rangle\langle\vec{r}, z\uparrow| + \int d^3\vec{r}\,|\vec{r}, z\downarrow\rangle\langle\vec{r}, z\downarrow|\,\right] = \hat{1}$$

Particles with Spin 1/2: Hilbert Space

$\vec{r}$

Consider the quantum state: $\left|\phi\right\rangle = \left|f\right\rangle \otimes \left[\alpha\left|z\uparrow\right\rangle + \beta\left|z\downarrow\right\rangle\right]$

What is the probability of the particle being at location $\vec{r}$ with spin-up?

$$\left|\left(\left\langle z\uparrow\right| \otimes \left\langle \vec{r}\right|\right)\left|\phi\right\rangle\right|^2 = \left|\left\langle \vec{r}, z\uparrow \middle| \phi\right\rangle\right|^2$$

$$= \left|\left\langle \vec{r} \middle| f\right\rangle \left\langle z\uparrow\right|\left[\alpha\left|z\uparrow\right\rangle + \beta\left|z\downarrow\right\rangle\right]\right|^2$$

$$= \left|f\left(\vec{r}\right)\alpha\right|^2$$

What is the probability of the particle being at location $\vec{r}$ with spin-down?

$$\left|\left(\left\langle z\downarrow\right| \otimes \left\langle \vec{r}\right|\right)\left|\phi\right\rangle\right|^2 = \left|\left\langle \vec{r}, z\downarrow \middle| \phi\right\rangle\right|^2$$

$$= \left|f\left(\vec{r}\right)\beta\right|^2$$

What is the probability of the particle being at location $\vec{r}$ (irrespective of the spin)?

$$\left|f\left(\vec{r}\right)\right|^2\left[\left|\alpha\right|^2 + \left|\beta\right|^2\right]$$

Particles with Spin 1/2: Two-Component Wavefunctions

$\vec{r}$

Consider the quantum state: $\left|\phi\right\rangle = \left|f\right\rangle \otimes \left[\alpha\left|z\uparrow\right\rangle + \beta\left|z\downarrow\right\rangle\right]$

What if one writes: $\left\langle \vec{r}\,\middle|\,\phi\right\rangle$

The above expression is meaningless by the strict rules we have established because one is taking an inner product between two different Hilbert spaces (one of particle with spin, and one of particle without spin)

However, it is often interpreted to mean the following:

$$\left\langle \vec{r}\,\middle|\,\phi\right\rangle = \phi(\vec{r}) = \left\langle \vec{r}\,\middle|\,f\right\rangle\left[\alpha\left|z\uparrow\right\rangle + \beta\left|z\downarrow\right\rangle\right] = f(\vec{r})\left[\alpha\left|z\uparrow\right\rangle + \beta\left|z\downarrow\right\rangle\right]$$

In column vector representation for spin:

$$\phi(\vec{r}) = f(\vec{r})\left[\alpha\left|z\uparrow\right\rangle + \beta\left|z\downarrow\right\rangle\right] = f(\vec{r})\left[\alpha\begin{bmatrix}1\\0\end{bmatrix} + \beta\begin{bmatrix}0\\1\end{bmatrix}\right] = \begin{bmatrix}\alpha f(\vec{r})\\ \beta f(\vec{r})\end{bmatrix}$$

Two-component wavefunction of the particle

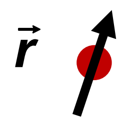
Probability of the electron at location $\vec{r}$ (irrespective of the spin):

$$\phi^H(\vec{r})\phi(\vec{r}) = \begin{bmatrix}\alpha^* f^*(\vec{r}) & \beta^* f^*(\vec{r})\end{bmatrix}\begin{bmatrix}\alpha f(\vec{r})\\ \beta f(\vec{r})\end{bmatrix} = \left|f(\vec{r})\right|^2\left[\left|\alpha\right|^2 + \left|\beta\right|^2\right]$$

Normalization: $\int d^3\vec{r}\;\phi^H(\vec{r})\phi(\vec{r}) = 1$

Particles with Spin 1/2 : Unentangled Spin-Space States

Consider the quantum state of an electron – a particle with spin 1/2:



The quantum states, <u>including the spin degree of freedom</u>, can be of the form:

$$\left|\phi\right\rangle = \left|f\right\rangle \otimes \left|z\uparrow\right\rangle$$

$$\left|\phi\right\rangle = \left|f\right\rangle \otimes \left[\alpha\left|z\uparrow\right\rangle + \beta\left|z\downarrow\right\rangle\right]$$

Two examples!

The two-component wavefunction for the state is:

$$\begin{aligned}\phi(\vec{r}) = \left\langle \vec{r}\middle|\phi\right\rangle &= \left\langle \vec{r}\middle|f\right\rangle\left[\alpha\left|z\uparrow\right\rangle + \beta\left|z\downarrow\right\rangle\right] \\ &= f(\vec{r})\left[\alpha\left|z\uparrow\right\rangle + \beta\left|z\downarrow\right\rangle\right] \\ &= \begin{bmatrix}\alpha f(\vec{r}) \\ \beta f(\vec{r})\end{bmatrix}\end{aligned}$$

Particles with Spin 1/2 : Entangled Spin-Space States

Consider the quantum state of an electron – a particle with spin 1/2:

$\vec{r}$ 

Consider the state:

$$\left|\phi\right\rangle = \left|f\right\rangle \otimes \left|z\uparrow\right\rangle + \left|g\right\rangle \otimes \left|z\downarrow\right\rangle$$

The spin-up component has the spatial part $\left|f\right\rangle$ and the spin-down component has the spatial part $\left|g\right\rangle$

The two-component wavefunction for the state is:

$$\begin{aligned}
\phi(\vec{r}) = \langle\vec{r}|\phi\rangle &= \langle\vec{r}|f\rangle\left|z\uparrow\right\rangle + \langle\vec{r}|g\rangle\left|z\downarrow\right\rangle \\
&= f(\vec{r})\left|z\uparrow\right\rangle + g(\vec{r})\left|z\downarrow\right\rangle \\
&= f(\vec{r})\begin{bmatrix}1\\0\end{bmatrix} + g(\vec{r})\begin{bmatrix}0\\1\end{bmatrix} \\
&= \begin{bmatrix}f(\vec{r})\\ g(\vec{r})\end{bmatrix}
\end{aligned}$$

Switch the spin basis representation to column vectors

Particles with Spin 1 and Three-Component Wavefunctions

Consider the quantum state of a meson with spin 1:

$\vec{r}$

There are now three spin states in the Hilbert space: $|z\uparrow\rangle \qquad |z\,0\rangle \qquad |z\downarrow\rangle$

They are all eigenstates of the z-component of the spin angular momentum:

$$\hat{S}_z|z\uparrow\rangle = +\hbar|z\uparrow\rangle$$

$$\hat{S}_z|z\,0\rangle = 0|z\uparrow\rangle_A = 0$$

$$\hat{S}_z|z\downarrow\rangle = -\hbar|z\downarrow\rangle$$

The complete quantum states, including the spin degree of freedom, are of the form:

$$|\phi\rangle = |f\rangle \otimes |z\uparrow\rangle$$

$$|\phi\rangle = |f\rangle \otimes \left[\alpha|z\uparrow\rangle + \beta|z\,0\rangle + \gamma|z\downarrow\rangle\right]$$

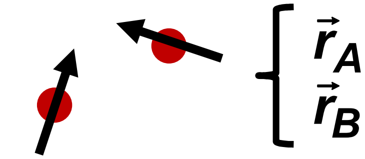
Two examples!

The three-component wavefunction for the last state above is:

$$\langle\vec{r}|\phi\rangle = \phi(\vec{r}) = \langle\vec{r}|f\rangle\left[\alpha|z\uparrow\rangle + \beta|z\,0\rangle + \gamma|z\downarrow\rangle\right] = f(\vec{r})\left[\alpha\begin{bmatrix}1\\0\\0\end{bmatrix} + \beta\begin{bmatrix}0\\1\\0\end{bmatrix} + \gamma\begin{bmatrix}0\\0\\1\end{bmatrix}\right] = \begin{bmatrix}\alpha f(\vec{r})\\ \beta f(\vec{r})\\ \gamma f(\vec{r})\end{bmatrix}$$

Two Indistinguishable Particles with Spin: Quantum State Symmetry

Consider two indistinguishable particles (with spin now):



1) The quantum state is, $|\psi\rangle = |\phi\rangle_A \otimes |\chi\rangle_B$

Spin is included — Spin is included

No special symmetry (in labels A and B)

2) The quantum state is, $|\psi\rangle = \frac{1}{\sqrt{2}}\left[|\phi\rangle_A \otimes |\chi\rangle_B + |\chi\rangle_A \otimes |\phi\rangle_B\right]$

Spin is included — Spin is included

Symmetric state (in labels A and B)

3) The quantum state is, $|\psi\rangle = \frac{1}{\sqrt{2}}\left[|\phi\rangle_A \otimes |\chi\rangle_B - |\chi\rangle_A \otimes |\phi\rangle_B\right]$

Spin is included — Spin is included

Antisymmetric state in labels A and B

Example, for two spin-half electron, the states $|\phi\rangle$ and $|\chi\rangle$ above could be:

$$|\phi\rangle_A = |f\rangle_A \otimes |z\uparrow\rangle_A \qquad |\chi\rangle_A = |g\rangle_A \otimes |z\downarrow\rangle_A$$

$$|\phi\rangle_B = |f\rangle_B \otimes |z\uparrow\rangle_B \qquad |\chi\rangle_B = |g\rangle_B \otimes |z\downarrow\rangle_B$$

Are all three types of states possible ?

The Spin-Statistics Theorem

The spin-statistics theorem states that:

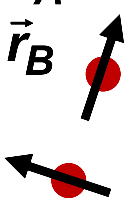
● The quantum state of a system of identical integer spin particles is symmetric under the exchange of any two particles. Particles with a symmetric state under such an exchange are called Bosons.

● The quantum state of a system of identical half-integer spin particles is antisymmetric under the exchange of any two particles. Particles with an antisymmetric state under such an exchange are called Fermions.

Markus Fierz
1912-2006

Wolfgang Ernst Pauli
1900-1958
Nobel Prize 1945

$$\left\{ \begin{array}{l} \vec{r}_A \\ \vec{r}_B \end{array} \right.$$



$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|\phi\rangle_A \otimes |\chi\rangle_B + |\chi\rangle_A \otimes |\phi\rangle_B\right]$$

Integer spin Bosons ⇨ Symmetric state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|\phi\rangle_A \otimes |\chi\rangle_B - |\chi\rangle_A \otimes |\phi\rangle_B\right]$$

Half-integer spin Fermions ⇨ Antisymmetric state

The Spin-Statistics Theorem Explained ……….NOT!

Usually the rule of thumb in sciences is:

"The more general the result, the easier the proof"

Or in Richard Feynman's words:

"Simple results have simple explanations"

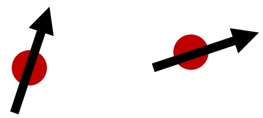
But thus far, an elementary explanation/proof for the spin-statistics theorem cannot be given despite the fact that the theorem is so general and so simple to state

In the Feynman Lectures on Physics, Richard Feynman said that this probably means that "we do not have a complete understanding of the fundamental principle involved"

Spin-statistics theorem can be shown to hold using relativistic quantum field theory

Fermion States and Pauli's Exclusion Principle: An Example

A system of two indistinguishable Fermions with spin 1/2:



Coordinates: $\vec{r}_A, \vec{r}_B$

Suppose: $$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|\phi\rangle_A \otimes |\chi\rangle_B - |\chi\rangle_A \otimes |\phi\rangle_B\right]$$

$$\begin{cases} |\phi\rangle_A = |f\rangle_A \otimes |z\uparrow\rangle_A \\ |\phi\rangle_B = |f\rangle_B \otimes |z\uparrow\rangle_B \\ |\chi\rangle_A = |g\rangle_A \otimes |z\uparrow\rangle_A \\ |\chi\rangle_B = |g\rangle_B \otimes |z\uparrow\rangle_B \end{cases}$$

Case 1:

$$\Rightarrow |\psi\rangle = \frac{1}{\sqrt{2}}\left[|\phi\rangle_A \otimes |\chi\rangle_B - |\chi\rangle_A \otimes |\phi\rangle_B\right]$$

$$= \underbrace{\frac{1}{\sqrt{2}}\left[|f\rangle_A |g\rangle_B - |g\rangle_A |f\rangle_B\right]}_{\text{Spatial part antisymmetric in labels A and B}} \otimes \underbrace{\left[|z\uparrow\rangle_A |z\uparrow\rangle_B\right]}_{\text{Spin part symmetric in labels A and B}}$$

The wavefunction is:

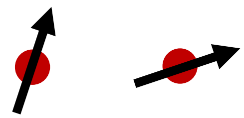
$$\langle \vec{r}_A, \vec{r}_B | \psi\rangle = \psi(\vec{r}_A, \vec{r}_B) = \frac{1}{\sqrt{2}}\left[f(\vec{r}_A)g(\vec{r}_B) - g(\vec{r}_A)f(\vec{r}_B)\right]\left[|z\uparrow\rangle_A |z\uparrow\rangle_B\right]$$

Note: $$\Rightarrow \psi(\vec{r}_A = \vec{r}_B = \vec{r}) = \frac{1}{\sqrt{2}}\left[f(\vec{r})g(\vec{r}) - g(\vec{r})f(\vec{r})\right]\left[|z\uparrow\rangle_A |z\uparrow\rangle_B\right] = 0$$

The probability of finding two Fermions with the same spin at the same location is 0

Fermion States and Pauli's Exclusion Principle

A system of two indistinguishable Fermions with spin 1/2:



Coordinates: $\vec{r}_A, \vec{r}_B$

Suppose: $$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|\phi\rangle_A \otimes |\chi\rangle_B - |\chi\rangle_A \otimes |\phi\rangle_B\right]$$

$$\begin{cases} |\phi\rangle_A = |f\rangle_A \otimes |z\uparrow\rangle_A \\ |\phi\rangle_B = |f\rangle_B \otimes |z\uparrow\rangle_B \\ |\chi\rangle_A = |f\rangle_A \otimes |z\downarrow\rangle_A \\ |\chi\rangle_B = |f\rangle_B \otimes |z\downarrow\rangle_B \end{cases}$$

Case 2:

$$\Rightarrow |\psi\rangle = \frac{1}{\sqrt{2}}\left[|\phi\rangle_A \otimes |\chi\rangle_B - |\chi\rangle_A \otimes |\phi\rangle_B\right]$$

$$= \underbrace{\left[|f\rangle_A |f\rangle_B\right]}_{\text{Spatial part symmetric}} \otimes \underbrace{\frac{1}{\sqrt{2}}\left[|z\uparrow\rangle_A |z\downarrow\rangle_B - |z\downarrow\rangle_A |z\uparrow\rangle_B\right]}_{\text{Spin part antisymmetric}}$$

The wavefunction is:

$$\Rightarrow \langle \vec{r}_A, \vec{r}_B | \psi\rangle = \psi(\vec{r}_A, \vec{r}_B) = \left[f(\vec{r}_A) f(\vec{r}_B)\right] \frac{1}{\sqrt{2}}\left[|z\uparrow\rangle_A |z\downarrow\rangle_B - |z\downarrow\rangle_A |z\uparrow\rangle_B\right]$$

Note: $$\Rightarrow \psi(\vec{r}_A = \vec{r}_B = \vec{r}) = \left[f(\vec{r}) f(\vec{r})\right] \frac{1}{\sqrt{2}}\left[|z\uparrow\rangle_A |z\downarrow\rangle_B - |z\downarrow\rangle_A |z\uparrow\rangle_B\right]$$

The probability of finding two Fermions with different spins at the same location is not 0

Pauli's Exclusion Principle

Consider the following CSCO for fermions:

$$\left\{\hat{O},\hat{A},\hat{B},\hat{H},.......\right\}$$

The formal statement of the Pauli's Exclusion Principle, which follows from the anti-symmetry of the Fermion quantum state, can be stated as follows:

No quantum state of two or more Fermions can exist in which two or more Fermions have the same eigenvalues for all the CSCO operators

In more simpler words:

No two Fermions can have the same values for all the observables

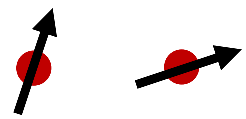
Proof: consider the antisymmetric state of two fermions:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|\phi\rangle_A \otimes |\chi\rangle_B - |\chi\rangle_A \otimes |\phi\rangle_B\right]$$

If the two fermions have the same eigenvalues for all the CSCO operators then this can only happen iff $|\phi\rangle = |\chi\rangle$ but then $|\psi\rangle = 0$ and the quantum states does not exist

Singlet and Triplet States of Spin 1/2 Particles

A system of two indistinguishable Fermions:



Coordinates: $\vec{r}_A, \vec{r}_B$

1) Consider the following <u>three</u> states in which the spin part of the state is symmetric in labels A and B:

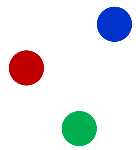
$$\left.\begin{aligned}
|\psi\rangle &= \frac{1}{\sqrt{2}}\Big[|f\rangle_A|g\rangle_B - |g\rangle_A|f\rangle_B\Big]\otimes|z\uparrow\rangle_A|z\uparrow\rangle_B \\
|\psi\rangle &= \frac{1}{\sqrt{2}}\Big[|f\rangle_A|g\rangle_B - |g\rangle_A|f\rangle_B\Big]\otimes\frac{1}{\sqrt{2}}\Big[|z\uparrow\rangle_A|z\downarrow\rangle_B + |z\downarrow\rangle_A|z\uparrow\rangle_B\Big] \\
|\psi\rangle &= \frac{1}{\sqrt{2}}\Big[|f\rangle_A|g\rangle_B - |g\rangle_A|f\rangle_B\Big]\otimes|z\downarrow\rangle_A|z\downarrow\rangle_B
\end{aligned}\right\}\text{Triplet states}$$

2) Consider the following single state in which the spin part of the state is anti-symmetric in labels A and B:

$$\left.\begin{aligned}
|\psi\rangle &= \frac{1}{\sqrt{2}}\Big[|f\rangle_A|g\rangle_B + |g\rangle_A|f\rangle_B\Big]\otimes\frac{1}{\sqrt{2}}\Big[|z\uparrow\rangle_A|z\downarrow\rangle_B - |z\downarrow\rangle_A|z\uparrow\rangle_B\Big] \\
|\psi\rangle &= |f\rangle_A|f\rangle_B\otimes\frac{1}{\sqrt{2}}\Big[|z\uparrow\rangle_A|z\downarrow\rangle_B - |z\downarrow\rangle_A|z\uparrow\rangle_B\Big]
\end{aligned}\right\}\text{Singlet states}$$

Three Distinguishable Particles of Spin Zero (Spinless Particles)

Consider a system of 3 distinguishable particles:



Coordinates: $\vec{r}_A, \vec{r}_B, \vec{r}_C$

Let the quantum state of the system be $|\psi\rangle$

Suppose:

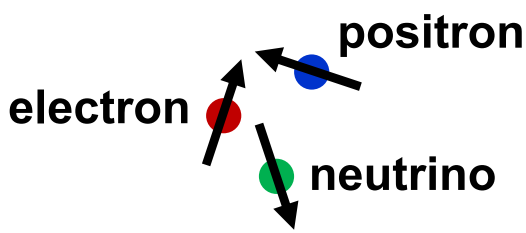
$$|\psi\rangle = |\phi\rangle_A \, |\chi\rangle_B \, |\omega\rangle_C$$

This implies that the wavefunction of the system of particles is:

$$\langle \vec{r}_A, \vec{r}_B, \vec{r}_C | \psi \rangle = \psi(\vec{r}_A, \vec{r}_B, \vec{r}_C) = \phi(\vec{r}_A)\,\chi(\vec{r}_B)\,\omega(\vec{r}_C)$$

Three Distinguishable Particles of Spin 1/2

Consider a system of 3 distinguishable Fermions:



Coordinates: $\vec{r}_A, \vec{r}_B, \vec{r}_C$

Let the quantum state of the system be $|\psi\rangle$

Suppose:

$$|\psi\rangle = |\phi\rangle_A \, |\chi\rangle_B \, |\omega\rangle_C$$

Spin is included

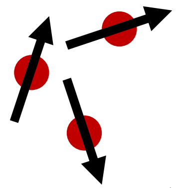
Spin is included

Spin is included

Note that the state need not be symmetric or anti-symmetric!

Example: Three Indistinguishable Bosons

Consider a system of 3 indistinguishable Bosons:



Coordinates: $\vec{r}_A, \vec{r}_B, \vec{r}_C$

Let the quantum state of the system be $|\psi\rangle$

Suppose:

$$|\psi\rangle = \frac{1}{\sqrt{3!}}\begin{bmatrix} |\phi\rangle_A |\chi\rangle_B |\omega\rangle_C + |\phi\rangle_A |\omega\rangle_B |\chi\rangle_C + |\omega\rangle_A |\chi\rangle_B |\phi\rangle_C \\ + |\omega\rangle_A |\phi\rangle_B |\chi\rangle_C + |\chi\rangle_A |\phi\rangle_B |\omega\rangle_C + |\chi\rangle_A |\omega\rangle_B |\phi\rangle_C \end{bmatrix}$$

Full symmetric state under the exchange of any two particles (i.e. any two labels)

Total of 3! terms

Spin is included

Spin is included

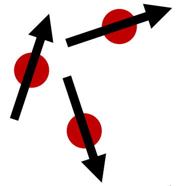
Spin is included

A good way to write fully symmetric quantum states is using Slater permanents:

$$|\psi\rangle = \frac{1}{\sqrt{3!}}\begin{bmatrix} |\phi\rangle_A & |\phi\rangle_B & |\phi\rangle_C \\ |\chi\rangle_A & |\chi\rangle_B & |\chi\rangle_C \\ |\omega\rangle_A & |\omega\rangle_B & |\omega\rangle_C \end{bmatrix}_+$$

Example: Three Indistinguishable Bosons, All in the Same State!

Consider a system of 3 indistinguishable Bosons:



Coordinates: $\vec{r}_A, \vec{r}_B, \vec{r}_C$

Let the quantum state of the system be $|\psi\rangle$

Suppose:

$$|\psi\rangle = |\chi\rangle_A |\chi\rangle_B |\chi\rangle_C$$

Spin is included

Spin is included

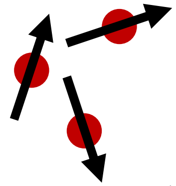
Spin is included

Full symmetric state under the exchange of any two particles (i.e. <u>any two labels</u>)
<u>But here all three particles have the same exact state!</u>

In a Bose-Einstein condensate, all bosons are in the same state!

Example: Three Indistinguishable Fermions

Consider a system of 3 indistinguishable Fermions:



Coordinates: $\vec{r}_A, \vec{r}_B, \vec{r}_C$

Let the quantum state of the system be $|\psi\rangle$

Suppose:

$$|\psi\rangle = \frac{1}{\sqrt{3!}}\begin{bmatrix} |\phi\rangle_A|\chi\rangle_B|\omega\rangle_C - |\phi\rangle_A|\omega\rangle_B|\chi\rangle_C - |\omega\rangle_A|\chi\rangle_B|\phi\rangle_C \\ +|\omega\rangle_A|\phi\rangle_B|\chi\rangle_C - |\chi\rangle_A|\phi\rangle_B|\omega\rangle_C + |\chi\rangle_A|\omega\rangle_B|\phi\rangle_C \end{bmatrix}$$

Full antisymmetric state under the exchange of any two particles (i.e. any two labels)

Total of 3! terms

Spin is included

Spin is included

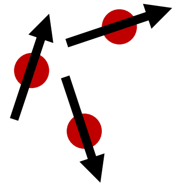
Spin is included

A good way to write fully antisymmetric quantum states is using Slater determinants:

$$|\psi\rangle = \frac{1}{\sqrt{3!}}\begin{bmatrix} |\phi\rangle_A & |\phi\rangle_B & |\phi\rangle_C \\ |\chi\rangle_A & |\chi\rangle_B & |\chi\rangle_C \\ |\omega\rangle_A & |\omega\rangle_B & |\omega\rangle_C \end{bmatrix}_-$$

Fermion States and Pauli's Exclusion Principle and Entanglement

Consider a system of three indistinguishable Fermions:



Coordinates: $\vec{r}_A, \vec{r}_B, \vec{r}_C$

Let the quantum state of the system be $|\psi\rangle$

Suppose:

$$|\psi\rangle = \frac{1}{\sqrt{3!}}\begin{bmatrix} |\phi\rangle_A|\chi\rangle_B|\omega\rangle_C - |\phi\rangle_A|\omega\rangle_B|\chi\rangle_C - |\omega\rangle_A|\chi\rangle_B|\phi\rangle_C \\ +|\omega\rangle_A|\phi\rangle_B|\chi\rangle_C - |\chi\rangle_A|\phi\rangle_B|\omega\rangle_C + |\chi\rangle_A|\omega\rangle_B|\phi\rangle_C \end{bmatrix}$$

Full antisymmetric state under the exchange of any two particles

Total of 3! terms

Spin is included

Spin is included

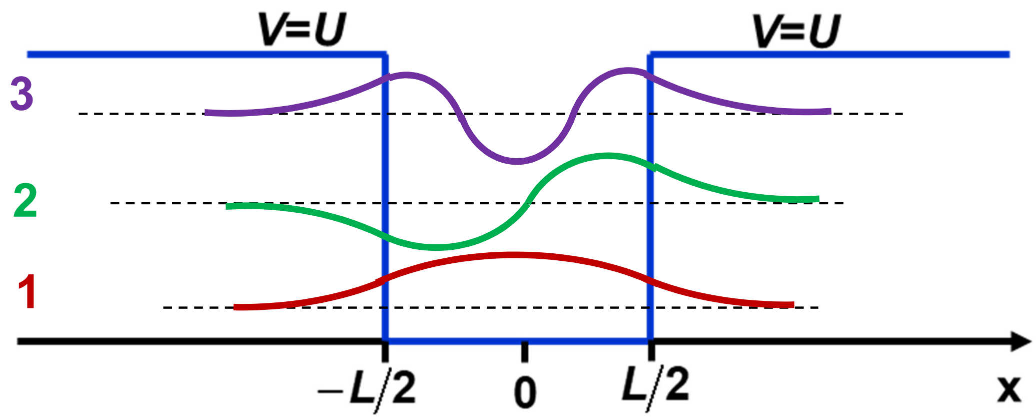
Spin is included

Suppose $|\chi\rangle = |\omega\rangle$ **(i.e. two of the three particles have the exact same quantum state)**

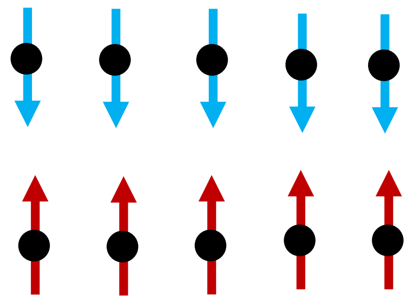
$$|\psi\rangle = \frac{1}{\sqrt{3!}}\begin{bmatrix} |\phi\rangle_A|\chi\rangle_B|\chi\rangle_C - |\phi\rangle_A|\chi\rangle_B|\chi\rangle_C - |\chi\rangle_A|\chi\rangle_B|\phi\rangle_C \\ +|\chi\rangle_A|\phi\rangle_B|\chi\rangle_C - |\chi\rangle_A|\phi\rangle_B|\chi\rangle_C + |\chi\rangle_A|\chi\rangle_B|\phi\rangle_C \end{bmatrix} = 0$$

The quantum state does not exist!
For the quantum state to exist, all Fermions must have different quantum states in $|\psi\rangle$
In other words, no two Fermions can have the same quantum state in $|\psi\rangle$

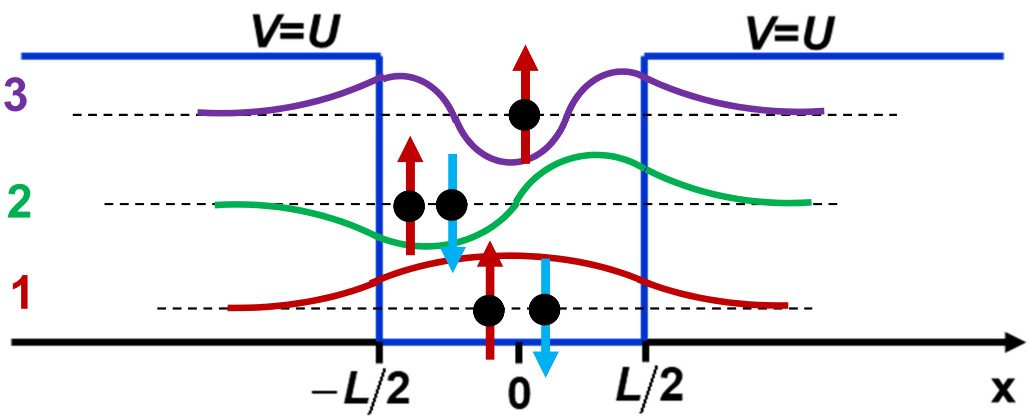
Electron Filling and Pauli's Exclusion Principle



Fill the quantum well energy eigenstates with <u>five</u> electrons such that the system has the lowest possible total energy!

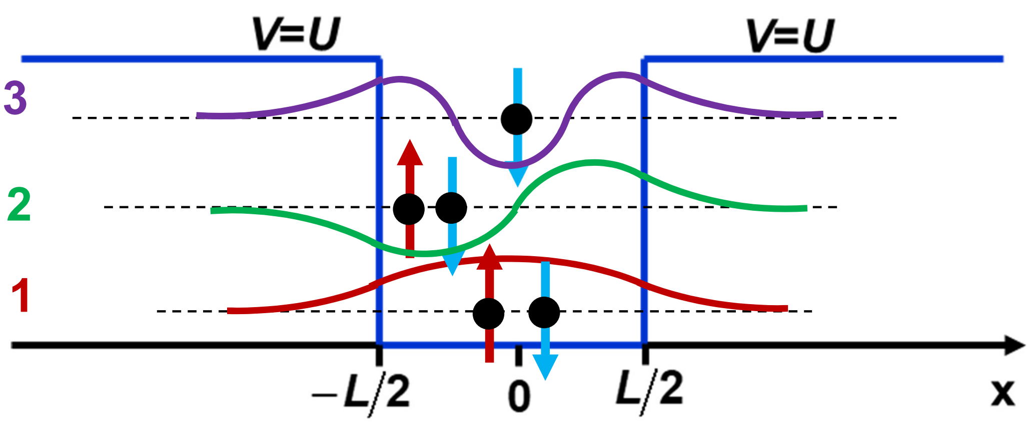


Electron Filling and Pauli's Exclusion Principle



No two electrons can have the exact same quantum state!!

Or



No two electrons can have the exact same quantum state!!