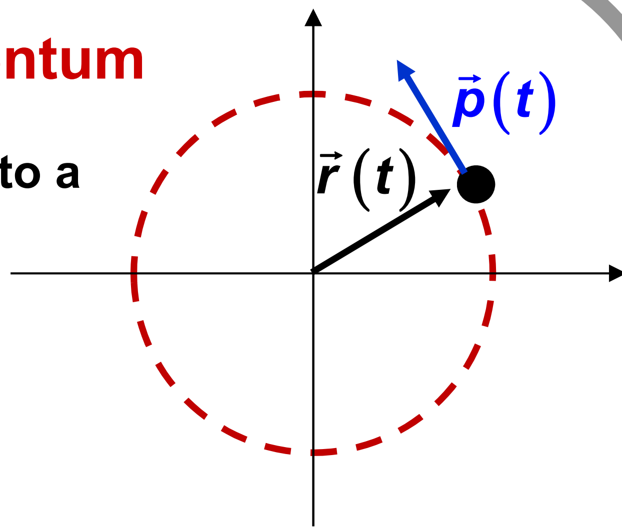
Lecture 24

Orbital Angular Momentum
And Spin Angular Momentum

In this lecture you will learn:

- Orbital angular momemtum
- Spin angular momentum
- Orbital angular momentum eigenstates and eigenvalues
- Spin angular momentum eigenstates and eigenvalues
- Spinor wavefunctions

Classical Orbital Angular Momentum



The **classical angular momentum** of a particle with respect to a point $\vec{r}_o$ is defined as:

$$\vec{L}(t) = \left[\vec{r}(t) - \vec{r}_o\right] \times \vec{p}(t)$$

Usually the point $\vec{r}_o$ is taken to be the origin and so:

$$\vec{L}(t) = \vec{r}(t) \times \vec{p}(t)$$

Angular momentum is a vector with **three components**:

$$\vec{L} = L_x\hat{x} + L_y\hat{y} + L_z\hat{z}$$

where:

$$L_x = yp_z - zp_y$$

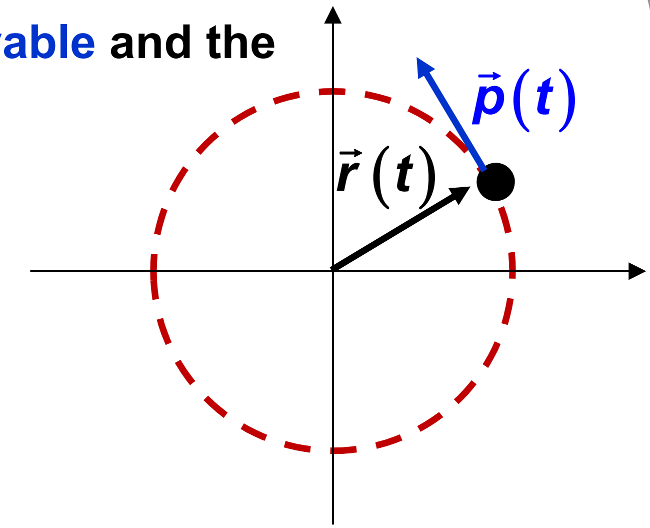
$$L_y = zp_x - xp_z$$

$$L_z = xp_y - yp_x$$

Finally the **squared magnitude of the angular momentum** is:

$$L^2 = \vec{L}\,.\,\vec{L} = L_x^2 + L_y^2 + L_z^2$$

Quantum Orbital Angular Momentum Operator



In quantum mechanics, the **angular momentum is an observable** and the corresponding operator is:

$$\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}} = -\hat{\vec{p}} \times \hat{\vec{r}}$$

It is a **vector operator** with three components:

$$\hat{\vec{L}} = \hat{L}_x \hat{x} + \hat{L}_y \hat{y} + \hat{L}_z \hat{z}$$

where:

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y$$

$$\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z$$

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$$

Finally the **squared magnitude of the angular momentum** is also an operator:

$$\hat{L}^2 = \hat{\vec{L}} \,.\, \hat{\vec{L}} = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$$

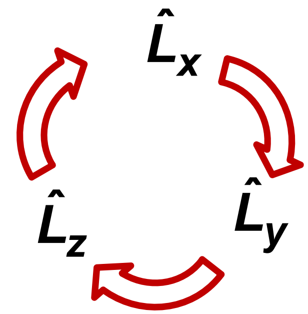
Orbital Angular Momentum Commutation Relations

It is not difficult to show (Homework 4) the following commutation relations:

$$\left[\hat{L}_x,\hat{L}_y\right]=i\hbar\hat{L}_z$$

$$\left[\hat{L}_y,\hat{L}_z\right]=i\hbar\hat{L}_x$$

$$\left[\hat{L}_z,\hat{L}_x\right]=i\hbar\hat{L}_y$$



Also, the $\hat{L}^2$ operator commutes with all three components of the angular momentum:

$$\left[\hat{L}^2,\hat{L}_x\right]=\left[\hat{L}^2,\hat{L}_y\right]=\left[\hat{L}^2,\hat{L}_z\right]=0$$

How do we find states of definite angular momentum? What do the commutation relations tell us?

- Eigenstates of one component of the angular momentum will likely NOT be eigenstates of the other two components of the angular momentum

- We can find eigenstates of one component of the angular momentum that will also be eigenstates of the $\hat{L}^2$ operator

Eigenstates and Eigenvalues of $\hat{L}_z$

We first try to find the eigenstates and eigenvalues of the z-component of the angular momentum operator

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$$

Let $|\psi\rangle$ be an eigenstate of $\hat{L}_z$ with eigenvalue λ:

$$\hat{L}_z|\psi\rangle = \lambda|\psi\rangle$$

In position basis, this becomes:

$$\langle\vec{r}|\hat{L}_z|\psi\rangle = \lambda\langle\vec{r}|\psi\rangle$$

$$\Rightarrow \langle x,y,z|\hat{x}\hat{p}_y - \hat{y}\hat{p}_x|\psi\rangle = \lambda\langle x,y,z|\psi\rangle$$

$$\Rightarrow \frac{\hbar}{i}\left[x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right]\psi(x,y,z) = \lambda\psi(x,y,z)$$

The equation becomes simpler in spherical coordinates:

$$\frac{\hbar}{i}\frac{\partial}{\partial\phi}\psi(r,\phi,\theta) = \lambda\psi(r,\phi,\theta)$$

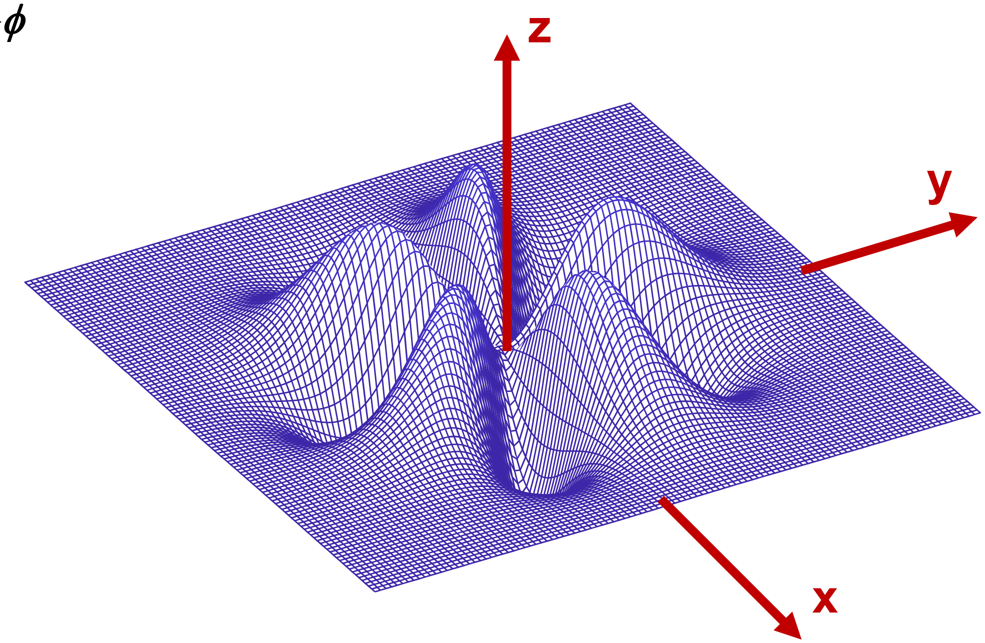
$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

Eigenstates and Eigenvalues of $\hat{L}_z$

$$\frac{\hbar}{i}\frac{\partial}{\partial\phi}\psi(r,\phi,\theta)=\lambda\psi(r,\phi,\theta)$$

Solution will give us only the ϕ-dependence of the eigenfunction:

$$\psi(r,\phi,\theta)=f(r,\theta)e^{i\frac{\lambda}{\hbar}\phi}$$



The circularly travelling wave must be in phase with itself after one complete roundtrip:

$$\Rightarrow\psi(r,\phi+2\pi,\theta)=\psi(r,\phi,\theta)$$

$$\Rightarrow e^{i\frac{\lambda}{\hbar}(\phi+2\pi)}=e^{i\frac{\lambda}{\hbar}\phi}$$

$$\Rightarrow e^{i\frac{\lambda}{\hbar}2\pi}=1$$

$$\Rightarrow\frac{\lambda}{\hbar}=m \qquad \{m=.....-3,-2,-1,0,+1,+2,+3,......$$

The eigenvalue of $\hat{L}_z$ must be an integral multiple of $\hbar$

$$\lambda=m\hbar \qquad \{m=.....-3,-2,-1,0,+1,+2,+3,......$$

Eigenstates and Eigenvalues of $\hat{L}_z$

The eigenstates of $\hat{L}_z$ can be labeled by the corresponding eigenvalue:

$$\hat{L}_z |m\rangle = m\hbar |m\rangle \qquad \{m = \ldots\ldots -2, -1, 0, +1, +2, \ldots\ldots$$

Since: $\left[\tilde{L}^2, \hat{L}_z\right] = 0$

the eigenstates $|m\rangle$ of $\hat{L}_z$ will also be eigenstates of $\tilde{L}^2$:

$$\tilde{L}^2 |m\rangle \propto |m\rangle$$

How do we find all the eigenstates and eigenvalues of $\tilde{L}^2$?

Eigenstates and Eigenvalues of $\tilde{L}^2$

Let $|\psi\rangle$ be an eigenstate of $\tilde{L}^2$ with eigenvalue λ :

$$\tilde{L}^2|\psi\rangle = \lambda|\psi\rangle$$

In position basis, this becomes:

$$\langle\vec{r}|\tilde{L}^2|\psi\rangle = \lambda\langle\vec{r}|\psi\rangle$$

$$\Rightarrow \langle r,\phi,\theta|\tilde{L}^2|\psi\rangle = \lambda\langle r,\phi,\theta|\psi\rangle$$

$$\Rightarrow \left(\frac{\hbar}{i}\right)^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi(r,\phi,\theta) = \lambda\psi(r,\phi,\theta)$$

This looks too complicated !!

Orbital Angular Momentum and Commutation Relations

The components of the angular momentum operator have the following commutation relations:

$$\hat{\vec{L}} = \hat{L}_x \hat{x} + \hat{L}_y \hat{y} + \hat{L}_z \hat{z}$$

$$\left[\hat{L}_x, \hat{L}_y\right] = i\hbar \hat{L}_z$$

$$\left[\hat{L}_y, \hat{L}_z\right] = i\hbar \hat{L}_x$$

$$\left[\hat{L}_z, \hat{L}_x\right] = i\hbar \hat{L}_y$$

$$\hat{L}^2 = \hat{\vec{L}} \,.\, \hat{\vec{L}} = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$$

$$\left[\hat{L}^2, \hat{L}_x\right] = \left[\hat{L}^2, \hat{L}_y\right] = \left[\hat{L}^2, \hat{L}_z\right] = 0$$

Eigenvalues and Eigenstates of the Angular Momentum

Whatever we do next will apply to both spin and orbital angular momentum

We need to find the eigenvalues of the angular momentum

Since $\left[\hat{L}^2, \hat{L}_z\right] = 0$, we seek states that are eigenstates of both $\hat{L}^2$ and $\hat{L}_z$:

$$\hat{L}_z |m\rangle = m\hbar |m\rangle$$
$$\hat{L}^2 |m\rangle = \lambda\hbar^2 |m\rangle$$

m and λ are some unknown numbers (not assuming anything here)

We can say the following:

$$|\omega\rangle = \hat{L}_x |\psi\rangle \quad \Rightarrow \quad \langle\omega|\omega\rangle = \langle\psi|\hat{L}_x^2|\psi\rangle \geq 0$$
$$|\chi\rangle = \hat{L}_y |\psi\rangle \quad \Rightarrow \quad \langle\chi|\chi\rangle = \langle\psi|\hat{L}_y^2|\psi\rangle \geq 0$$
$$|\phi\rangle = \hat{L}_z |\psi\rangle \quad \Rightarrow \quad \langle\phi|\phi\rangle = \langle\psi|\hat{L}_z^2|\psi\rangle \geq 0$$

It follows that all the eigenvalues $\hat{L}^2$ of must be positive semi-definite :

$$\hat{L}^2 |m\rangle = \lambda\hbar^2 |m\rangle$$
$$\Rightarrow \langle m|\hat{L}^2|m\rangle = \lambda\hbar^2 \geq 0$$
$$\Rightarrow \lambda \geq 0$$

Eigenvalues and Eigenstates of the Angular Momentum

$$\hat{L}_z |m\rangle = m\hbar |m\rangle$$
$$\hat{L}^2 |m\rangle = \lambda\hbar^2 |m\rangle$$

m and λ are some unknown numbers and λ is ≥ 0

We write λ as $\ell(\ell+1)$ where ℓ is some number that is also ≥ 0 (for convenience only):

$$\hat{L}_z |m\rangle = m\hbar |m\rangle$$
$$\hat{L}^2 |m\rangle = \ell(\ell+1)\hbar^2 |m\rangle$$

Define two new operators as:

$$\hat{L}_+ = \hat{L}_x + i\hat{L}_y \qquad \hat{L}_- = \hat{L}_x - i\hat{L}_y$$

The new operators have the following commutation relations:

$$\left[\hat{L}_z, \hat{L}_\pm\right] = \pm\hbar\hat{L}_\pm \qquad \left[\hat{L}^2, \hat{L}_\pm\right] = 0$$

And:

$$\hat{L}_+\hat{L}_- = \hat{L}_x^2 + \hat{L}_y^2 - i\left[\hat{L}_x, \hat{L}_y\right] = \hat{L}^2 - \hat{L}_z^2 + \hbar\hat{L}_z$$
$$\hat{L}_-\hat{L}_+ = \hat{L}_x^2 + \hat{L}_y^2 + i\left[\hat{L}_x, \hat{L}_y\right] = \hat{L}^2 - \hat{L}_z^2 - \hbar\hat{L}_z$$

Eigenvalues and Eigenstates of the Angular Momentum: $\hat{L}_+$

Start from an eigenstate of $\hat{L}_z$ and $\hat{L}^2$:

$$\hat{L}_z|m\rangle = m\hbar|m\rangle$$

$$\hat{L}^2|m\rangle = \ell(\ell+1)\hbar^2|m\rangle$$

m and *ℓ* are some unknown numbers and $\ell \geq 0$

And then consider the state:

$$\hat{L}_+|m\rangle$$

1) Apply $\hat{L}_z$ operator to it:

$$\hat{L}_z\hat{L}_+|m\rangle = \left(\left[\hat{L}_z,\hat{L}_+\right]+\hat{L}_+\hat{L}_z\right)|m\rangle = \left(\hbar\hat{L}_+ + \hat{L}_+ m\hbar\right)|m\rangle$$

$$= (m+1)\hbar \ \ \hat{L}_+|m\rangle$$

This means $\hat{L}_+|m\rangle$ is also an eigenstate of $\hat{L}_z$ with eigenvalue $(m+1)\hbar$

2) Apply $\hat{L}^2$ operator to it:

$$\hat{L}^2\hat{L}_+|m\rangle = \left(\left[\hat{L}^2,\hat{L}_+\right]+\hat{L}_+\hat{L}^2\right)|m\rangle = \left(0+\hat{L}_+\ell(\ell+1)\hbar^2\right)|m\rangle = \ell(\ell+1)\hbar^2 \ \ \hat{L}_+|m\rangle$$

This means $\hat{L}_+|m\rangle$ is also an eigenstate of $\hat{L}^2$ with the same eigenvalue $\ell(\ell+1)\hbar^2$

Eigenvalues and Eigenstates of the Angular Momentum: $\hat{L}_-$

Start from an eigenstate of $\hat{L}_z$ and $\hat{L}^2$:

$$\hat{L}_z|m\rangle = m\hbar|m\rangle$$

$$\hat{L}^2|m\rangle = \ell(\ell+1)\hbar^2|m\rangle$$

m and ℓ are some unknown numbers and $\ell \geq 0$

And then consider the state:

$$\hat{L}_-|m\rangle$$

1) Apply $\hat{L}_z$ operator to it:

$$\hat{L}_z\hat{L}_-|m\rangle = \left(\left[\hat{L}_z,\hat{L}_-\right]+\hat{L}_-\hat{L}_z\right)|m\rangle = \left(-\hbar\hat{L}_- + \hat{L}_- m\hbar\right)|m\rangle$$

$$= (m-1)\hbar \;\; \hat{L}_-|m\rangle$$

This means $\hat{L}_-|m\rangle$ is also an eigenstate of $\hat{L}_z$ with eigenvalue $(m-1)\hbar$

2) Apply $\hat{L}^2$ operator to it:

$$\hat{L}^2\hat{L}_-|m\rangle = \left(\left[\hat{L}^2,\hat{L}_-\right]+\hat{L}_-\hat{L}^2\right)|m\rangle = \left(0+\hat{L}_-\ell(\ell+1)\hbar^2\right)|m\rangle = \ell(\ell+1)\hbar^2 \;\; \hat{L}_-|m\rangle$$

This means $\hat{L}_-|m\rangle$ is also an eigenstate of $\hat{L}^2$ with the same eigenvalue $\ell(\ell+1)\hbar^2$

Eigenvalues and Eigenstates of the Angular Momentum

This means $\left(\hat{L}_+\right)^p |m\rangle$ is also an eigenstate of $\hat{L}_z$ with eigenvalue $(m+p)\hbar$ and it is also an eigenstate of $\hat{L}^2$ with eigenvalue $\ell(\ell+1)\hbar^2$

This means $\left(\hat{L}_-\right)^p |m\rangle$ is also an eigenstate of $\hat{L}_z$ with eigenvalue $(m-p)\hbar$ and it is also an eigenstate of $\hat{L}^2$ with eigenvalue $\ell(\ell+1)\hbar^2$

We can write:

$$\hat{L}_+|m\rangle \propto |m+1\rangle$$

$$\Rightarrow \hat{L}_+|m\rangle = A|m+1\rangle$$

$$\Rightarrow \langle m|\hat{L}_-\hat{L}_+|m\rangle = |A|^2 \langle m+1|m+1\rangle$$

$$\Rightarrow \langle m|\hat{L}^2 - \hat{L}_z^2 - \hbar\hat{L}_z|m\rangle = |A|^2$$

$$\Rightarrow |A| = \hbar\sqrt{\ell(\ell+1) - m(m+1)}$$

$$\Rightarrow \hat{L}_+|m\rangle = \hbar\sqrt{\ell(\ell+1) - m(m+1)}\,|m+1\rangle$$

Similarly:

$$\hat{L}_-|m\rangle = \hbar\sqrt{\ell(\ell+1) - m(m-1)}\,|m-1\rangle$$

Eigenvalues and Eigenstates of the Angular Momentum

1) Consider the state:

$$\hat{L}_-|m\rangle = \hbar\sqrt{\ell(\ell+1)-m(m-1)}\,|m-1\rangle$$

The inner product of this state with itself must be non-negative:

$$\langle m|\hat{L}_+\hat{L}_-|m\rangle = \hbar\left[\ell(\ell+1)-m(m-1)\right] \geq 0$$

$$\boxed{\Rightarrow -\ell \leq m \leq \ell+1}$$

2) Consider the state:

$$\hat{L}_+|m\rangle = \hbar\sqrt{\ell(\ell+1)-m(m+1)}\,|m+1\rangle$$

The inner product of this state with itself must be non-negative:

$$\langle m|\hat{L}_-\hat{L}_+|m\rangle = \hbar\left[\ell(\ell+1)-m(m+1)\right] \geq 0$$

$$\boxed{\Rightarrow -(\ell+1) \leq m \leq \ell}$$

(1) and (2) above give:

$$\boxed{-\ell \leq m \leq \ell}$$

Largest Possible Value of *m*

Suppose we have a state $|m\rangle$ with the largest possible value of m such that:

$$m \leq \ell$$

$$m + 1 > \ell$$

Recall that:

$$-\ell \leq m \leq \ell$$

Then application of $\hat{L}_+$ to this state should not give us another state, but we know that:

$$\hat{L}_+ |m\rangle = \hbar\sqrt{\ell(\ell+1) - m(m+1)}\,|m+1\rangle$$

The only way to ensure that we do not get a bonafide state by applying $\hat{L}_+$ to $|m\rangle$ is by requiring that the largest allowed value of m must be equal to ℓ, because then:

$$\hat{L}_+ |m\rangle = 0 \qquad \text{when } m = \ell$$

This means that the largest allowed value of m is EXACTLY equal to ℓ and:

$$\hat{L}_+ |m = \ell\rangle = 0$$

Smallest Possible Value of *m*

Suppose we have a state $|m\rangle$ with the smallest possible value of *m* such that:

$$-\ell \le m$$

$$-\ell > m - 1$$

Recall that:

$$-\ell \le m \le \ell$$

Then application of $\hat{L}_-$ to this state should not give us another state, but we know that:

$$\hat{L}_- |m\rangle = \hbar\sqrt{\ell(\ell+1) - m(m-1)}\,|m-1\rangle$$

The only way to ensure that we do not get a bonafide state by applying $\hat{L}_-$ to $|m\rangle$ is by requiring that the smallest allowed value of *m* must be equal to -*ℓ*, because then:

$$\hat{L}_- |m\rangle = 0 \qquad \text{when } m = -\ell$$

This means that the smallest allowed value of *m* is EXACTLY equal to –*ℓ* and:

$$\hat{L}_- |m = -\ell\rangle = 0$$

Eigenstates of $\hat{L}_z$ and $\hat{L}^2$

Eigenstates of $\hat{L}_z$ must be:

$$\left.\begin{array}{l} |m=\ell\rangle \\ |m=\ell-1\rangle \\ |m=\ell-2\rangle \\ \vdots \\ \vdots \\ |m=-\ell+2\rangle \\ |m=-\ell+1\rangle \\ |m=-\ell\rangle \end{array}\right\}$$

$$\left.\begin{array}{l} \hat{L}_z|m\rangle = m\hbar|m\rangle \\ \hat{L}^2|m\rangle = \ell(\ell+1)\hbar^2|m\rangle \end{array}\right\} -\ell \le m \le \ell$$

These states are all eigenstates of $\hat{L}^2$ with the same eigenvalue $\ell(\ell+1)\hbar^2$

It must be possible to reach $-\ell$ from $+\ell$ by subtracting a positive integer "p", i.e.:

$$\ell - p = -\ell$$
$$\Rightarrow 2\ell = p$$
$$\Rightarrow \ell = \frac{p}{2}$$

This means ℓ must be a positive integer or positive half-integer

But the requirement of having a single-valued wavefunction means ℓ must be a positive integer

Eigenstates of $\hat{L}_z$ and $\hat{L}^2$

One can classify the angular momentum eigenstates by the quantum numbers "ℓ" and "m" as follows:

$\boxed{\ell=0}$

$$\left|\ell=0,m=0\right\rangle$$

$$-0 \le m \le 0$$

$$\hat{L}_z\left|\ell=0,m=0\right\rangle = 0\left|\ell=0,m=0\right\rangle = 0$$

$$\hat{L}^2\left|\ell=0,m=0\right\rangle = 0(0+1)\hbar^2\left|\ell=0,m=0\right\rangle = 0$$

$\boxed{\ell=1}$

$$\left|\ell=1,m=+1\right\rangle$$

$$\left|\ell=1,m=0\right\rangle$$

$$\left|\ell=1,m=-1\right\rangle$$

$$-1 \le m \le 1$$

$$\hat{L}_z\left|\ell=1,m\right\rangle = m\hbar\left|\ell=1,m\right\rangle$$

$$\hat{L}^2\left|\ell=1,m\right\rangle = \ell(\ell+1)\hbar^2\left|\ell=1,m\right\rangle$$

$$= 1(1+1)\hbar^2\left|\ell=1,m\right\rangle = 2\hbar^2\left|\ell=1,m\right\rangle$$

Eigenstates of $\hat{L}_z$ and $\hat{L}^2$

One can classify the angular momentum eigenstates by the quantum numbers "*ℓ*" and "*m*" as follows:

ℓ=2

$$
\left.
\begin{array}{l}
|\ell = 2, m = +2\rangle \\
|\ell = 2, m = +1\rangle \\
|\ell = 2, m = 0\rangle \\
|\ell = 2, m = -1\rangle \\
|\ell = 2, m = -2\rangle
\end{array}
\right\}
$$

$$-2 \le m \le 2$$

$$\hat{L}_z |\ell = 2, m\rangle = m\hbar |\ell = 2, m\rangle$$

$$\hat{L}^2 |\ell = 2, m\rangle = \ell(\ell+1)\hbar^2 |\ell = 2, m\rangle$$

$$= 2(2+1)\hbar^2 |\ell = 2, m\rangle = 6\hbar^2 |\ell = 2, m\rangle$$

ℓ=3

$$\vdots$$

Eigenstates of $\hat{L}_z$ and $\hat{L}^2$: Orthogonality

Orthogonality:

Since the states $|\ell, m\rangle$ are eigenstates of the Hermitian operator $\hat{L}_z$, therefore:

$$\langle \ell', m' | \ell, m \rangle = \delta_{m,m'}\, \delta_{\ell,\ell'}$$

Wavefunctions for the Eigenstates of $\hat{L}^2$ and $\hat{L}_z$

Let $|\psi\rangle$ be an eigenstate of $\hat{L}^2$:

$$\hat{L}^2 |\psi\rangle = \ell(\ell+1)\hbar^2 |\psi\rangle$$

In position basis, this becomes:

$$\langle \vec{r} | \hat{L}^2 | \psi\rangle = \ell(\ell+1)\hbar^2 \langle \vec{r} | \psi\rangle$$

$$\Rightarrow \langle r,\phi,\theta | \hat{L}^2 | \psi\rangle = \ell(\ell+1)\hbar^2 \langle r,\phi,\theta | \psi\rangle$$

$$\Rightarrow \left(\frac{\hbar}{i}\right)^2 \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi(r,\phi,\theta) = \ell(\ell+1)\hbar^2\psi(r,\phi,\theta)$$

Let: $\psi(r,\phi,\theta) = e^{im\phi} f(\theta) g(r)$

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi(r,\phi,\theta) = -\ell(\ell+1)\psi(r,\phi,\theta)$$

$$\Rightarrow \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{m^2}{\sin^2\theta}\right] f(\theta) = -\ell(\ell+1) f(\theta)$$

Spherical Harmonics

Let $|\psi\rangle$ be an eigenstate of $\hat{L}^2$: $\quad \hat{L}^2|\psi\rangle = \ell(\ell+1)\hbar^2|\psi\rangle$

Let: $\psi(r,\phi,\theta) = e^{im\phi} f(\theta) g(r)$

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{m^2}{\sin^2\theta}\right] f(\theta) = -\ell(\ell+1) f(\theta)$$

Solutions are: $\quad f(\theta) = \underbrace{P_\ell^m(\cos\theta)}_{\text{Associated Legendre polynomial}} \qquad \{ -\ell \le m \le \ell$

$$\Rightarrow \psi(r,\phi,\theta) = g(r)\underbrace{P_\ell^m(\cos\theta)e^{im\phi}}_{\text{Spherical harmonic } \propto Y_\ell^m(\theta,\phi)}$$

$$Y_\ell^m(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\; P_\ell^m(\cos\theta)e^{im\phi}$$

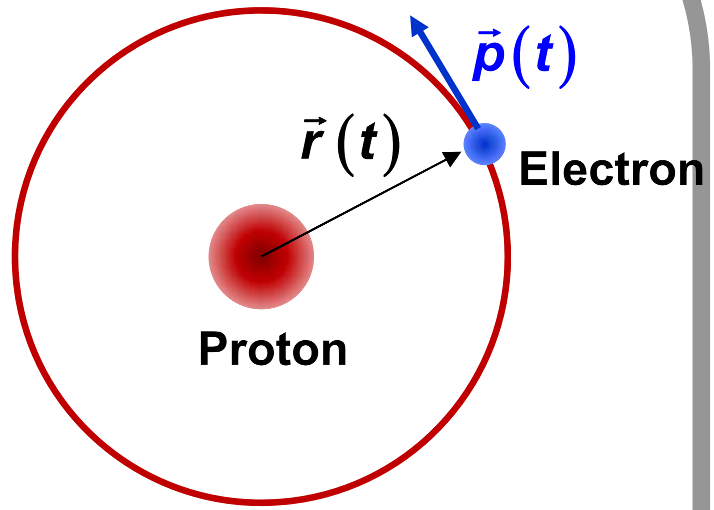
Usually one writes the wavefunctions for the eigenstates of $\hat{L}^2$ and $\hat{L}_z$ as:

$$\boxed{\psi(r,\phi,\theta) = \chi(r)Y_\ell^m(\theta,\phi) = \langle r,\phi,\theta|\ell,m\rangle}$$

Orbital Angular Momentum and the Hydrogen Atom

The proton is ~1837 times more massive than an electron

As a crude approximation one may consider the proton to be stationary and focus just on the electron



The electron Hamiltonian is:

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\vec{r})$$

$$V(\hat{\vec{r}}) = -\frac{e^2}{4\pi\varepsilon_o \left|\hat{\vec{r}}\right|}$$

Coulomb potential

We need to find the energy eigenstates and the energy eigenvalues:

$$\hat{H}|\phi\rangle = E|\phi\rangle$$

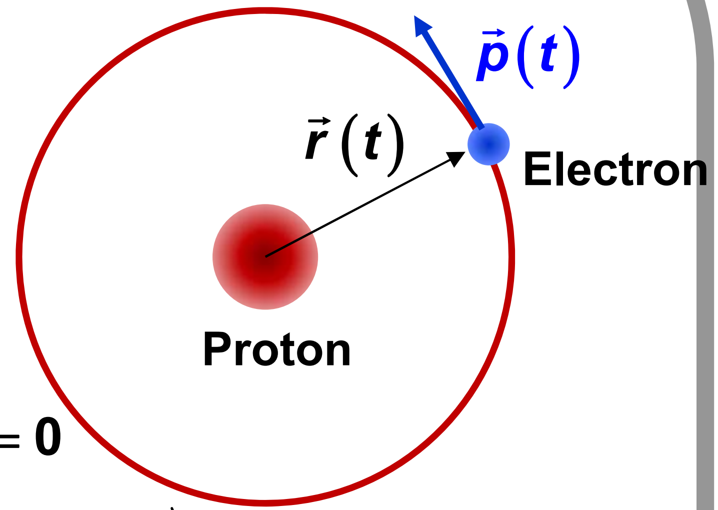
We also know from the spherical symmetry of the potential that:

$$\left[\hat{H}, \hat{L}^2\right] = \left[\hat{H}, \hat{L}_z\right] = 0$$

This means that the angular momentum eigenstates could also be the energy eigenstates!!

Orbital Angular Momentum and the Hydrogen Atom

$$\left[\hat{H},\hat{L}^2\right]=\left[\hat{H},\hat{L}_z\right]=0$$



The angular momentum eigenstates could also be the energy eigenstates!!

ℓ=0

$$\left|\ell=0,m=0\right\rangle$$

$$\hat{L}_z\left|\ell=0,m=0\right\rangle=0\left|\ell=0,m=0\right\rangle=0$$

$$\hat{L}^2\left|\ell=0,m=0\right\rangle=0(0+1)\hbar^2\left|\ell=0,m=0\right\rangle=0$$

S-orbitals of the hydrogen atom (1s, 2s, 3s, 4s, …..)

ℓ=1

$$\left|\ell=1,m=+1\right\rangle$$
$$\left|\ell=1,m=0\right\rangle$$
$$\left|\ell=1,m=-1\right\rangle$$

$$-1\le m\le 1$$

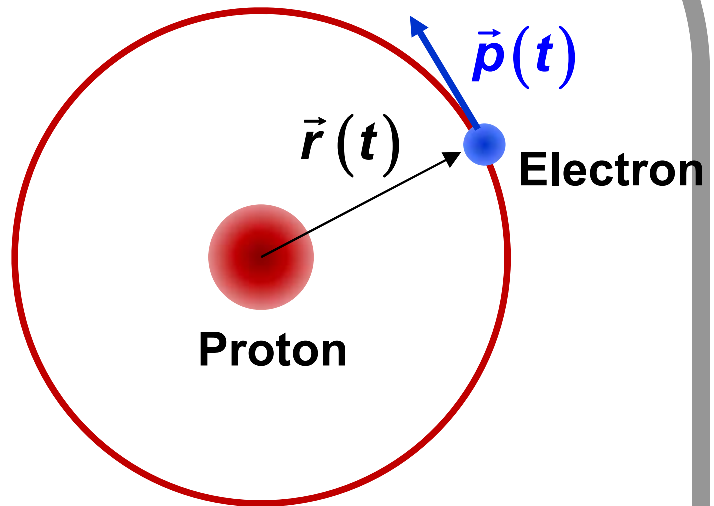
$$\hat{L}_z\left|\ell=1,m\right\rangle=m\hbar\left|\ell=1,m\right\rangle$$

$$\hat{L}^2\left|\ell=1,m\right\rangle=\ell(\ell+1)\hbar^2\left|\ell=1,m\right\rangle$$
$$=1(1+1)\hbar^2\left|\ell=1,m\right\rangle=2\hbar^2\left|\ell=1,m\right\rangle$$

P-orbitals of the hydrogen atom (2p, 3p, 4p, ….)

Orbital Angular Momentum and the Hydrogen Atom

$$\left[\hat{H},\hat{L}^2\right]=\left[\hat{H},\hat{L}_z\right]=0$$



The angular momentum eigenstates could also be the energy eigenstates!!

ℓ=2

$$\left.\begin{array}{l}|\ell=2,m=+2\rangle\\ |\ell=2,m=+1\rangle\\ |\ell=2,m=0\rangle\\ |\ell=2,m=-1\rangle\\ |\ell=2,m=-2\rangle\end{array}\right]$$

$$-2\le m\le 2$$

$$\hat{L}_z|\ell=2,m\rangle=m\hbar|\ell=2,m\rangle$$

$$\hat{L}^2|\ell=2,m\rangle=\ell(\ell+1)\hbar^2|\ell=2,m\rangle$$

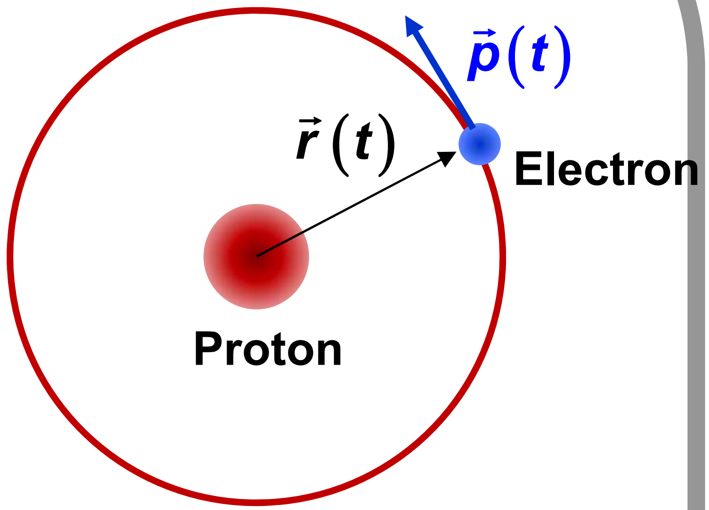
$$=2(2+1)\hbar^2|\ell=2,m\rangle=6\hbar^2|\ell=2,m\rangle$$

D-orbitals of the hydrogen atom (3d, 4d, 5d, …..)

CSCO for the Hydrogen Atom

Knowing the angular momentum numbers, m and ℓ, are not enough to uniquely specify all states of the Hydrogen atom.

In other words, $\hat{L}_z, \hat{L}^2$ by themselves do not form a CSCO



The electron Hamiltonian is:

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\vec{r})$$

$$V(\hat{\vec{r}}) = -\frac{e^2}{4\pi\varepsilon_o |\hat{\vec{r}}|}$$

Coulomb potential

There are many different states with the same m, ℓ values but they do have different energy eigenvalues

And since $\hat{H}, \hat{L}_z, \hat{L}^2$ all commute, we can form a CSCO with these three operators!

Spin Angular Momentum of Massive Particles

The components of the spin angular momentum operator have the same commutation relations as the components of the orbital angular momentum operator:

$$\hat{\vec{S}} = \hat{S}_x \hat{x} + \hat{S}_y \hat{y} + \hat{S}_z \hat{z}$$

$$\left[\hat{S}_x, \hat{S}_y\right] = i\hbar \hat{S}_z$$

$$\left[\hat{S}_y, \hat{S}_z\right] = i\hbar \hat{S}_x$$

$$\left[\hat{S}_z, \hat{S}_x\right] = i\hbar \hat{S}_y$$

In addition if we define $\hat{S}^2$ operator as:

$$\hat{S}^2 = \hat{\vec{S}}.\hat{\vec{S}} = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2$$

Then:

$$\left[\hat{S}^2, \hat{S}_x\right] = \left[\hat{S}^2, \hat{S}_y\right] = \left[\hat{S}^2, \hat{S}_z\right] = 0$$

Eigenstates and eigenvalues of the z-component:

$$\hat{S}_z |\psi\rangle = ?|\psi\rangle$$

No wavefunction for spin states

$$\hat{\vec{L}} = \hat{L}_x \hat{x} + \hat{L}_y \hat{y} + \hat{L}_z \hat{z}$$

$$\left[\hat{L}_x, \hat{L}_y\right] = i\hbar \hat{L}_z$$

$$\left[\hat{L}_y, \hat{L}_z\right] = i\hbar \hat{L}_x$$

$$\left[\hat{L}_z, \hat{L}_x\right] = i\hbar \hat{L}_y$$

$$\hat{L}^2 = \hat{\vec{L}}\,.\,\hat{\vec{L}} = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$$

$$\left[\hat{L}^2, \hat{L}_x\right] = \left[\hat{L}^2, \hat{L}_y\right] = \left[\hat{L}^2, \hat{L}_z\right] = 0$$

$$\hat{L}_z |\psi\rangle = m\hbar |\psi\rangle$$

$$\{m = \ldots\ldots -2, -1, 0, +1, +2, \ldots\ldots$$

This came from the wavefunction being single-valued

Eigenvalues and Eigenstates of the Spin Angular Momentum

Whatever we do next will applies to both spin and orbital angular momentum

We need to find the eigenvalues of the angular momentum

Since $\left[\hat{S}^2, \hat{S}_z\right] = 0$, we seek states that are eigenstates of both $\hat{S}^2$ and $\hat{S}_z$:

$$\hat{S}_z |m\rangle = m\hbar |m\rangle$$

$$\hat{S}^2 |m\rangle = \lambda\hbar^2 |m\rangle$$

m and λ are some unknown numbers

We can say the following:

$$|\omega\rangle = \hat{S}_x |\psi\rangle \quad \Rightarrow \quad \langle\omega|\omega\rangle = \langle\psi|\hat{S}_x^2|\psi\rangle \geq 0$$

$$|\chi\rangle = \hat{S}_y |\psi\rangle \quad \Rightarrow \quad \langle\chi|\chi\rangle = \langle\psi|\hat{S}_y^2|\psi\rangle \geq 0$$

$$|\phi\rangle = \hat{S}_z |\psi\rangle \quad \Rightarrow \quad \langle\phi|\phi\rangle = \langle\psi|\hat{S}_z^2|\psi\rangle \geq 0$$

It follows that all the eigenvalues $\hat{S}^2$ of must be positive semi-definite :

$$\hat{S}^2 |m\rangle = \lambda\hbar^2 |m\rangle$$

$$\Rightarrow \langle m|\hat{S}^2|m\rangle = \lambda\hbar^2 \geq 0$$

$$\Rightarrow \lambda \geq 0$$

Eigenvalues and Eigenstates of the Spin Angular Momentum

$$\hat{S}_z \left| m \right\rangle = m\hbar \left| m \right\rangle$$
$$\hat{S}^2 \left| m \right\rangle = \lambda\hbar^2 \left| m \right\rangle$$

m and λ are some unknown numbers and λ is ≥ 0

We write λ as $s(s+1)$ where "s" is some number that is also ≥ 0 (for convenience only):

$$\hat{S}_z \left| m \right\rangle = m\hbar \left| m \right\rangle$$
$$\hat{S}^2 \left| m \right\rangle = s\left(s+1\right)\hbar^2 \left| m \right\rangle$$

Define two new operators as:

$$\hat{S}_+ = \hat{S}_x + i\hat{S}_y \qquad \hat{S}_- = \hat{S}_x - i\hat{S}_y$$

The new operators have the following commutation relations:

$$\left[\hat{S}_z, \hat{S}_\pm\right] = \pm\hbar\hat{S}_\pm \qquad \left[\hat{S}^2, \hat{S}_\pm\right] = 0$$

And:

$$\hat{S}_+\hat{S}_- = \hat{S}_x^2 + \hat{S}_y^2 - i\left[\hat{S}_x, \hat{S}_y\right] = \hat{S}^2 - \hat{S}_z^2 + \hbar\hat{S}_z$$
$$\hat{S}_-\hat{S}_+ = \hat{S}_x^2 + \hat{S}_y^2 + i\left[\hat{S}_x, \hat{S}_y\right] = \hat{S}^2 - \hat{S}_z^2 - \hbar\hat{S}_z$$

Eigenvalues and Eigenstates of the Spin Angular Momentum: $\hat{S}_+$

Start from an eigenstate of $\hat{S}_z$ and $\hat{S}^2$:

$$\hat{S}_z |m\rangle = m\hbar |m\rangle$$

$$\hat{S}^2 |m\rangle = s(s+1)\hbar^2 |m\rangle$$

m and *s* are some unknown numbers and $s \geq 0$

And then consider the state:

$$\hat{S}_+ |m\rangle$$

1) Apply $\hat{S}_z$ operator to it:

$$\hat{S}_z \hat{S}_+ |m\rangle = \left(\left[\hat{S}_z, \hat{S}_+\right] + \hat{S}_+ \hat{S}_z\right)|m\rangle = \left(\hbar \hat{S}_+ + \hat{S}_+ m\hbar\right)|m\rangle$$

$$= (m+1)\hbar \;\; \hat{S}_+ |m\rangle$$

This means $\hat{S}_+ |m\rangle$ is also an eigenstate of $\hat{S}_z$ with eigenvalue $(m+1)\hbar$

2) Apply $\hat{S}^2$ operator to it:

$$\hat{S}^2 \hat{S}_+ |m\rangle = \left(\left[\hat{S}^2, \hat{S}_+\right] + \hat{S}_+ \hat{S}^2\right)|m\rangle = \left(0 + \hat{S}_+ s(s+1)\hbar^2\right)|m\rangle = s(s+1)\hbar^2 \;\; \hat{S}_+ |m\rangle$$

This means $\hat{S}_+ |m\rangle$ is also an eigenstate of $\hat{S}^2$ with the same eigenvalue $s(s+1)\hbar^2$

Eigenvalues and Eigenstates of the Spin Angular Momentum: $\hat{S}_-$

Start from an eigenstate of $\hat{S}_z$ and $\hat{S}^2$:

$$\hat{S}_z|m\rangle = m\hbar|m\rangle$$
$$\hat{S}^2|m\rangle = s(s+1)\hbar^2|m\rangle$$

m and *s* are some unknown numbers and $s \geq 0$

And then consider the state:

$$\hat{S}_-|m\rangle$$

1) Apply $\hat{S}_z$ operator to it:

$$\hat{S}_z\hat{S}_-|m\rangle = \left(\left[\hat{S}_z,\hat{S}_-\right]+\hat{S}_+\hat{S}_z\right)|m\rangle = \left(-\hbar\hat{S}_- + \hat{S}_+ m\hbar\right)|m\rangle$$
$$= (m-1)\hbar \;\; \hat{S}_-|m\rangle$$

This means $\hat{S}_-|m\rangle$ is also an eigenstate of $\hat{S}_z$ with eigenvalue $(m-1)\hbar$

2) Apply $\hat{S}^2$ operator to it:

$$\hat{S}^2\hat{S}_-|m\rangle = \left(\left[\hat{S}^2,\hat{S}_-\right]+\hat{S}_-\hat{S}^2\right)|m\rangle = \left(0+\hat{S}_- s(s+1)\hbar^2\right)|m\rangle = s(s+1)\hbar^2 \;\; \hat{S}_-|m\rangle$$

This means $\hat{S}_-|m\rangle$ is also an eigenstate of $\hat{S}^2$ with the same eigenvalue $s(s+1)\hbar^2$

Eigenvalues and Eigenstates of the Spin Angular Momentum

This means $\left(\hat{S}_+\right)^p |m\rangle$ is also an eigenstate of $\hat{S}_z$ with eigenvalue $(m+p)\hbar$ and it is also an eigenstate of $\hat{S}^2$ with eigenvalue $s(s+1)\hbar^2$

This means $\left(\hat{S}_-\right)^p |m\rangle$ is also an eigenstate of $\hat{S}_z$ with eigenvalue $(m-p)\hbar$ and it is also an eigenstate of $\hat{S}^2$ with eigenvalue $s(s+1)\hbar^2$

We can write:

$$\hat{S}_+ |m\rangle \propto |m+1\rangle$$

$$\Rightarrow \hat{S}_+ |m\rangle = A|m+1\rangle$$

$$\Rightarrow \langle m|\hat{S}_-\hat{S}_+|m\rangle = |A|^2 \langle m+1|m+1\rangle$$

$$\Rightarrow \langle m|\hat{S}^2 - \hat{S}_z^2 - \hbar\hat{S}_z|m\rangle = |A|^2$$

$$\Rightarrow |A| = \hbar\sqrt{s(s+1) - m(m+1)}$$

$$\Rightarrow \hat{S}_+ |m\rangle = \hbar\sqrt{s(s+1) - m(m+1)}\,|m+1\rangle$$

Similarly:

$$\hat{S}_- |m\rangle = \hbar\sqrt{s(s+1) - m(m-1)}\,|m-1\rangle$$

Eigenvalues and Eigenstates of the Spin Angular Momentum

1) Consider the state:

$$\hat{S}_- |m\rangle = \hbar\sqrt{s(s+1)-m(m-1)}\,|m-1\rangle$$

The inner product of this state with itself must be non-negative:

$$\langle m|\hat{S}_+\hat{S}_-|m\rangle = \hbar\left[s(s+1)-m(m-1)\right] \geq 0$$

$$\boxed{\Rightarrow -s \leq m \leq s+1}$$

2) Consider the state:

$$\hat{S}_+ |m\rangle = \hbar\sqrt{s(s+1)-m(m+1)}\,|m+1\rangle$$

The inner product of this state with itself must be non-negative:

$$\langle m|\hat{S}_-\hat{S}_+|m\rangle = \hbar\left[s(s+1)-m(m+1)\right] \geq 0$$

$$\boxed{\Rightarrow -(s+1) \leq m \leq s}$$

(1) and (2) above give:

$$\boxed{-s \leq m \leq s}$$

Largest Possible Value of *m*

Suppose we have a state $|m\rangle$ with the largest possible value of m such that:

$$m \leq s$$

$$m + 1 > s$$

Recall that:

$$-s \leq m \leq s$$

Then application of $\hat{S}_+$ to this state should not give us another state, but we know that:

$$\hat{S}_+ |m\rangle = \hbar\sqrt{s(s+1) - m(m+1)}\,|m+1\rangle$$

The only way to ensure that we do not get a bonafide state by applying $\hat{S}_+$ to $|m\rangle$ is by requiring that the largest allowed value of m must be equal to s, because then:

$$\hat{S}_+ |m\rangle = 0 \qquad \text{when } m = s$$

This means that the largest allowed value of m is EXACTLY equal to s and:

$$\hat{S}_+ |m = s\rangle = 0$$

Smallest Possible Value of *m*

Suppose we have a state $|m\rangle$ with the smallest possible value of *m* such that:

$$-s \le m$$

$$-s > m-1$$

Recall that:

$$-s \le m \le s$$

Then application of $\hat{S}_-$ to this state should not give us another state, but we know that:

$$\hat{S}_- |m\rangle = \hbar\sqrt{s(s+1)-m(m-1)}\,|m-1\rangle$$

The only way to ensure that we do not get a bonafide state by applying $\hat{S}_-$ to $|m\rangle$ is by requiring that the smallest allowed value of *m* must be equal to -*s*, because then:

$$\hat{S}_- |m\rangle = 0 \qquad \text{when } m = -s$$

This means that the smallest allowed value of *m* is EXACTLY equal to –*s* and:

$$\hat{S}_- |m=-s\rangle = 0$$

Eigenstates of $\hat{S}_z$ and $\hat{S}^2$

Eigenstates of $\hat{S}_z$ must be:

$$\left.\begin{array}{l} |m = s\rangle \\ |m = s-1\rangle \\ |m = s-2\rangle \\ \vdots \\ \vdots \\ |m = -s+2\rangle \\ |m = -s+1\rangle \\ |m = -s\rangle \end{array}\right\}$$

$$\left.\begin{array}{l} \hat{S}_z|m\rangle = m\hbar|m\rangle \\ \hat{S}^2|m\rangle = s(s+1)\hbar^2|m\rangle \end{array}\right\} -s \le m \le s$$

These states are all eigenstates of $\hat{S}^2$ with the same eigenvalue $s(s+1)\hbar^2$

It must be possible to reach $-s$ from $+s$ by subtracting a positive integer "p", i.e.:

$$s - p = -s$$
$$\Rightarrow 2s = p$$
$$\Rightarrow s = \frac{p}{2}$$

This means s must be a positive integer or positive half-integer

Eigenstates of $\hat{S}_z$ and $\hat{S}^2$

One can classify the angular momentum eigenstates by the quantum numbers "*s*" and "*m*" as follows:

s=0

$$|s=0,m=0\rangle$$

$$-0 \le m \le 0$$

$$\hat{S}_z|s=0,m=0\rangle = 0|s=0,m=0\rangle = 0$$

$$\hat{S}^2|s=0,m=0\rangle = 0(0+1)\hbar^2|s=0,m=0\rangle = 0$$

s=1/2

$$|s=1/2,m=+1/2\rangle$$
$$|s=1/2,m=-1/2\rangle$$

$$-\frac{1}{2} \le m \le \frac{1}{2}$$

$$\hat{S}_z|s=1/2,m\rangle = m\hbar|s=1/2,m\rangle$$

$$\hat{S}^2|s=1/2,m\rangle = s(s+1)\hbar^2|s=1/2,m\rangle$$
$$= \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2|s=1/2,m\rangle = \frac{3}{4}\hbar^2|s=1/2,m\rangle$$

Eigenstates of $\hat{S}_z$ and $\hat{S}^2$

One can classify the angular momentum eigenstates by the quantum numbers "*s*" and "*m*" as follows:

s=1

$$\left.\begin{array}{l} |s=1, m=+1\rangle \\ |s=1, m=0\rangle \\ |s=1, m=-1\rangle \end{array}\right\}$$

$$-1 \leq m \leq 1$$

$$\hat{S}_z|s=1/2, m\rangle = m\hbar|s=1/2, m\rangle$$

$$\hat{S}^2|s=1/2, m\rangle = s(s+1)\hbar^2|s=1/2, m\rangle$$

$$= 1(1+1)\hbar^2|s=1/2, m\rangle = 2\hbar^2|s=1/2, m\rangle$$

s=3/2

$$\left.\begin{array}{l} |s=3/2, m=+3/2\rangle \\ |s=3/2, m=+1/2\rangle \\ |s=3/2, m=-1/2\rangle \\ |s=3/2, m=-3/2\rangle \end{array}\right\}$$

$$-3/2 \leq m \leq 3/2$$

$$\hat{S}_z|s=3/2, m\rangle = m\hbar|s=3/2, m\rangle$$

$$\hat{S}^2|s=3/2, m\rangle = s(s+1)\hbar^2|s=3/2, m\rangle$$

$$= \frac{3}{2}\left(\frac{3}{2}+1\right)\hbar^2|s=3/2, m\rangle = \frac{15}{4}\hbar^2|s=3/2, m\rangle$$

Fermions (with Mass)

Fermions are particles which are eigenstates of $\hat{S}^2$ with s equal to **half-integer:**

$$\hat{S}^2 |s,m\rangle = s(s+1)\hbar^2 |s,m\rangle \qquad \text{where} \quad s = \frac{1}{2} \text{ or } \frac{3}{2} \text{ or } \frac{5}{2} \ \ldots\ldots\ldots$$

s is called the spin of the particle

And where:

$$\hat{S}_z |s,m\rangle = m\hbar |s,m\rangle \qquad \left\{ -s \le m \le s \right.$$

Example: Electrons

For electrons s is equal to ½ and therefore eigenstates of $\hat{S}_z$ are:

$$\hat{S}_z \left|\frac{1}{2},+\frac{1}{2}\right\rangle = +\frac{\hbar}{2}\left|\frac{1}{2},+\frac{1}{2}\right\rangle$$

$$\hat{S}_z \left|\frac{1}{2},-\frac{1}{2}\right\rangle = -\frac{\hbar}{2}\left|\frac{1}{2},-\frac{1}{2}\right\rangle$$

Or if you prefer:

$$\hat{S}_z |z\uparrow\rangle = +\frac{\hbar}{2}|z\uparrow\rangle$$

$$\hat{S}_z |z\downarrow\rangle = -\frac{\hbar}{2}|z\downarrow\rangle$$

$$\hat{S}^2 \left|\frac{1}{2},m\right\rangle = \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 \left|\frac{1}{2},m\right\rangle = \frac{3}{4}\hbar^2 \left|\frac{1}{2},m\right\rangle$$

Bosons (with Mass)

Bosons are particles which are eigenstates of $\hat{S}^2$ with s equal to an integer:

$$\hat{S}^2|s,m\rangle = s(s+1)\hbar^2|s,m\rangle$$

where $s = 0$ or 1 or 2 ....

s is called the spin of the particle

And where:

$$\hat{S}_z|s,m\rangle = m\hbar|s,m\rangle \qquad \Big\{ -s \le m \le s$$

Example: W and Z vector bosons (that are responsible for the electroweak interactions)

For W and Z bosons, s is equal to 1

$$\hat{S}_z|1,+1\rangle = +\hbar|1,+1\rangle$$

$$\hat{S}_z|1,0\rangle = 0|1,0\rangle$$

$$\hat{S}_z|1,-1\rangle = -\hbar|1,-1\rangle$$

$$\hat{S}^2|1,m\rangle = 1(1+1)\hbar^2|1,m\rangle = 2\hbar^2|1,m\rangle$$

Spin vs Orbital Angular Momentum

Spin angular momentum and orbital angular momentum are similar in many ways:

$$\hat{L}^2|\ell,m\rangle=\ell(\ell+1)\hbar^2|\ell,m\rangle \quad \text{where} \quad \ell=0 \text{ or } 1 \text{ or } 2 \text{ ....}$$

$$\hat{L}_z|\ell,m\rangle=m\hbar|\ell,m\rangle \qquad -\ell\le m\le\ell$$

$$\hat{S}^2|s,m\rangle=s(s+1)\hbar^2|s,m\rangle \quad \text{where} \quad s=0 \text{ or } 1/2 \text{ or } 1 \text{ or } 3/2 \text{ ....}$$

$$\hat{S}_z|s,m\rangle=m\hbar|s,m\rangle \qquad -s\le m\le s$$

There is one big difference between the orbital and spin angular momentum characteristics:

For spin angular momentum: $s=0 \text{ or } \frac{1}{2} \text{ or } 1 \text{ or } \frac{3}{2} \text{ or } 2\text{..........}$

For orbital angular momentum: $\ell=0 \text{ or } 1 \text{ or } 2 \text{ ....}$

Why the difference? Recall that the values of *m* for orbital angular momentum must be integers (this follows from the requirement of the wavefunction being single-valued), but since there is no wavefunction associated with the spin state of a particle, values of *m* for spin angular momentum can be half-integers. Consequently, ℓ must only take positive integer values. But *s* can take both integer and half-integer values.

Quantum States of Particles with Spin

The spin degree of freedom is included in the quantum state by just enlarging the Hilbert space

So, for example, the quantum state of a spin-half electron with spin-up is written as,

$$|\psi\rangle = |\phi\rangle \otimes |z\uparrow\rangle = |\phi\rangle \otimes |m=+1/2\rangle = |\phi\rangle \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Spatial degrees of freedom **Spin degree of freedom**

Consider two states:

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\left[|\phi\rangle \otimes |z\uparrow\rangle + |\chi\rangle \otimes |z\downarrow\rangle\right]$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\left[|\omega\rangle \otimes |z\uparrow\rangle + |\beta\rangle \otimes |z\downarrow\rangle\right]$$

Their inner product will be:

$$\langle\psi_2|\psi_2\rangle = \frac{1}{2}\left[\langle\omega|\phi\rangle\langle z\uparrow|z\uparrow\rangle + \langle\omega|\chi\rangle\cancelto{0}{\langle z\uparrow|z\downarrow\rangle} + \langle\beta|\phi\rangle\cancelto{0}{\langle z\downarrow|z\uparrow\rangle} + \langle\beta|\chi\rangle\langle z\downarrow|z\downarrow\rangle\right]$$

$$= \frac{1}{2}\left[\langle\omega|\phi\rangle + \langle\beta|\chi\rangle\right]$$

Quantum States of Particles with Spin: Spinor Wavefunctions

Consider:

$$\left|\psi\right\rangle = \frac{1}{\sqrt{2}}\left[\left|\phi\right\rangle \otimes \left|z\uparrow\right\rangle + \left|\chi\right\rangle \otimes \left|z\downarrow\right\rangle\right] = \frac{1}{\sqrt{2}}\left[\left|\phi\right\rangle \otimes \begin{bmatrix}1\\0\end{bmatrix} + \left|\chi\right\rangle \otimes \begin{bmatrix}0\\1\end{bmatrix}\right]$$

If we need the probability of finding the electron at location $\vec{r}$ with spin-up for the state $\left|\psi\right\rangle$ we take the squared magnitude of the inner product:

$$\left|\left(\left\langle\vec{r}\right| \otimes \left\langle z\uparrow\right|\right)\left|\psi\right\rangle\right|^2 = \left|\frac{1}{\sqrt{2}}\left[\left\langle\vec{r}\middle|\phi\right\rangle\left\langle z\uparrow\middle|z\uparrow\right\rangle + \left\langle\vec{r}\middle|\chi\right\rangle\left\langle z\uparrow\middle|z\downarrow\right\rangle\right]\right|^2 = \frac{1}{2}\left|\phi(\vec{r})\right|^2$$

If we need the wavefunction, we can take a partial inner-product on the coordinate basis:

$$\begin{aligned}
\left\langle\vec{r}\middle|\psi\right\rangle &= \frac{1}{\sqrt{2}}\left[\left\langle\vec{r}\middle|\phi\right\rangle\left|z\uparrow\right\rangle + \left\langle\vec{r}\middle|\chi\right\rangle\left|z\downarrow\right\rangle\right] \\
&= \frac{1}{\sqrt{2}}\left[\phi(\vec{r})\left|z\uparrow\right\rangle + \chi(\vec{r})\left|z\downarrow\right\rangle\right] \\
&= \frac{1}{\sqrt{2}}\left[\phi(\vec{r})\begin{bmatrix}1\\0\end{bmatrix} + \chi(\vec{r})\begin{bmatrix}0\\1\end{bmatrix}\right] \\
&= \frac{1}{\sqrt{2}}\begin{bmatrix}\phi(\vec{r})\\ \chi(\vec{r})\end{bmatrix}
\end{aligned}$$

$\longrightarrow$ Two-component spinor wavefunction

Spinor States and Operators

Consider a spinor state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big[|\phi\rangle \otimes |z\uparrow\rangle + |\chi\rangle \otimes |z\downarrow\rangle\Big]$$

The wavefunction is:

$$\langle \vec{r}|\psi\rangle = \frac{1}{\sqrt{2}}\Big[\langle \vec{r}|\phi\rangle |z\uparrow\rangle + \langle \vec{r}|\chi\rangle |z\downarrow\rangle\Big] = \frac{1}{\sqrt{2}}\Big[\phi(\vec{r})|z\uparrow\rangle + \chi(\vec{r})|z\downarrow\rangle\Big] = \frac{1}{\sqrt{2}}\begin{bmatrix}\phi(\vec{r})\\ \chi(\vec{r})\end{bmatrix}$$

Suppose we act upon it with the orbital angular momentum operator $\hat{L}_z$:

$$\hat{L}_z|\psi\rangle = \frac{1}{\sqrt{2}}\Big[\big(\hat{L}_z|\phi\rangle\big) \otimes |z\uparrow\rangle + \big(\hat{L}_z|\chi\rangle\big) \otimes |z\downarrow\rangle\Big]$$

Suppose we act upon it with the spin angular momentum operator $\hat{S}_z$:

$$\hat{S}_z|\psi\rangle = \frac{1}{\sqrt{2}}\Big[|\phi\rangle \otimes \big(\hat{S}_z|z\uparrow\rangle\big) + |\chi\rangle \otimes \big(\hat{S}_z|z\downarrow\rangle\big)\Big]$$

$$= \left(\frac{\hbar}{2}\right)\frac{1}{\sqrt{2}}\Big[|\phi\rangle \otimes |z\uparrow\rangle - |\chi\rangle \otimes |z\downarrow\rangle\Big]$$

Point to note here: each operator acts in its own sub-space of the full Hilbert space of the particle