Lecture 22

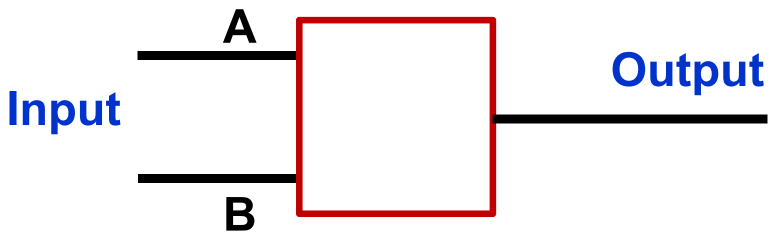
Interactions, Entanglement, and Two-Qubit Quantum Gates

In this lecture you will learn:

- Two-qubit quantum gates
- How interaction causes entanglement
- Coupled spin qubits
- Coupled SHO qubits and superconducting qubits
- C-NOT (Control-X) gate implementation

Two-Bit Classical Logic Gates

Classical two-bit logic gate takes two-bit input and typically produces a one-bit output:



NAND

Input	Output
00	1
01	1
10	1
11	0

NOR Gate

Input	Output
00	1
01	0
10	0
11	0

XOR Gate

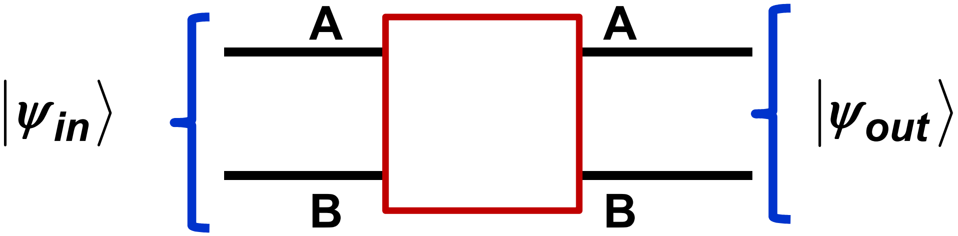
Input	Output
00	0
01	1
10	1
11	0

- **The gates AND and NOT taken together are functionally complete (i.e. any digital logic circuit can be implemented using just AND and NOT gates)**
- **The gate NAND by itself is functionally complete (i.e. any digital logic circuit can be implemented using just NAND gates)**
- **NAND is a universal gate!**

Two-input one-output two-bit classical logic gates are irreversible

Two-Qubit Quantum Gates

A two-qubit quantum gate operates on two qubits and the result is two qubits



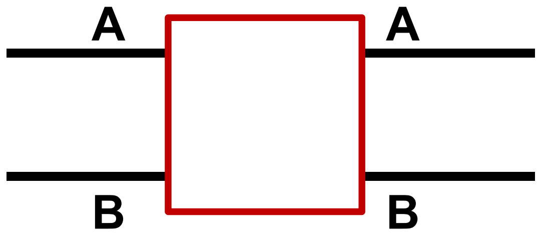
The input (or the output) state consists of a state belonging to the Hilbert state of two different qubits

Examples:

$$|\psi_{in}\rangle = |z\uparrow\rangle_A \otimes |z\downarrow\rangle_B$$

Unentangled

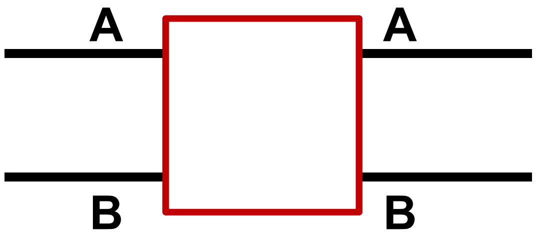
$$|z\uparrow\rangle_A \qquad |z\downarrow\rangle_B$$



$$|\psi_{in}\rangle = \frac{1}{\sqrt{2}}\Big[|z\uparrow\rangle_A + |z\downarrow\rangle_A\Big] \otimes |z\downarrow\rangle_B$$

Unentangled

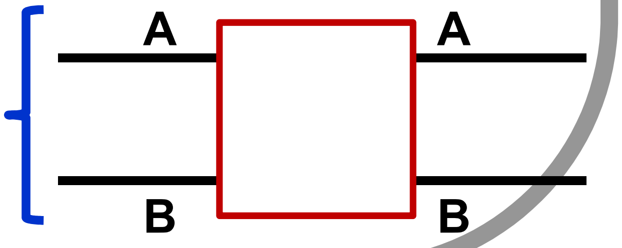
$$\frac{1}{\sqrt{2}}\Big[|z\uparrow\rangle_A + |z\downarrow\rangle_A\Big] \qquad |z\downarrow\rangle_B$$



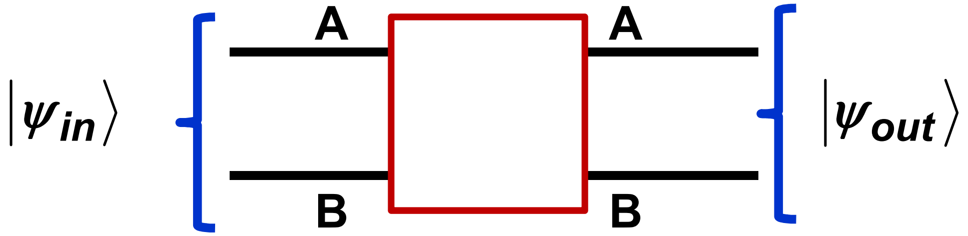
$$|\psi_{in}\rangle = \frac{1}{\sqrt{2}}\Big[|z\uparrow\rangle_A \otimes |z\downarrow\rangle_B + |z\downarrow\rangle_A \otimes |z\uparrow\rangle_B\Big]$$

Entangled

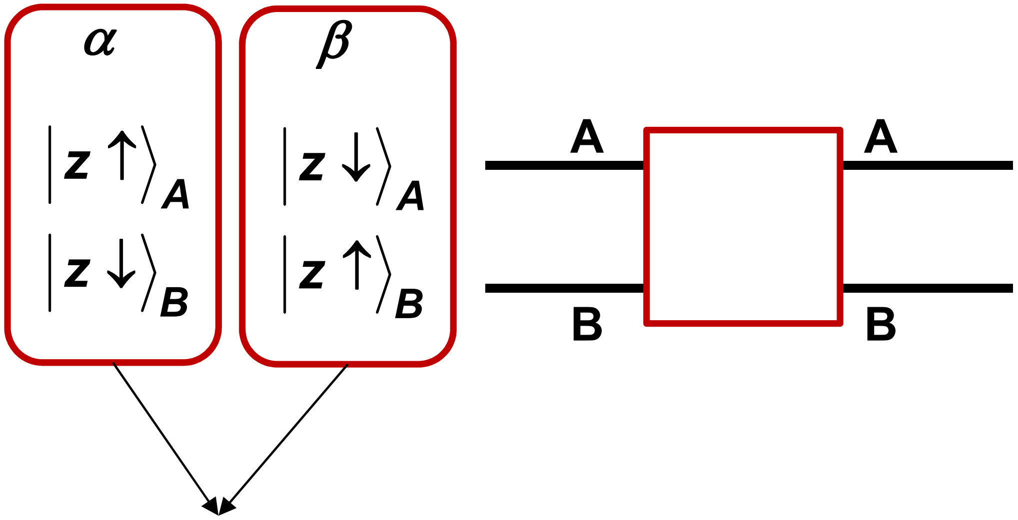
$$\frac{1}{\sqrt{2}}\Big[|z\uparrow\rangle_A \otimes |z\downarrow\rangle_B + |z\downarrow\rangle_A \otimes |z\uparrow\rangle_B\Big]$$



Two-Qubit Quantum Gate: Entangled State Input



$$|\psi_{in}\rangle = \alpha |z\uparrow\rangle_A \otimes |z\downarrow\rangle_B + \beta |z\downarrow\rangle_A \otimes |z\uparrow\rangle_B$$



These two realities are simultaneously going into the gate

Mapping Convention: Spins

Always remember the mapping convention between the physical quantum states of spins, their column vector representations, and the logical qubits:

$$|z\uparrow\rangle \leftrightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} \leftrightarrow |0\rangle$$

$$|z\downarrow\rangle \leftrightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} \leftrightarrow |1\rangle$$

Spin physical states

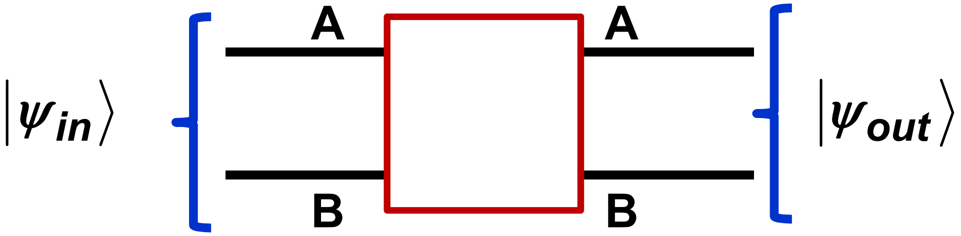
Mapping to col vector rep

Logical qubits

You are very likely going to make an error if you forget the above!

Two-Qubit Quantum Gates: Different Notation

$$\begin{cases} |z\uparrow\rangle \leftrightarrow |0\rangle \\ |z\downarrow\rangle \leftrightarrow |1\rangle \end{cases}$$

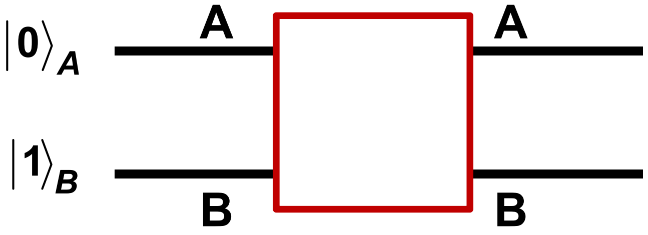


Even when one is working with a spin qubits, one often uses the logical 0 and logical 1 qubit notation:

Examples:

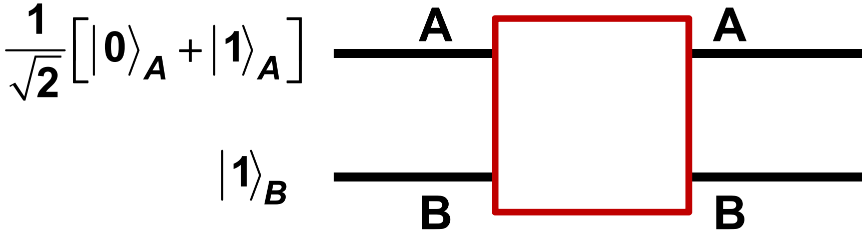
$$|\psi_{in}\rangle = |0\rangle_A |1\rangle_B$$

Unentangled



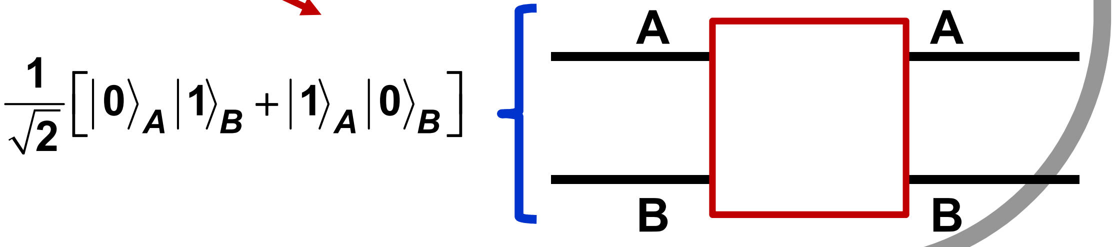
$$|\psi_{in}\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle_A + |1\rangle_A\right]|1\rangle_B$$

Unentangled

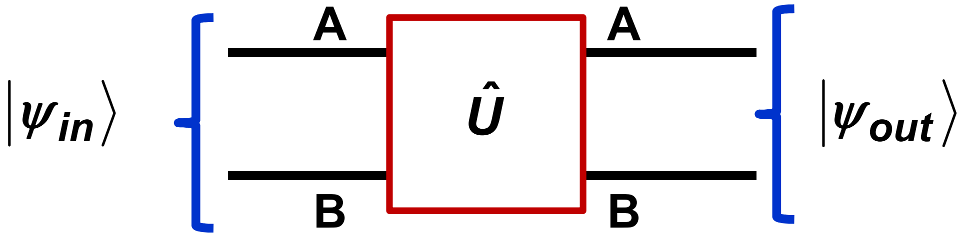


$$|\psi_{in}\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B\right]$$

Entangled



Two-Qubit Quantum Gates: Unitarity, Linearity, and Reversibility



Unitarity:

$|\psi_{out}\rangle = \hat{U}|\psi_{in}\rangle$ ⇒ $\hat{U}$ is a unitary operator

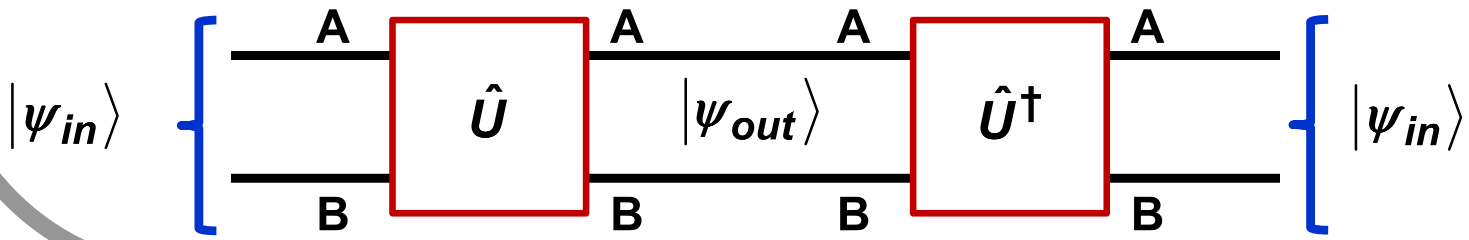
Linearity:

Suppose $|\chi_{out}\rangle = \hat{U}|\chi_{in}\rangle$ and $|\phi_{out}\rangle = \hat{U}|\phi_{in}\rangle$, then if $|\psi_{in}\rangle = \alpha|\chi_{in}\rangle + \beta|\phi_{in}\rangle$, the output is:

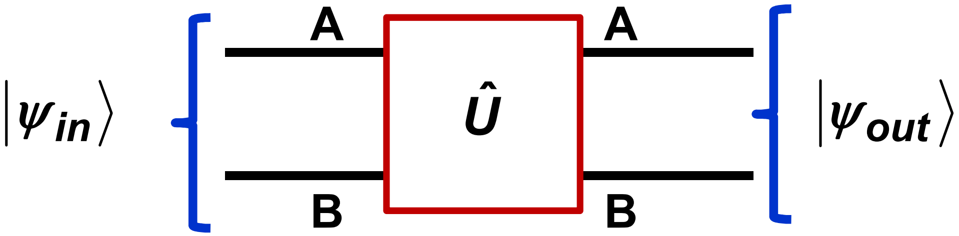
$$|\psi_{out}\rangle = \hat{U}|\psi_{in}\rangle = \hat{U}\left(\alpha|\chi_{in}\rangle + \beta|\phi_{in}\rangle\right) = \alpha\hat{U}|\chi_{in}\rangle + \beta\hat{U}|\phi_{in}\rangle = \alpha|\chi_{out}\rangle + \beta|\phi_{out}\rangle$$

Reversibility:

$|\psi_{out}\rangle = \hat{U}|\psi_{in}\rangle$ ⇔ $|\psi_{in}\rangle = \hat{U}^{\dagger}|\psi_{out}\rangle$



Two-Qubit Quantum Gates: Hilbert Space



The Hilbert space of two-qubits consists of the following four unentangled basis states:

$$|0\rangle_A|0\rangle_B \qquad |0\rangle_A|1\rangle_B \qquad |1\rangle_A|0\rangle_B \qquad |1\rangle_A|1\rangle_B$$

The operation of a quantum gate is fully specified by its actions on all basis states

Both the input and the output state can be written in terms of the above four states

$$|\psi_{in}\rangle = \alpha|0\rangle_A|0\rangle_B + \beta|0\rangle_A|1\rangle_B + \gamma|1\rangle_A|0\rangle_B + \delta|1\rangle_A|1\rangle_B$$

$$\begin{aligned}|\psi_{out}\rangle &= \hat{U}|\psi_{in}\rangle \\ &= a|0\rangle_A|0\rangle_B + b|0\rangle_A|1\rangle_B + c|1\rangle_A|0\rangle_B + d|1\rangle_A|1\rangle_B\end{aligned}$$

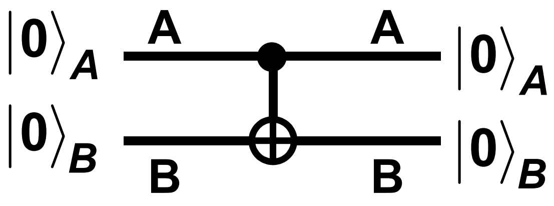
2-Qubut Gate	
Input	Output
$|0\rangle_A|0\rangle_B$	
$|0\rangle_A|1\rangle_B$	
$|1\rangle_A|0\rangle_B$	
$|1\rangle_A|1\rangle_B$	

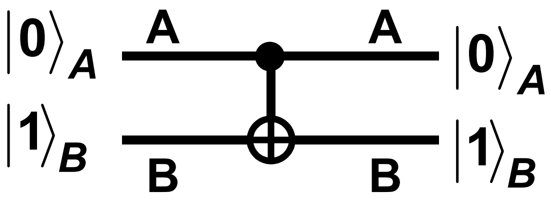
The Control-X (C-NOT) Gate

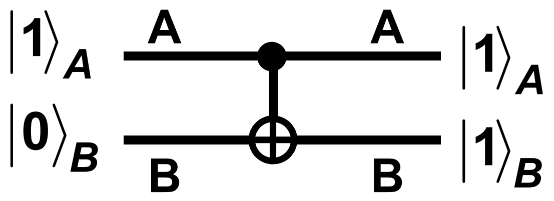
The Hilbert space of two-qubits consists of the following four unentangled basis states:

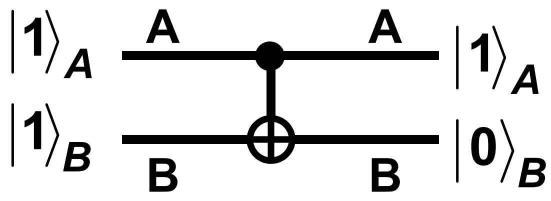
$$|0\rangle_A|0\rangle_B \qquad |0\rangle_A|1\rangle_B \qquad |1\rangle_A|0\rangle_B \qquad |1\rangle_A|1\rangle_B$$

The operation of a quantum gate is fully specified by its actions on all basis states







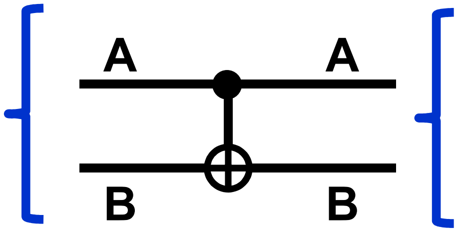


Control-X or C-NOT	
Input	Output
$|0\rangle_A|0\rangle_B$	$|0\rangle_A|0\rangle_B$
$|0\rangle_A|1\rangle_B$	$|0\rangle_A|1\rangle_B$
$|1\rangle_A|0\rangle_B$	$|1\rangle_A|1\rangle_B$
$|1\rangle_A|1\rangle_B$	$|1\rangle_A|0\rangle_B$

Depending on the value of the control qubit (i.e. A), the qubit of the second input (B) is flipped (i.e. acted upon by the $\hat{\sigma}_x$ operator)

Use <u>linearity</u> of quantum gates to figure out what happens when the input state is a superposition of different basis states:

$$\alpha|0\rangle_A|0\rangle_B+\beta|0\rangle_A|1\rangle_B \\ +\gamma|1\rangle_A|0\rangle_B+\delta|1\rangle_A|1\rangle_B$$



$$\alpha|0\rangle_A|0\rangle_B+\beta|0\rangle_A|1\rangle_B \\ +\gamma|1\rangle_A|1\rangle_B+\delta|1\rangle_A|0\rangle_B$$

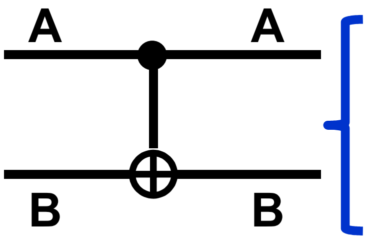
The Control-X (C-NOT) Gate can Entangle

$$
\begin{aligned}
|\psi_{in}\rangle &= \frac{1}{\sqrt{2}}\Big[|0\rangle_A + |1\rangle_A\Big]|1\rangle_B \\
&= \frac{1}{\sqrt{2}}\Big[|0\rangle_A|1\rangle_B + |1\rangle_A|1\rangle_B\Big]
\end{aligned}
$$

$$|\psi_{out}\rangle = \frac{1}{\sqrt{2}}\Big[|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B\Big]$$

$$\frac{1}{\sqrt{2}}\Big[|0\rangle_A + |1\rangle_A\Big]$$

$$|1\rangle_B$$



$$\left\{ \frac{1}{\sqrt{2}}\Big[|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B\Big] \right.$$

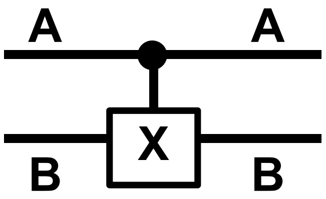
Unentangled input

Entangled output

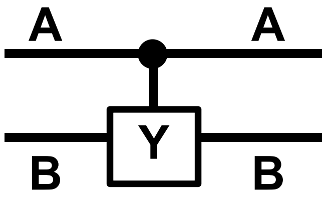
Universal Set of Quantum Gates

The set of all single-qubit quantum gates together with the two-qubit CNOT gate form a widely used universal set of quantum gates

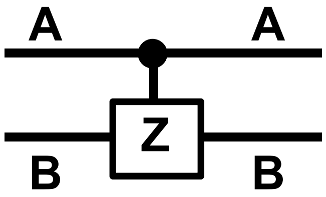
Other Common Two-qubit Control Gates



Control-X (CNOT)



Control-Y



Control-Z

Control-X or CNOT	
Input	Output
$|0\rangle_A|0\rangle_B$	$|0\rangle_A|0\rangle_B$
$|0\rangle_A|1\rangle_B$	$|0\rangle_A|1\rangle_B$
$|1\rangle_A|0\rangle_B$	$|1\rangle_A|1\rangle_B$
$|1\rangle_A|1\rangle_B$	$|1\rangle_A|0\rangle_B$

Control-Y	
Input	Output
$|0\rangle_A|0\rangle_B$	$|0\rangle_A|0\rangle_B$
$|0\rangle_A|1\rangle_B$	$|0\rangle_A|1\rangle_B$
$|1\rangle_A|0\rangle_B$	$-i|1\rangle_A|1\rangle_B$
$|1\rangle_A|1\rangle_B$	$i|1\rangle_A|0\rangle_B$

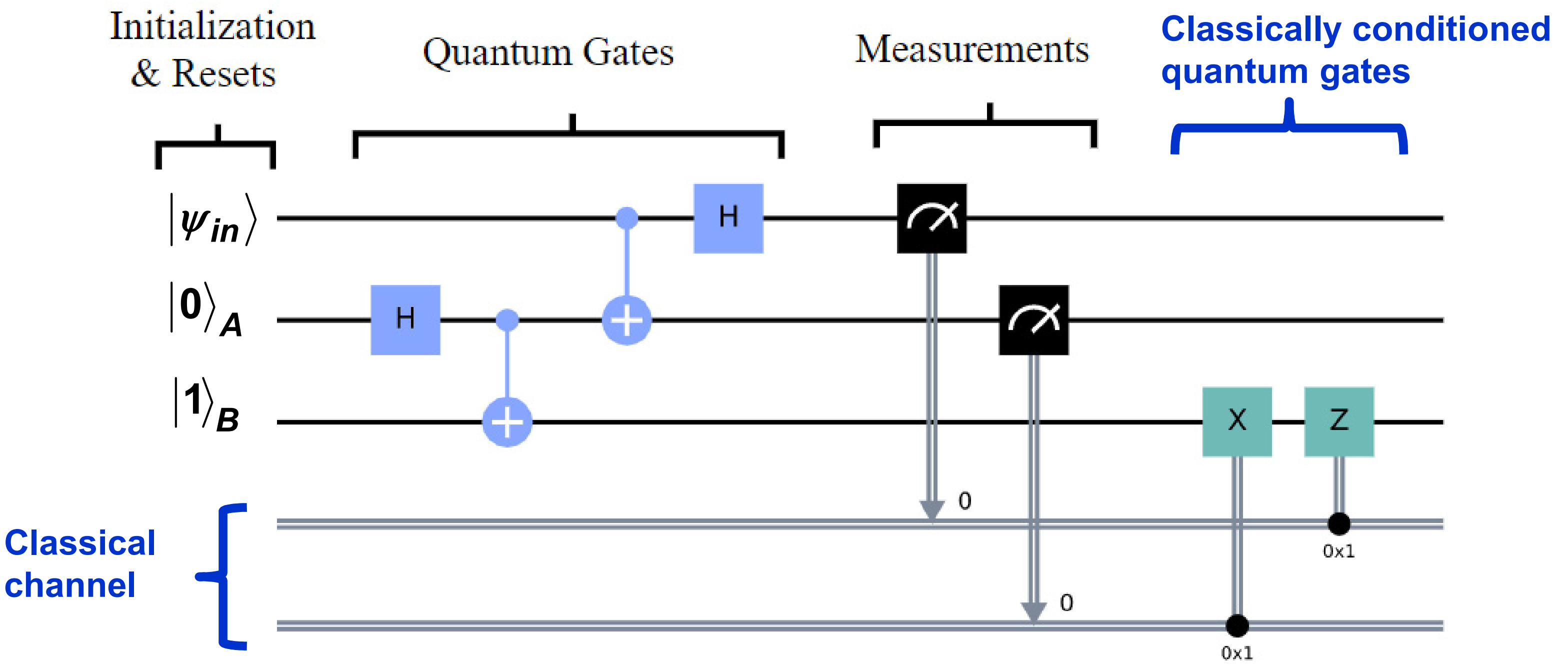
Control-Z	
Input	Output
$|0\rangle_A|0\rangle_B$	$|0\rangle_A|0\rangle_B$
$|0\rangle_A|1\rangle_B$	$|0\rangle_A|1\rangle_B$
$|1\rangle_A|0\rangle_B$	$|1\rangle_A|0\rangle_B$
$|1\rangle_A|1\rangle_B$	$-|1\rangle_A|1\rangle_B$

Quantum Circuits

What is a Quantum Circuit?

A quantum circuit is a computational routine consisting of coherent quantum operations on quantum data, such as qubits, and concurrent real-time classical computation

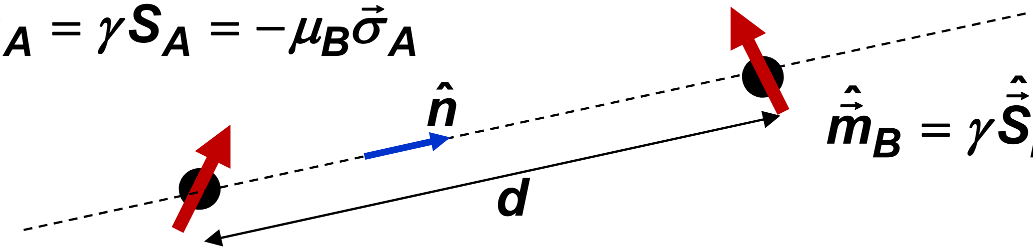
It is an ordered sequence of quantum gates, measurements and resets, all of which may be conditioned on and use data from the real-time classical computation



Interaction and Entanglement: Coupled Spins

Consider two spin 1/2 particles located as shown below:

$$\hat{\vec{m}}_A = \gamma \hat{\vec{S}}_A = -\mu_B \hat{\vec{\sigma}}_A$$



$$\hat{\vec{m}}_B = \gamma \hat{\vec{S}}_B = -\mu_B \hat{\vec{\sigma}}_B$$

The magnetic dipole potential energy between them can be written in terms of their magnetic moments:

$$\hat{H} = \frac{\mu_o}{4\pi d^3}\left[\hat{\vec{m}}_A . \hat{\vec{m}}_B - 3\left(\hat{\vec{m}}_A . \hat{n}\right)\left(\hat{\vec{m}}_B . \hat{n}\right)\right] = \frac{\mu_o \mu_B^2}{4\pi d^3}\left[\hat{\vec{\sigma}}_A . \hat{\vec{\sigma}}_B - 3\left(\hat{\vec{\sigma}}_A . \hat{n}\right)\left(\hat{\vec{\sigma}}_B . \hat{n}\right)\right]$$

In materials, the exchange energy is much larger than the magnetic dipole energy and the resulting Hamiltonian usually takes the form:

$$\hat{H} = J_x \hat{\sigma}_A^x \otimes \hat{\sigma}_B^x + J_y \hat{\sigma}_A^y \otimes_B^y + J_z \hat{\sigma}_A^z \otimes \hat{\sigma}_B^z$$

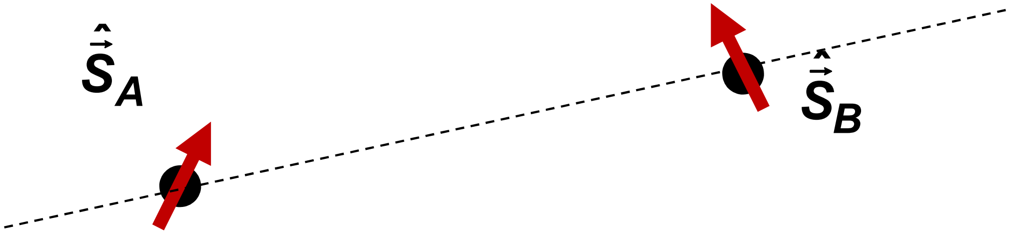
$$= J_x \hat{\sigma}_A^x \hat{\sigma}_B^x + J_y \hat{\sigma}_A^y \hat{\sigma}_B^y + J_z \hat{\sigma}_A^z \hat{\sigma}_B^z$$

$$= J \hat{\vec{\sigma}}_A . \hat{\vec{\sigma}}_B \qquad \left\{ \text{If } J_x = J_y = J_z = J \right.$$

$\Big\{$ The value of J is generally a decreasing function of the distance between the spins

Interaction and Entanglement: Coupled Spins

Consider two spins 1/2 as shown below:



Suppose the Hamiltonian is:

$$\hat{H} = J_x \hat{\sigma}_A^x \hat{\sigma}_B^x = \frac{4}{\hbar^2} J_x \hat{S}_A^x \hat{S}_B^x$$

What about the eigenstates of the Hamiltonian? They can be the entangled states:

$$\hat{H}\left|v_j\right\rangle = E_j\left|v_j\right\rangle$$

$$\left|v_1\right\rangle = \frac{1}{\sqrt{2}}\left[\left|z\uparrow\right\rangle_A\left|z\downarrow\right\rangle_B - \left|z\downarrow\right\rangle_A\left|z\uparrow\right\rangle_B\right] \qquad E_1 = -J_x$$

$$\left|v_2\right\rangle = \frac{1}{\sqrt{2}}\left[\left|z\uparrow\right\rangle_A\left|z\downarrow\right\rangle_B + \left|z\downarrow\right\rangle_A\left|z\uparrow\right\rangle_B\right] \qquad E_2 = J_x$$

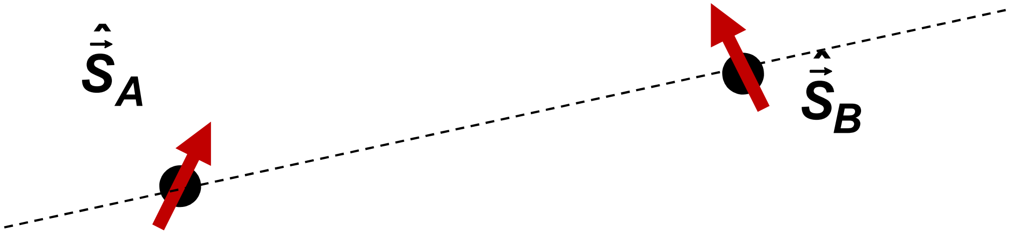
$$\left|v_3\right\rangle = \frac{1}{\sqrt{2}}\left[\left|z\uparrow\right\rangle_A\left|z\uparrow\right\rangle_B + \left|z\downarrow\right\rangle_A\left|z\downarrow\right\rangle_B\right] \qquad E_3 = J_x$$

$$\left|v_4\right\rangle = \frac{1}{\sqrt{2}}\left[\left|z\uparrow\right\rangle_A\left|z\uparrow\right\rangle_B - \left|z\downarrow\right\rangle_A\left|z\downarrow\right\rangle_B\right] \qquad E_4 = -J_x$$

The degenerate eigenstates can be combined to generate unentangled eigenstates

Interaction and Entanglement: Coupled Spins

Consider two spins 1/2 as shown below:



The Hamiltonian is:

$$\hat{H} = J_x \hat{\sigma}_A^x \hat{\sigma}_B^x = \frac{4}{\hbar^2} J_x \hat{S}_A^x \hat{S}_B^x$$

The unentangled eigenstates are:

$$\hat{H}\left|v_j\right\rangle = E_j \left|v_j\right\rangle$$

$$\left|v_1\right\rangle = \left|x\uparrow\right\rangle_A \left|x\downarrow\right\rangle_B \qquad E_1 = -J_x$$

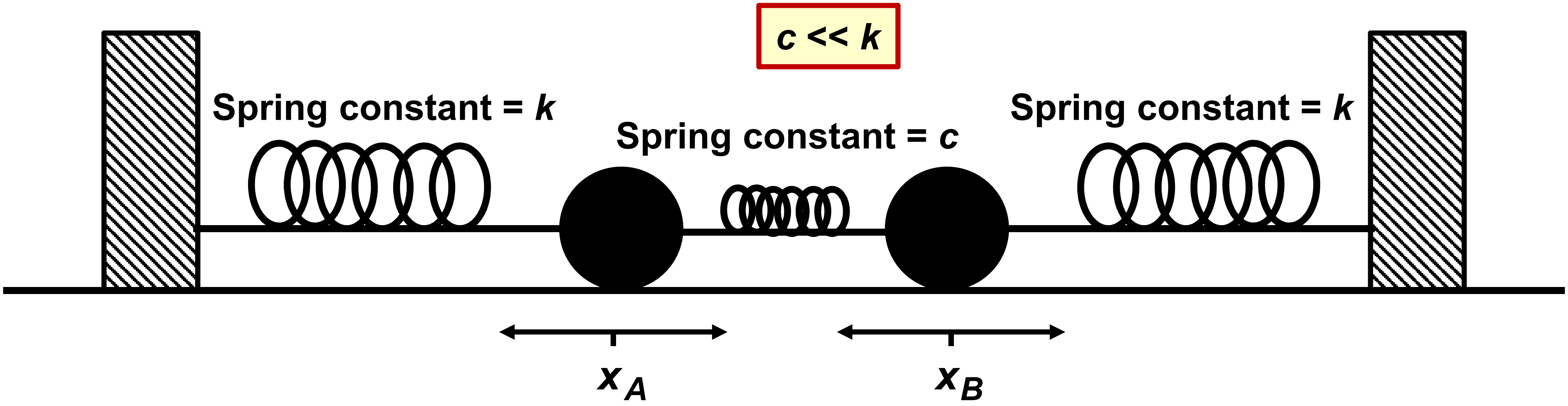
$$\left|v_2\right\rangle = \left|x\uparrow\right\rangle_A \left|x\uparrow\right\rangle_B \qquad E_2 = J_x$$

$$\left|v_3\right\rangle = \left|x\downarrow\right\rangle_A \left|x\downarrow\right\rangle_B \qquad E_3 = J_x$$

$$\left|v_4\right\rangle = \left|x\downarrow\right\rangle_A \left|x\uparrow\right\rangle_B \qquad E_4 = -J_x$$

Interaction and Entanglement: Coupled Quantum SHOs

Consider two coupled quantum SHOs:



$$\hat{H} = \frac{\hat{p}_A^2}{2m} + \frac{1}{2}k\hat{x}_A^2 + \frac{\hat{p}_B^2}{2m} + \frac{1}{2}k\hat{x}_B^2 + \frac{1}{2}c\left(\hat{x}_A - \hat{x}_B\right)^2$$

$$= \left[\frac{\hat{p}_A^2}{2m} + \frac{1}{2}(k+c)\,\hat{x}_A^2\right] + \left[\frac{\hat{p}_B^2}{2m} + \frac{1}{2}(k+c)\,\hat{x}_B^2\right] - c\,\hat{x}_A\hat{x}_B$$

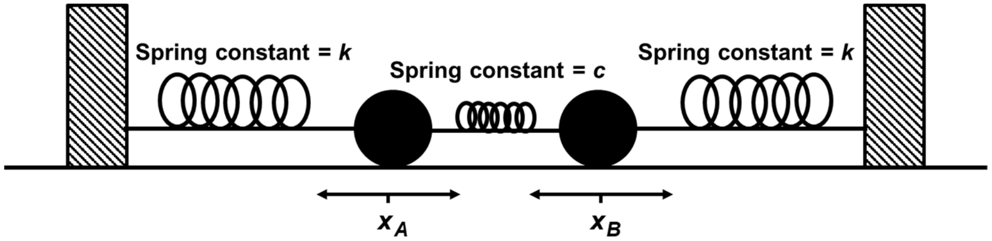
$$= \hat{H}_A + \hat{H}_B + \hat{H}_{AB}$$

$$= \hbar\omega_o\left[\hat{a}^\dagger\hat{a} + \frac{1}{2}\right] + \hbar\omega_o\left[\hat{b}^\dagger\hat{b} + \frac{1}{2}\right] + \hat{H}_{AB}$$

$$\left\{\quad \omega_o = \sqrt{\frac{k+c}{m}}\right.$$

Interaction and Entanglement: Coupled Quantum SHOs

Consider two coupled quantum SHOs:



<u>Restrict the Hilbert space of each SHO to just the first two states:</u>

$$|n=0\rangle_A \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad |n=1\rangle_A \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\hat{H}_A = E_0 |0\rangle_{A\ A}\langle 0| + E_1 |1\rangle_{A\ A}\langle 1|$$

$$= \frac{E_0 + E_1}{2} + \frac{\hbar\omega_o}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \frac{E_0 + E_1}{2} + \frac{\hbar\omega_o}{2}\hat{\sigma}_A^z$$

$$\approx \frac{\hbar\omega_o}{2}\hat{\sigma}_A^z$$

$$\hat{H}_B \approx \frac{\hbar\omega_o}{2}\hat{\sigma}_B^z$$

$$\hat{H}_{AB} = -c\hat{x}_A\hat{x}_B$$

$$= -c\frac{\hbar}{2m\omega_o}\left(\hat{a}+\hat{a}^\dagger\right)\left(\hat{b}+\hat{b}^\dagger\right)$$

$$= -c\frac{\hbar}{2m\omega_o}\hat{\sigma}_A^x\hat{\sigma}_B^x$$

Recall from HW

$$= J_x\ \hat{\sigma}_A^x\hat{\sigma}_B^x$$

$$\boxed{\hat{H} = \frac{\hbar\omega_o}{2}\left[\hat{\sigma}_A^z + \hat{\sigma}_B^z\right] + J_x\ \hat{\sigma}_A^x\hat{\sigma}_B^x}$$

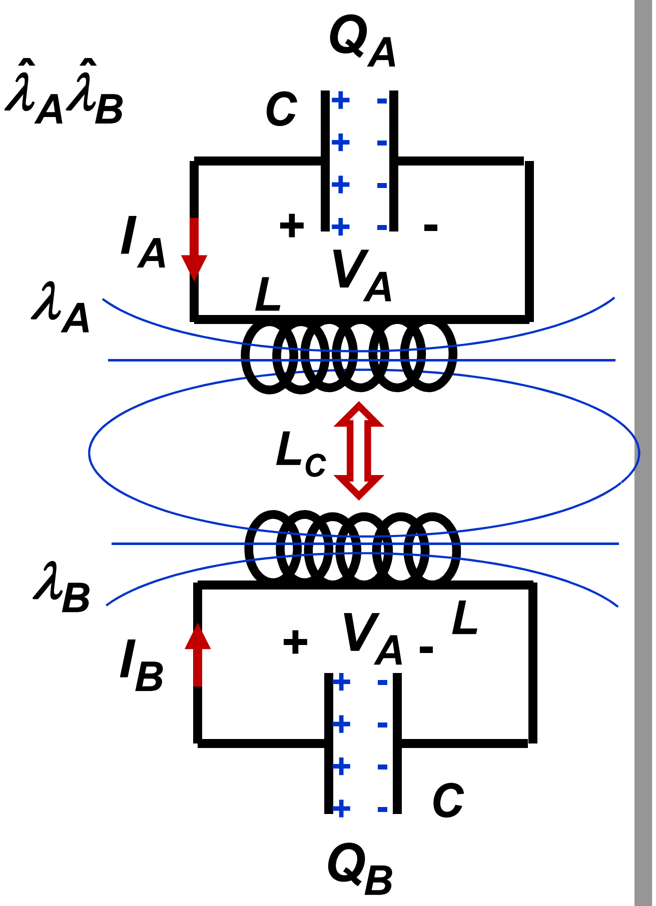
Interaction and Entanglement: Coupled Superconducting Qubits

Consider two coupled quantum superconductive qubits:

$$\hat{H} = \left[\frac{\left(\hat{Q}_A\right)^2}{2C} + \frac{1}{2} C\omega_o^2 \left(\hat{\lambda}_A\right)^2 \right] + \left[\frac{\left(\hat{Q}_B\right)^2}{2C} + \frac{1}{2} C\omega_o^2 \left(\hat{\lambda}_B\right)^2 \right] + \frac{L_c}{L^2} \hat{\lambda}_A \hat{\lambda}_B$$

$$= \hat{H}_A + \hat{H}_B + \hat{H}_{AB}$$

$$= \hbar\omega_o \left[\hat{a}^\dagger \hat{a} + \frac{1}{2} \right] + \hbar\omega_o \left[\hat{b}^\dagger \hat{b} + \frac{1}{2} \right] + \hat{H}_{AB}$$



Restrict the Hilbert space of each qubit to just the first two states:

$$\hat{H}_A \approx \frac{\hbar\omega_o}{2} \hat{\sigma}_A^z$$

$$\hat{H}_B \approx \frac{\hbar\omega_o}{2} \hat{\sigma}_B^z$$

$$\hat{H}_{AB} = \frac{L_c}{L^2} \hat{\lambda}_A \hat{\lambda}_B$$

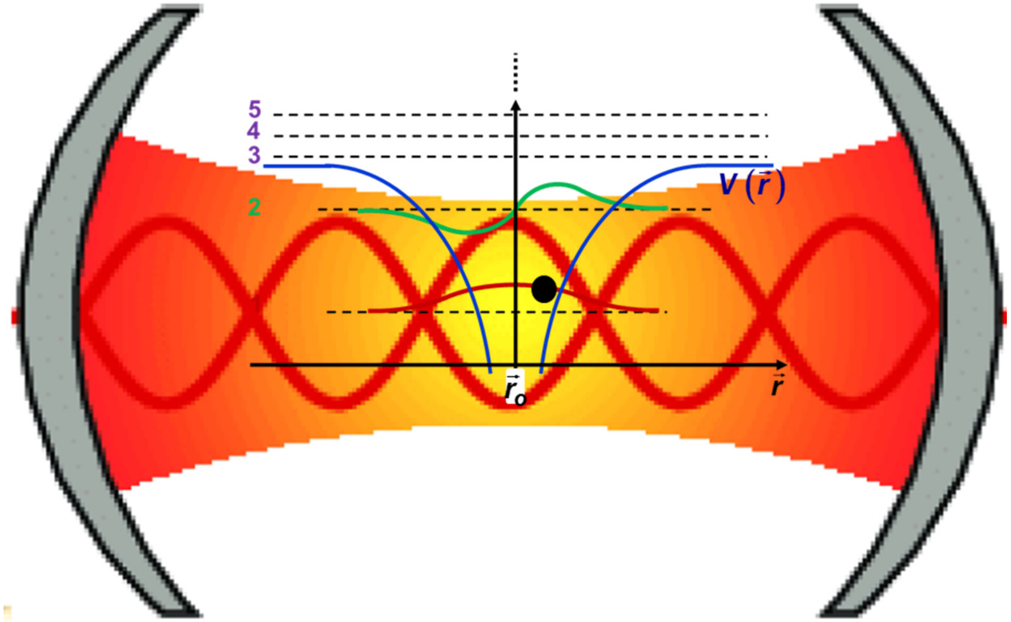
$$= \frac{L_c}{L^2} \frac{\hbar}{2C\omega_o} \left(\hat{a} + \hat{a}^\dagger \right) \left(\hat{b} + \hat{b}^\dagger \right)$$

$$= \frac{L_c}{L^2} \frac{\hbar}{2C\omega_o} \hat{\sigma}_A^x \hat{\sigma}_B^x$$

$$= J_x \, \hat{\sigma}_A^x \hat{\sigma}_B^x$$

$$\boxed{\hat{H} = \frac{\hbar\omega_o}{2} \left[\hat{\sigma}_A^z + \hat{\sigma}_B^z \right] + J_x \, \hat{\sigma}_A^x \hat{\sigma}_B^x}$$

Interaction and Entanglement: Atom in an Optical Cavity



System A:
The electron in the two-level atom

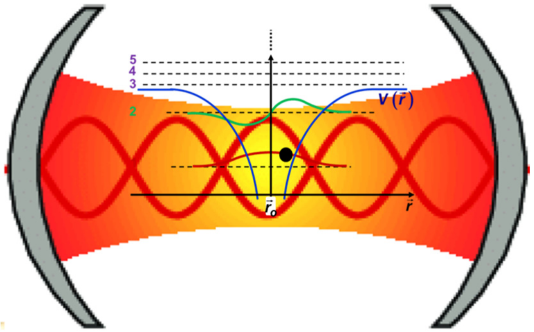
System B:
The cavity mode (Handout 13)

$$\hat{H} = E_1|e_1\rangle\langle e_1| + E_2|e_2\rangle\langle e_2| + \hbar\omega_o\hat{a}^\dagger\hat{a} - q\hat{E}(\vec{r}_o)\,d\left[|e_1\rangle\langle e_2| + |e_2\rangle\langle e_1|\right]$$

$$= E_1|e_1\rangle\langle e_1| + E_2|e_2\rangle\langle e_2| + \hbar\omega_o\hat{a}^\dagger\hat{a} + g\left(\hat{a} - \hat{a}^\dagger\right)\left[|e_1\rangle\langle e_2| + |e_2\rangle\langle e_1|\right]$$

$$\left\{ g = iq\sqrt{\frac{\hbar\omega_o}{2\varepsilon_o}}U(\vec{r}_o)d \right.$$

Interaction and Entanglement: Atom in an Optical Cavity



System A:
The electron in the two-level atom

System B:
The cavity mode (Handout 13)

Joint Hilbert space consists of all states of the form:

$$|e_1\rangle \otimes |n\rangle \quad \text{or} \quad |e_2\rangle \otimes |n\rangle \qquad \{\ n = 0,1,2,3.............$$

Eigenstates and eigenvalues of the Hamiltonian when $\hbar\omega_o = E_2 - E_1$ **:**

$$\lambda_1^n = E_2 + n\hbar\omega_o + |g|\sqrt{n+1}$$

$$\lambda_2^n = E_2 + n\hbar\omega_o - |g|\sqrt{n+1} \qquad \{\ n = 0,1,2,3.............$$

$$|v_1^n\rangle = \frac{1}{\sqrt{2}}\left[|e_2\rangle \otimes |n\rangle + \frac{g^*}{|g|}|e_1\rangle \otimes |n+1\rangle\right]$$

$$|v_2^n\rangle = \frac{1}{\sqrt{2}}\left[|e_2\rangle \otimes |n\rangle - \frac{g^*}{|g|}|e_1\rangle \otimes |n+1\rangle\right]$$

Entangled states!

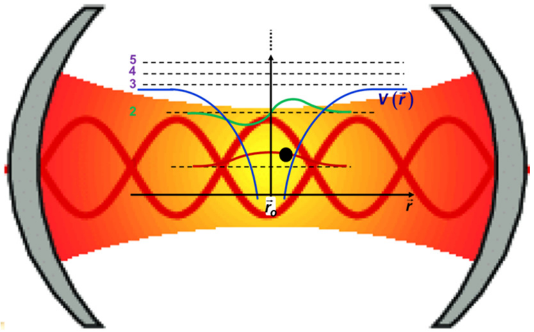
One final eigenstate is:

$$|e_1\rangle \otimes |0\rangle$$

with eigenvalue:

$$E_1$$

Interaction and Entanglement: Atom in an Optical Cavity



System A:
The electron in the two-level atom

System B:
The cavity mode (Handout 13)

Suppose we restrict the Hilbert space to just these two states in each system:

$|e_1\rangle$ **and** $|e_2\rangle$ $\longrightarrow$ **System A**

$|0\rangle$ **and** $|1\rangle$ $\longrightarrow$ **System B (0 and 1 photon states only)**

And assume zero detuning:

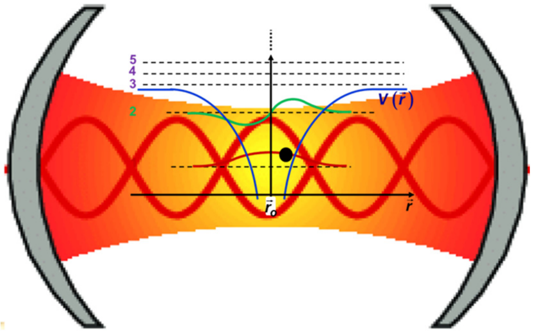
$$E_2 - E_1 = \hbar\omega_o$$

Perform the following mapping of each (atom and cavity mode) Hilbert space to the 2D Hilbert space of a spin 1/2:

$$|e_1\rangle \rightarrow |z\downarrow\rangle_A = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad |e_2\rangle \rightarrow |z\uparrow\rangle_A = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow \text{System A}$$

$$|0\rangle \rightarrow |z\downarrow\rangle_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad |1\rangle \rightarrow |z\uparrow\rangle_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow \text{System B}$$

Interaction and Entanglement: Atom in an Optical Cavity



System A:
The electron in the two-level atom

System B:
The cavity mode (Handout 13)

Then under this mapping:

$$\left[|e_1\rangle\langle e_2| + |e_2\rangle\langle e_1|\right] \rightarrow \hat{\sigma}_x^A$$

$$\left(\hat{a} + \hat{a}^\dagger\right) \rightarrow \hat{\sigma}_x^B$$

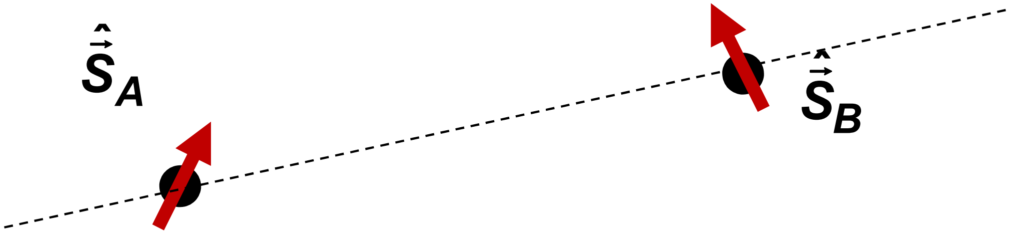
$$\left(\hat{a} - \hat{a}^\dagger\right) \rightarrow -i\hat{\sigma}_y^B$$

And the Hamiltonian in this restricted subspace becomes:

$$\begin{aligned}\hat{H} &= E_1|e_1\rangle\langle e_1| + E_2|e_2\rangle\langle e_2| + \hbar\omega_o\hat{a}^\dagger\hat{a} - q\hat{E}(\vec{r}_o)d\left[|e_1\rangle\langle e_2| + |e_2\rangle\langle e_1|\right] \\ &= E_1|e_1\rangle\langle e_1| + E_2|e_2\rangle\langle e_2| + \hbar\omega_o\hat{a}^\dagger\hat{a} + g\left(\hat{a} - \hat{a}^\dagger\right)\left[|e_1\rangle\langle e_2| + |e_2\rangle\langle e_1|\right] \\ &= \underbrace{\left(\frac{E_1+E_2}{2} + \frac{\hbar\omega_o}{2}\right)}_{\text{offset}} + \frac{\hbar\omega_o}{2}\left(\hat{\sigma}_z^A + \hat{\sigma}_z^B\right) - ig\hat{\sigma}_x^A\hat{\sigma}_y^B\end{aligned}$$

Interaction and Entanglement: Coupled Spins

Consider two 1/2 spins located not far from each other, as shown below:



Suppose the interaction Hamiltonian is:

$$\hat{H} = J_x \hat{\sigma}_A^x \hat{\sigma}_B^x = \frac{4}{\hbar^2} J_x \hat{S}_A^x \hat{S}_B^x$$

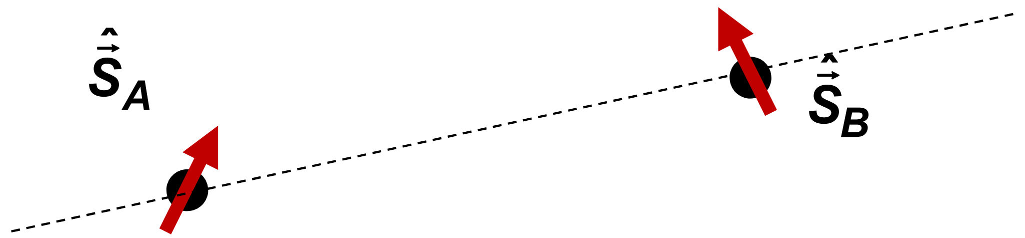
Suppose:

$$\left|\psi(t=0)\right\rangle = \left|z\uparrow\right\rangle_A \left|z\downarrow\right\rangle_B$$

What is the state at time *t*>0:

$$\left|\psi(t)\right\rangle = e^{-\frac{i}{\hbar}\hat{H}t}\left|\psi(t=0)\right\rangle = e^{-\frac{i}{\hbar}\hat{H}t}\left|z\uparrow\right\rangle_A \left|z\downarrow\right\rangle_B$$

Interaction and Entanglement: Coupled Spins



What is the state at time *t*>0:

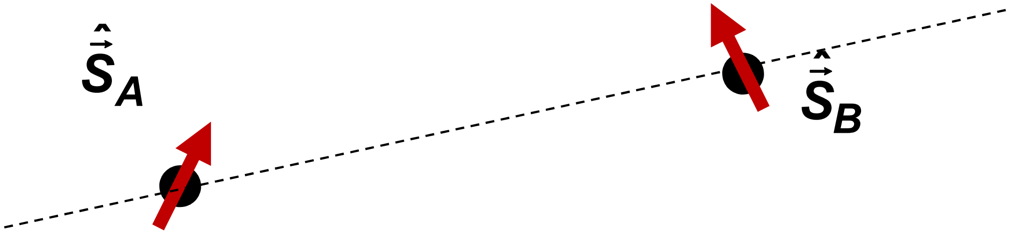
$$\left|\psi(t)\right\rangle = e^{-\frac{i}{\hbar}\hat{H}t}\left|z\uparrow\right\rangle_A\left|z\downarrow\right\rangle_B = e^{-\frac{i}{\hbar}J_x\hat{\sigma}_A^x\hat{\sigma}_B^x t}\left|z\uparrow\right\rangle_A\left|z\downarrow\right\rangle_B$$

$$= \left[\hat{1}\cos\left(\frac{J_x t}{\hbar}\right) - i\hat{\sigma}_A^x\hat{\sigma}_B^x\sin\left(\frac{J_x t}{\hbar}\right)\right]\left|z\uparrow\right\rangle_A\left|z\downarrow\right\rangle_B$$

$$= \cos\left(\frac{J_x t}{\hbar}\right)\left|z\uparrow\right\rangle_A\left|z\downarrow\right\rangle_B - i\sin\left(\frac{J_x t}{\hbar}\right)\left|z\downarrow\right\rangle_A\left|z\uparrow\right\rangle_B$$

$$\boxed{e^{i\alpha\left(\hat{n}.\hat{\vec{\sigma}}_A\right)\left(\hat{m}.\hat{\vec{\sigma}}_B\right)} = \left[\hat{1}\cos(\alpha) + i\left(\hat{n}.\hat{\vec{\sigma}}_A\right)\left(\hat{m}.\hat{\vec{\sigma}}_B\right)\sin(\alpha)\right]}$$

Interaction and Entanglement: Coupled Spins



Consider the state at time $J_x t/\hbar = \pi/4$:

$$\left|\psi(t)\right\rangle = e^{-\frac{i\pi}{4}\hat{\sigma}_A^x\hat{\sigma}_B^x}\left|z\uparrow\right\rangle_A\left|z\downarrow\right\rangle_B$$

$$= \frac{1}{\sqrt{2}}\left[\left|z\uparrow\right\rangle_A\left|z\downarrow\right\rangle_B - i\left|z\downarrow\right\rangle_A\left|z\uparrow\right\rangle_B\right]$$

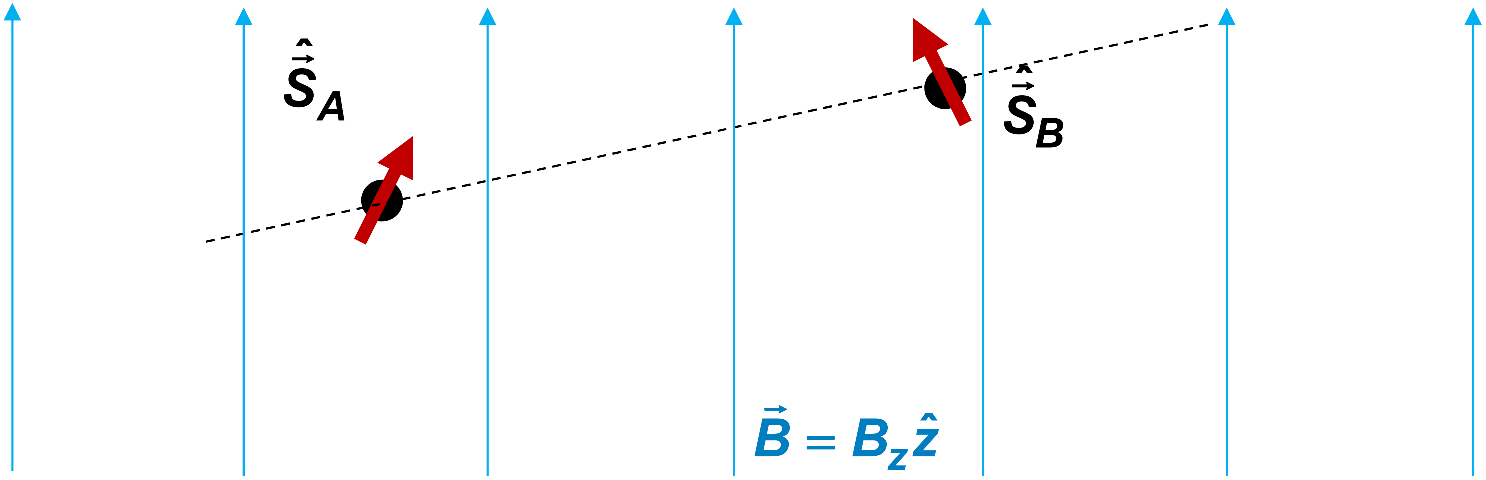
Entangled state!!

Two lessons (or rules of thumb):

- Interactions between two systems result in them getting entangled
- Interactions between two systems have equal and opposite effects on the systems involved

Interaction and Entanglement: Coupled Spins

Now consider two spins 1/2 in a DC magnetic field as shown below:



Each spin will now experience Zeeman splitting due to the DC z-directed magnetic field

$$\hat{H} = \underbrace{\frac{\Delta}{2}\hat{\sigma}_A^z + \frac{\Delta}{2}\hat{\sigma}_B^z}_{\text{Separable part}} + \underbrace{J_x\hat{\sigma}_A^x\hat{\sigma}_B^x}_{\text{Interaction part (involves operators from both systems A and B)}}$$

$$\hat{H} = \hat{H}_A + \hat{H}_B + \hat{H}_{AB}$$

Some Useful Relations

$$\left|\psi(t)\right\rangle = e^{-i\left[\omega_o\left[\frac{\hat{\sigma}_A^z+\hat{\sigma}_B^z}{2}\right]+\frac{J_x}{\hbar}\,\hat{\sigma}_A^x\hat{\sigma}_B^x\right]t}\left|z\uparrow\right\rangle_A\left|z\uparrow\right\rangle_B$$

$$= \left[\hat{1}\cos\left(\sqrt{\omega_o^2+\left(J_x/\hbar\right)^2}\;t\right) - i\sin\left(\sqrt{\omega_o^2+\left(J_x/\hbar\right)^2}\;t\right)\frac{\omega_o\left[\frac{\hat{\sigma}_A^z+\hat{\sigma}_B^z}{2}\right]+\frac{J_x}{\hbar}\,\hat{\sigma}_A^x\hat{\sigma}_B^x}{\sqrt{\omega_o^2+\left(J_x/\hbar\right)^2}}\right]\left|z\uparrow\right\rangle_A\left|z\uparrow\right\rangle_B$$

$$\left|\psi(t)\right\rangle = e^{-i\left[\omega_o\left[\frac{\hat{\sigma}_A^z+\hat{\sigma}_B^z}{2}\right]+\frac{J_x}{\hbar}\,\hat{\sigma}_A^x\hat{\sigma}_B^x\right]t}\left|z\downarrow\right\rangle_A\left|z\downarrow\right\rangle_B$$

$$= \left[\hat{1}\cos\left(\sqrt{\omega_o^2+\left(J_x/\hbar\right)^2}\;t\right) - i\sin\left(\sqrt{\omega_o^2+\left(J_x/\hbar\right)^2}\;t\right)\frac{\omega_o\left[\frac{\hat{\sigma}_A^z+\hat{\sigma}_B^z}{2}\right]+\frac{J_x}{\hbar}\,\hat{\sigma}_A^x\hat{\sigma}_B^x}{\sqrt{\omega_o^2+\left(J_x/\hbar\right)^2}}\right]\left|z\downarrow\right\rangle_A\left|z\downarrow\right\rangle_B$$

$$\left|\psi(t)\right\rangle = e^{-i\left[\omega_o\left[\frac{\hat{\sigma}_A^z+\hat{\sigma}_B^z}{2}\right]+\frac{J_x}{\hbar}\,\hat{\sigma}_A^x\hat{\sigma}_B^x\right]t}\left|z\uparrow\right\rangle_A\left|z\downarrow\right\rangle_B$$

$$= \left[\hat{1}\cos\left(\left(J_x/\hbar\right)t\right) - i\sin\left(\left(J_x/\hbar\right)t\right)\hat{\sigma}_A^x\hat{\sigma}_B^x\right]\left|z\uparrow\right\rangle_A\left|z\downarrow\right\rangle_B$$

$$\left|\psi(t)\right\rangle = e^{-i\left[\omega_o\left[\frac{\hat{\sigma}_A^z+\hat{\sigma}_B^z}{2}\right]+\frac{J_x}{\hbar}\,\hat{\sigma}_A^x\hat{\sigma}_B^x\right]t}\left|z\downarrow\right\rangle_A\left|z\uparrow\right\rangle_B$$

$$= \left[\hat{1}\cos\left(\left(J_x/\hbar\right)t\right) - i\sin\left(\left(J_x/\hbar\right)t\right)\hat{\sigma}_A^x\hat{\sigma}_B^x\right]\left|z\downarrow\right\rangle_A\left|z\uparrow\right\rangle_B$$

Mapping Convention: SHO

Always remember the mapping convention between the physical quantum states of SHOs, their representations, and the logical qubits:

$$\left|n=1\right\rangle \leftrightarrow \left|z\uparrow\right\rangle \leftrightarrow \begin{bmatrix}1\\0\end{bmatrix} \leftrightarrow \left|0\right\rangle$$

$$\left|n=0\right\rangle \leftrightarrow \left|z\downarrow\right\rangle \leftrightarrow \begin{bmatrix}0\\1\end{bmatrix} \leftrightarrow \left|1\right\rangle$$

SHO physical states

Mapping to spin qubit rep

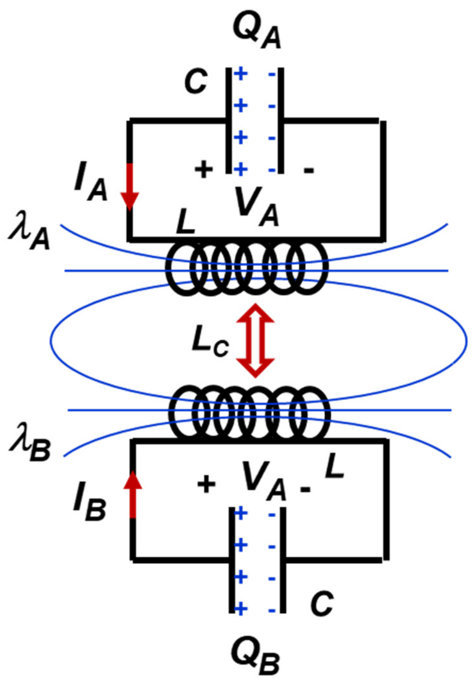
Mapping to col vector rep

Logical qubits

You are very likely going to make an error if you forget the above!

Interaction and Entanglement: Coupled Superconducting Qubits

Two coupled superconducting qubits:



$$\hat{H} = \frac{\hbar\omega_o}{2}\left[\hat{\sigma}_A^z + \hat{\sigma}_B^z\right] + J_x\ \hat{\sigma}_A^x\hat{\sigma}_B^x$$

Suppose:

$$\left|\psi(t=0)\right\rangle = \left|n=1\right\rangle_A \left|n=0\right\rangle_B = \left|z\uparrow\right\rangle_A \left|z\downarrow\right\rangle_B$$

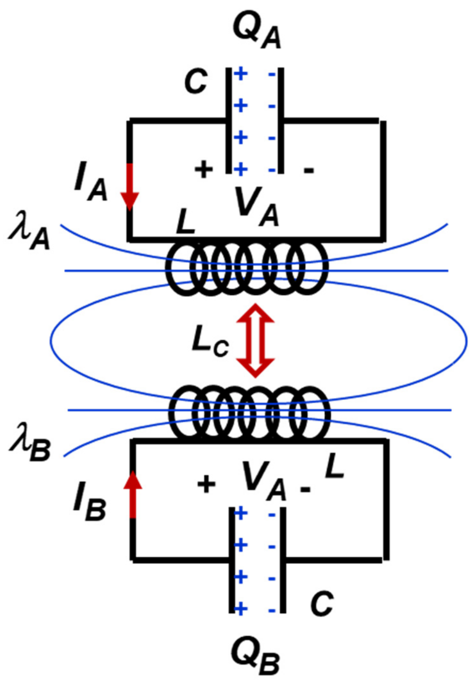
What is the state at time *t*>0:

$$\left|\psi(t)\right\rangle = e^{-\frac{i}{\hbar}\hat{H}t}\left|\psi(t=0)\right\rangle = e^{-\frac{i}{\hbar}\hat{H}t}\left|z\uparrow\right\rangle_A \left|z\downarrow\right\rangle_B$$

Write the initial state in the spin representation

Interaction and Entanglement: Coupled Quantum SHOs

Two coupled superconductive qubits:



$$\hat{H} = \frac{\hbar\omega_o}{2}\left[\hat{\sigma}_A^z + \hat{\sigma}_B^z\right] + J_x\ \hat{\sigma}_A^x\hat{\sigma}_B^x$$

What is the state at time *t*>0:

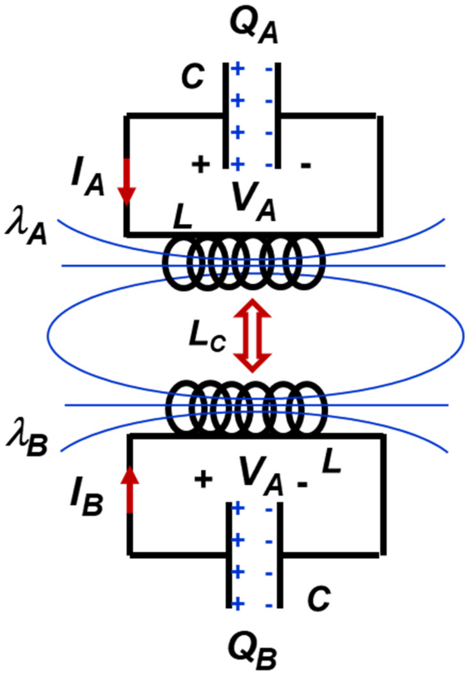
$$\left|\psi(t)\right\rangle = e^{-\frac{i}{\hbar}\left[\frac{\hbar\omega_o}{2}\left[\hat{\sigma}_A^z+\hat{\sigma}_B^z\right]+J_x\ \hat{\sigma}_A^x\hat{\sigma}_B^x\right]t}\left|z\uparrow\right\rangle_A\left|z\downarrow\right\rangle_B$$

$$= e^{-\frac{i}{\hbar}\left[J_x\ \hat{\sigma}_A^x\hat{\sigma}_B^x\right]t}\left|z\uparrow\right\rangle_A\left|z\downarrow\right\rangle_B$$

$$\left\{\left[\hat{\sigma}_A^z + \hat{\sigma}_B^z\right]\left|z\uparrow\right\rangle_A\left|z\downarrow\right\rangle_B = 0\right.$$

$$= \cos\left(\frac{J_x t}{\hbar}\right)\left|z\uparrow\right\rangle_A\left|z\downarrow\right\rangle_B - i\sin\left(\frac{J_x t}{\hbar}\right)\left|z\downarrow\right\rangle_A\left|z\uparrow\right\rangle_B$$

Interaction and Entanglement: Coupled Quantum SHOs



And find the state at time $J_x t/\hbar = \pi/4$:

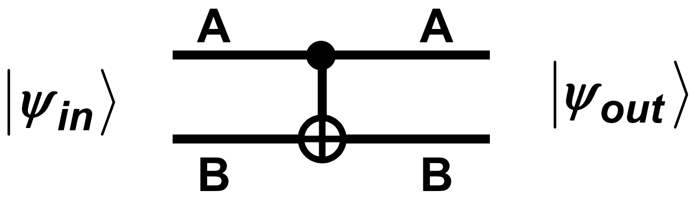
$$\left|\psi(t)\right\rangle = e^{-i\left[\frac{\omega_o t}{2}\left[\hat{\sigma}_A^z+\hat{\sigma}_B^z\right]+\frac{\pi}{4}\hat{\sigma}_A^x\hat{\sigma}_B^x\right]}\left|z\uparrow\right\rangle_A\left|z\downarrow\right\rangle_B$$

$$=\frac{1}{\sqrt{2}}\left[\left|z\uparrow\right\rangle_A\left|z\downarrow\right\rangle_B - i\left|z\downarrow\right\rangle_A\left|z\uparrow\right\rangle_B\right]$$

$$=\frac{1}{\sqrt{2}}\left[\left|n=1\right\rangle_A\left|n=0\right\rangle_B - i\left|n=0\right\rangle_A\left|n=1\right\rangle_B\right]$$

Entangled state!!

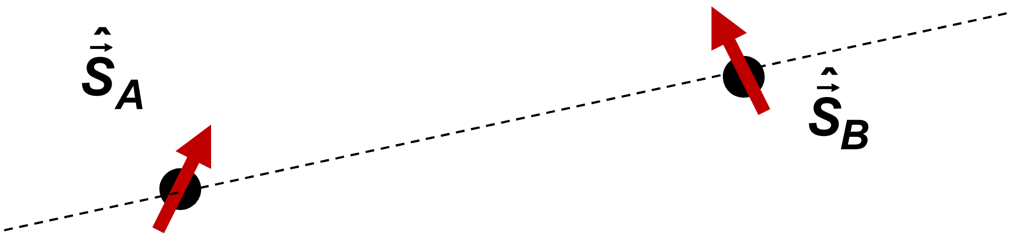
Implementation of a CNOT Gate



Control-X or C-NOT	
Input	Output
$|0\rangle_A|0\rangle_B$	$|0\rangle_A|0\rangle_B$
$|0\rangle_A|1\rangle_B$	$|0\rangle_A|1\rangle_B$
$|1\rangle_A|0\rangle_B$	$|1\rangle_A|1\rangle_B$
$|1\rangle_A|1\rangle_B$	$|1\rangle_A|0\rangle_B$

$$\begin{cases} |z\uparrow\rangle \leftrightarrow |0\rangle \\ |z\downarrow\rangle \leftrightarrow |1\rangle \end{cases}$$

We need to implement a CNOT gate using spin qubits:



Two assumptions:

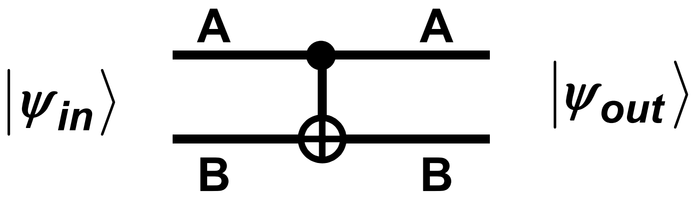
a) We can perform any single-qubit operation separately on each of the spins separately

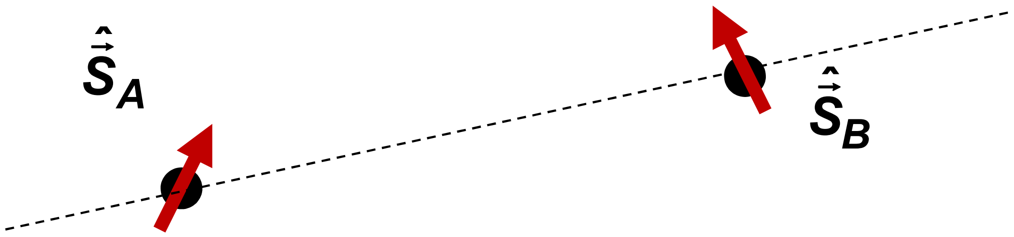
b) We can bring the spins together and implement the following interaction Hamiltonian:

$$\hat{H} = J_x \hat{\sigma}_A^x \hat{\sigma}_B^x$$

$$|\psi_{out}\rangle = e^{-\frac{i}{\hbar} J_x \hat{\sigma}_A^x \hat{\sigma}_B^x} |\psi_{in}\rangle = \hat{U} |\psi_{in}\rangle$$

Implementation of a CNOT Gate





Control-X or C-NOT	
Input	Output
$\|0\rangle_A\|0\rangle_B$	$\|0\rangle_A\|0\rangle_B$
$\|0\rangle_A\|1\rangle_B$	$\|0\rangle_A\|1\rangle_B$
$\|1\rangle_A\|0\rangle_B$	$\|1\rangle_A\|1\rangle_B$
$\|1\rangle_A\|1\rangle_B$	$\|1\rangle_A\|0\rangle_B$

One relation that can come in handy is the following:

$$\left[\hat{\sigma}^y , \hat{\sigma}^z\right] = 2i\hat{\sigma}^x \quad \Longrightarrow \quad \boxed{e^{-i\frac{\pi}{4}\hat{\sigma}^y}\,\hat{\sigma}^z\,e^{+i\frac{\pi}{4}\hat{\sigma}^y} = \hat{\sigma}^x}$$

The above can be used to show that:

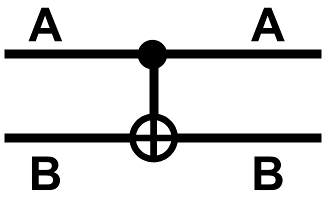
$$e^{-i\frac{\pi}{4}\hat{\sigma}^y}\left(\hat{\sigma}^z\right)^m e^{+i\frac{\pi}{4}\hat{\sigma}^y} = \left(\hat{\sigma}^x\right)^m \qquad \{m = 0,1,2,3.........$$

$$e^{-i\frac{\pi}{4}\hat{\sigma}^y}\,e^{+i\alpha\hat{\sigma}^z}\,e^{+i\frac{\pi}{4}\hat{\sigma}^y} = e^{+i\alpha\hat{\sigma}^x}$$

$e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}$
only if
$\left[\hat{A},\hat{B}\right] = 0$

Implementation of the CNOT Gate

$|\psi_{in}\rangle$



$|\psi_{out}\rangle$

Control-X or C-NOT	
Input	Output
$\|z\uparrow\rangle_A\|z\uparrow\rangle_B$	$\|z\uparrow\rangle_A\|z\uparrow\rangle_B$
$\|z\uparrow\rangle_A\|z\downarrow\rangle_B$	$\|z\uparrow\rangle_A\|z\downarrow\rangle_B$
$\|z\downarrow\rangle_A\|z\uparrow\rangle_B$	$\|z\downarrow\rangle_A\|z\downarrow\rangle_B$
$\|z\downarrow\rangle_A\|z\downarrow\rangle_B$	$\|z\downarrow\rangle_A\|z\uparrow\rangle_B$

$$\left\{\begin{array}{l} |z\uparrow\rangle \leftrightarrow |0\rangle \\ |z\downarrow\rangle \leftrightarrow |1\rangle \end{array}\right.$$

What we need is the following operation:

$$|\psi_{out}\rangle = \hat{U}|\psi_{in}\rangle = \underbrace{e^{-i\frac{\pi}{4}\left(1-\hat{\sigma}_A^z\right)}}_{\text{If spin-A is down, removes the unwanted phase of }\pi/2}\underbrace{e^{i\frac{\pi}{4}\left(1-\hat{\sigma}_A^z\right)\sigma_B^x}}_{\text{If spin-A is down, flips spin-B and adds an unwanted phase of }\pi/2}|\psi_{in}\rangle$$

$$= e^{-i\frac{\pi}{4}}e^{i\frac{\pi}{4}\hat{\sigma}_A^z}e^{i\frac{\pi}{4}\hat{\sigma}_B^x}e^{-i\frac{\pi}{4}\hat{\sigma}_A^z\sigma_B^x}|\psi_{in}\rangle$$

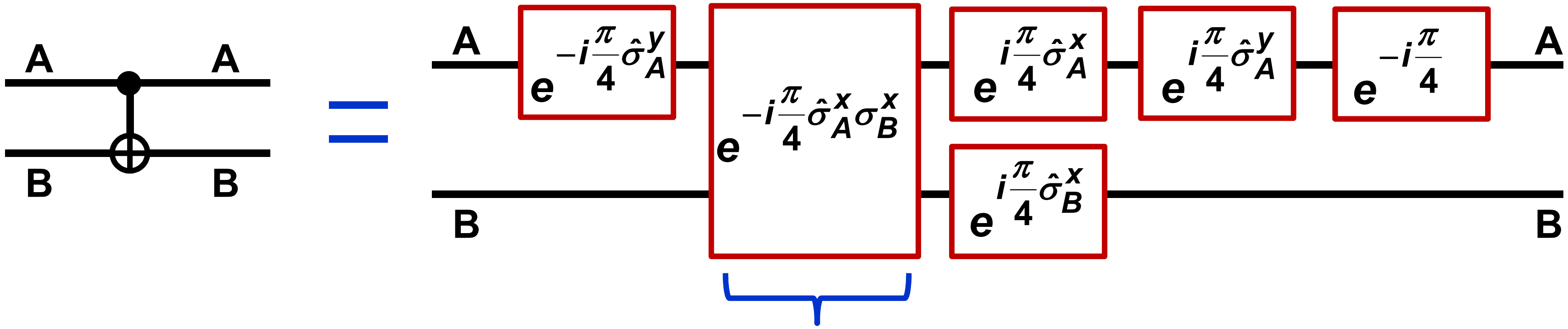
$$= e^{-i\frac{\pi}{4}}\overset{\text{add}}{\boxed{e^{+i\frac{\pi}{4}\hat{\sigma}_A^y}e^{-i\frac{\pi}{4}\hat{\sigma}_A^y}}}e^{i\frac{\pi}{4}\hat{\sigma}_A^z}e^{i\frac{\pi}{4}\hat{\sigma}_B^x}e^{-i\frac{\pi}{4}\hat{\sigma}_A^z\sigma_B^x}\overset{\text{add}}{\boxed{e^{+i\frac{\pi}{4}\hat{\sigma}_A^y}e^{-i\frac{\pi}{4}\hat{\sigma}_A^y}}}|\psi_{in}\rangle$$

Use the relation on previous slide

$$= \underbrace{e^{-i\frac{\pi}{4}}e^{+i\frac{\pi}{4}\hat{\sigma}_A^y}e^{i\frac{\pi}{4}\hat{\sigma}_A^x}e^{i\frac{\pi}{4}\hat{\sigma}_B^x}}_{\text{Single qubit gates}}\underbrace{e^{-i\frac{\pi}{4}\hat{\sigma}_A^x\sigma_B^x}}_{\text{Two-qubit gate}}\underbrace{e^{-i\frac{\pi}{4}\hat{\sigma}_A^y}}_{\text{Single qubit gate}}|\psi_{in}\rangle$$

Implementation of a CNOT Gate

$$\left|\psi_{out}\right\rangle = e^{-i\frac{\pi}{4}} e^{+i\frac{\pi}{4}\hat{\sigma}_A^y} e^{i\frac{\pi}{4}\hat{\sigma}_A^x} e^{i\frac{\pi}{4}\hat{\sigma}_B^x} e^{-i\frac{\pi}{4}\hat{\sigma}_A^x\sigma_B^x} e^{-i\frac{\pi}{4}\hat{\sigma}_A^y} \left|\psi_{in}\right\rangle$$

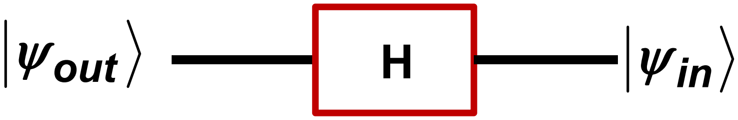


Two-qubit gate

The $e^{-i\frac{\pi}{4}\hat{\sigma}_A^x\sigma_B^x}$ two-qubit gate can be implemented using any of the techniques discussed earlier in this lecture

The Hadamard Gate Revisited

The H-gate:



$$|\psi_{out}\rangle = \hat{U}|\psi_{in}\rangle = \frac{1}{\sqrt{2}}\left[\hat{\sigma}_z + \hat{\sigma}_x\right]|\psi_{in}\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}|\psi_{in}\rangle$$

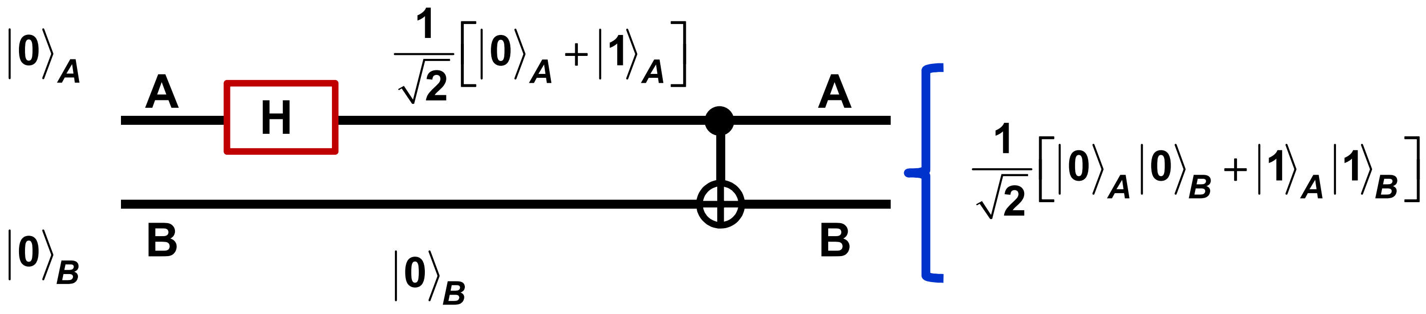
$$|\psi_{in}\rangle = |0\rangle = |z\uparrow\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|\psi_{out}\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right] = \frac{1}{\sqrt{2}}\left[|z\uparrow\rangle + |z\downarrow\rangle\right] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$|\psi_{in}\rangle = |1\rangle = |z\downarrow\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$|\psi_{out}\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle - |1\rangle\right] = \frac{1}{\sqrt{2}}\left[|z\uparrow\rangle - |z\downarrow\rangle\right] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The Control-X (C-NOT) Gate as an Entangler: The Bell Circuit



$|0\rangle_A$

$|0\rangle_B$

$\frac{1}{\sqrt{2}}\left[|0\rangle_A + |1\rangle_A\right]$

$|0\rangle_B$

$\frac{1}{\sqrt{2}}\left[|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B\right]$

Unentangled input state

Unentangled state

Entangled output state

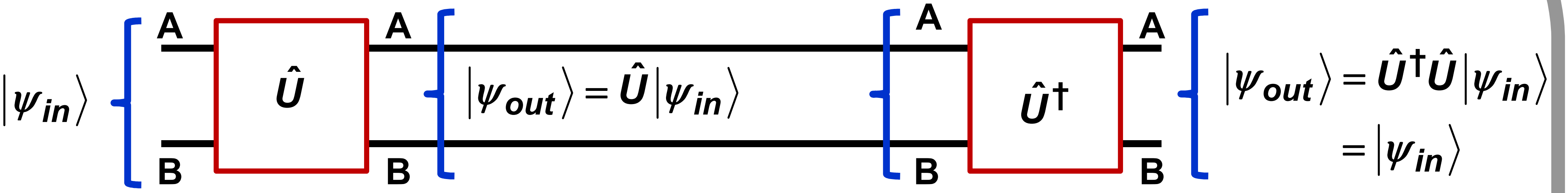
Being able to implement C-NOT gate reliably in any technology is the key to achieving quantum information processing ability in that technology

The Bell Circuit

Input	Output
$\|0\rangle_A\|0\rangle_B$	$\frac{1}{\sqrt{2}}\left[\|0\rangle_A\|0\rangle_B + \|1\rangle_A\|1\rangle_B\right]$
$\|0\rangle_A\|1\rangle_B$	$\frac{1}{\sqrt{2}}\left[\|0\rangle_A\|1\rangle_B + \|1\rangle_A\|0\rangle_B\right]$
$\|1\rangle_A\|0\rangle_B$	$\frac{1}{\sqrt{2}}\left[\|0\rangle_A\|0\rangle_B - \|1\rangle_A\|1\rangle_B\right]$
$\|1\rangle_A\|1\rangle_B$	$\frac{1}{\sqrt{2}}\left[\|0\rangle_A\|1\rangle_B - \|1\rangle_A\|0\rangle_B\right]$

The set of four possible output states are called Bell states

Two-Qubit Quantum Gates and Circuits are Reversible

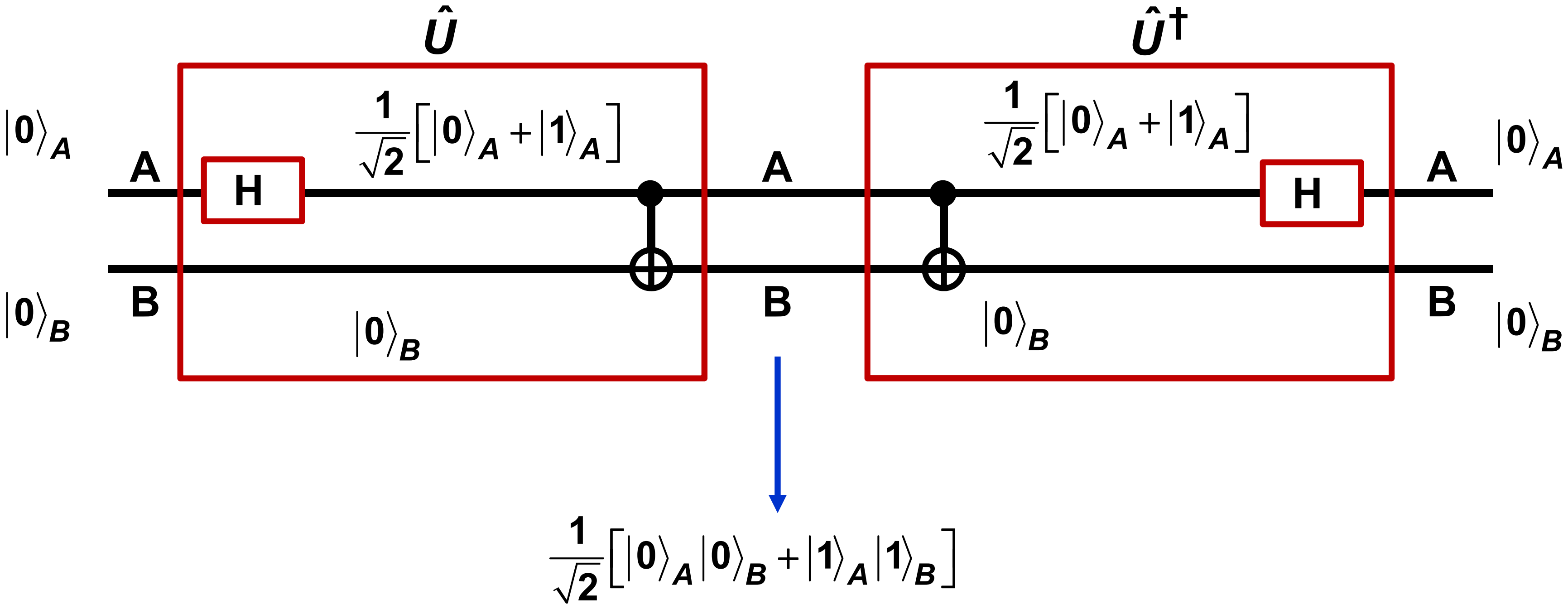


$|\psi_{in}\rangle$

$|\psi_{out}\rangle = \hat{U}|\psi_{in}\rangle$

$$|\psi_{out}\rangle = \hat{U}^\dagger \hat{U}|\psi_{in}\rangle$$
$$= |\psi_{in}\rangle$$

Example: **Reversing a Bell circuit**



$|0\rangle_A$

$|0\rangle_B$

$\frac{1}{\sqrt{2}}\left[|0\rangle_A + |1\rangle_A\right]$

$|0\rangle_B$

$\frac{1}{\sqrt{2}}\left[|0\rangle_A + |1\rangle_A\right]$

$|0\rangle_B$

$|0\rangle_A$

$|0\rangle_B$

$$\frac{1}{\sqrt{2}}\left[|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B\right]$$

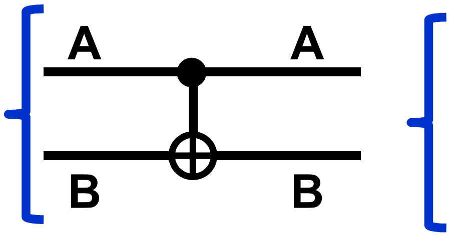
Another Hilbert Space Mapping: Change of Representation

The Hilbert space of two-qubits consists of the following four unentangled basis states that can be represented by the four dimensional Hilbert space of column vectors:

$$|0\rangle_A |0\rangle_B \rightarrow \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad |0\rangle_A |1\rangle_B \rightarrow \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \quad |1\rangle_A |0\rangle_B \rightarrow \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \quad |1\rangle_A |1\rangle_B \rightarrow \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

Then the action of the C-NOT gate (or the Control-X gate) can be represented by the 4x4 matrix acting in the new Hilbert space:

$|\psi_{in}\rangle$



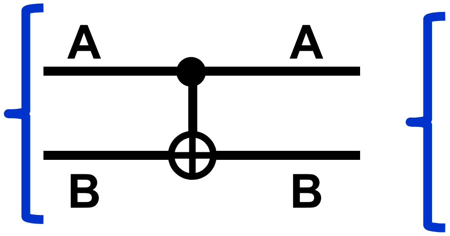
$$|\psi_{out}\rangle = \hat{U}|\psi_{in}\rangle \quad \Longrightarrow \quad \hat{U} = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & \boxed{\begin{matrix}0 & 1\\1 & 0\end{matrix}}\end{bmatrix}$$

$\hat{\sigma}_x$

The Control-Z and Control-Y Gates

The Control-X (C-NOT) gate is:

$|\psi_{in}\rangle$



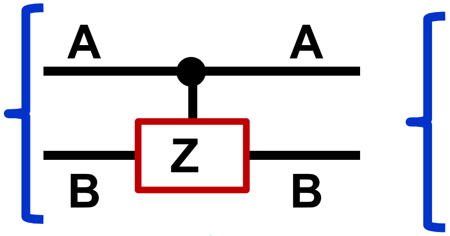
$$|\psi_{out}\rangle = \hat{U}|\psi_{in}\rangle \Longrightarrow \hat{U} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\hat{\sigma}_x$

Depending on the value of the control qubit (i.e. A), the other qubit (B) gets operated upon by $\hat{\sigma}_x$

The Control-Z gate is:

$|\psi_{in}\rangle$



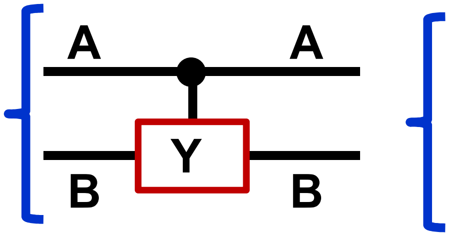
$$|\psi_{out}\rangle = \hat{U}|\psi_{in}\rangle \Longrightarrow \hat{U} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$\hat{\sigma}_z$

Depending on the value of the control qubit (i.e. A), the other qubit (B) gets operated upon by $\hat{\sigma}_z$

The Control-Y gate is:

$|\psi_{in}\rangle$



$$|\psi_{out}\rangle = \hat{U}|\psi_{in}\rangle \Longrightarrow \hat{U} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}$$

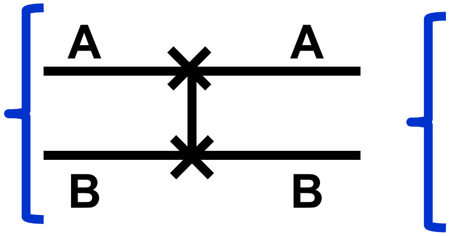
$\hat{\sigma}_y$

Depending on the value of the control qubit (i.e. A), the other qubit (B) gets operated upon by $\hat{\sigma}_y$

The SWAP Gate

$$|0\rangle_A|0\rangle_B \rightarrow \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad |0\rangle_A|1\rangle_B \rightarrow \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \quad |1\rangle_A|0\rangle_B \rightarrow \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \quad |1\rangle_A|1\rangle_B \rightarrow \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

$|\psi_{in}\rangle$



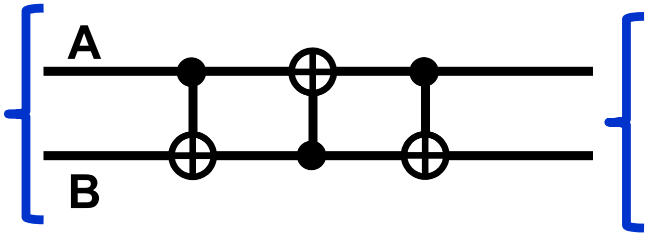
$$|\psi_{out}\rangle = \hat{U}|\psi_{in}\rangle \Longrightarrow \hat{U} = \begin{bmatrix}1&0&0&0\\0&\boxed{\begin{matrix}0&1\\1&0\end{matrix}}&0\\0&&&0\\0&0&0&1\end{bmatrix}$$

$\hat{\sigma}_x$

SWAP	
Input	Output
$\|0\rangle_A\|0\rangle_B$	$\|0\rangle_A\|0\rangle_B$
$\|0\rangle_A\|1\rangle_B$	$\|1\rangle_A\|0\rangle_B$
$\|1\rangle_A\|0\rangle_B$	$\|0\rangle_A\|1\rangle_B$
$\|1\rangle_A\|1\rangle_B$	$\|1\rangle_A\|1\rangle_B$

The Mystery Circuit

$|\psi_{in}\rangle$



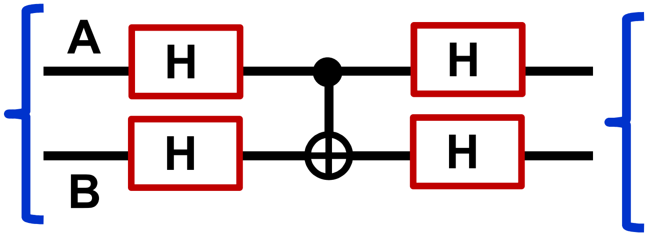
$|\psi_{out}\rangle = \hat{U}|\psi_{in}\rangle$

Mystery	
Input	Output
$|0\rangle_A|0\rangle_B$	
$|0\rangle_A|1\rangle_B$	
$|1\rangle_A|0\rangle_B$	
$|1\rangle_A|1\rangle_B$	

$$\hat{U} = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

The Mystery Circuit - II

$|\psi_{in}\rangle$



$|\psi_{out}\rangle = \hat{U}|\psi_{in}\rangle$

Mystery - II	
Input	Output
$\vert 0\rangle_A \vert 0\rangle_B$	
$\vert 0\rangle_A \vert 1\rangle_B$	
$\vert 1\rangle_A \vert 0\rangle_B$	
$\vert 1\rangle_A \vert 1\rangle_B$	

$$\hat{U} = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$