Lecture 2

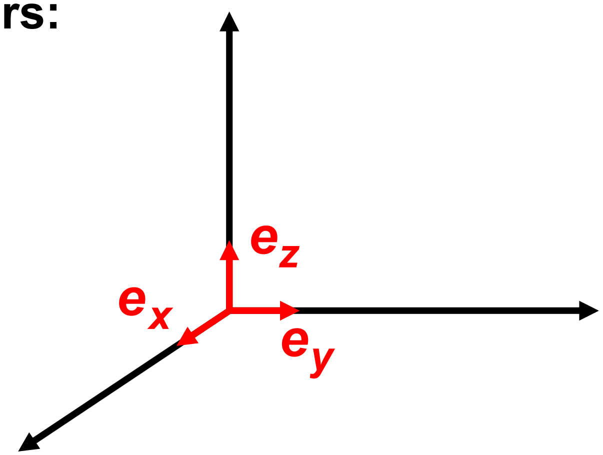
A Primer on Wave Phenomenon

In this lecture you will learn:

- **Waves and wave motion**
- **Wave equations**
- **Wave properties (wavelength, wave velocity, wave dispersion, wave frequency, etc)**
- **More wave properties (interference)**

A Word on Notation

Coordinate system and unit vectors:

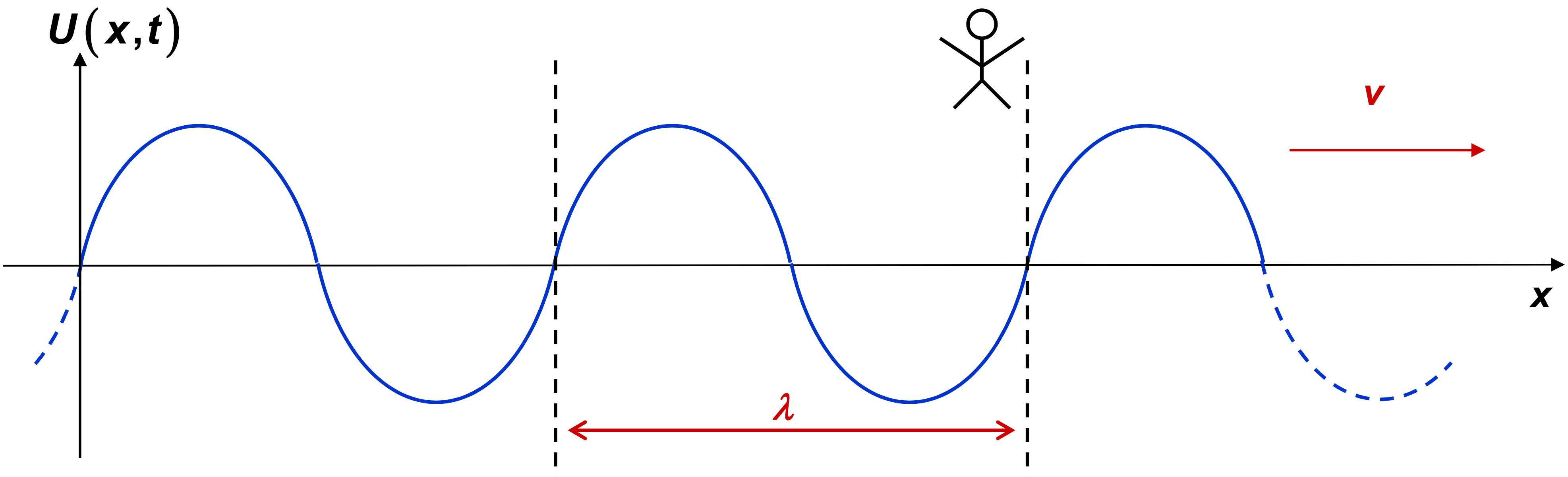


Vectors:

$$\vec{A} = A_x e_x + A_y e_y + A_z e_z$$

Basic Wave Motion

Consider this wave moving in the +*x*-direction:
The wave travels a distance equal to one wavelength in one time period *T*



v = velocity of wave propagation
λ = wavelength of the wave
f = frequency of the wave
T = period = $1/f$

$$v = \text{velocity} = \frac{\text{distance}}{\text{time}} = \frac{\lambda}{T} = f\lambda$$

Basic relation for wave motion: $\boxed{f\,\lambda = v}$

1D Wave Equation

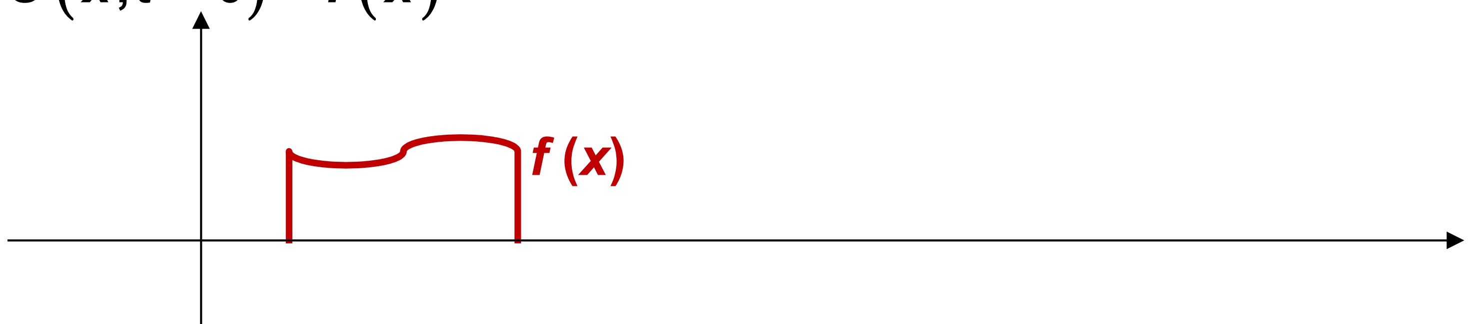
Most common linear waves are described by a differential equation that is second order in space and second order in time:

$$\frac{\partial^2 U(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 U(x,t)}{\partial t^2}$$

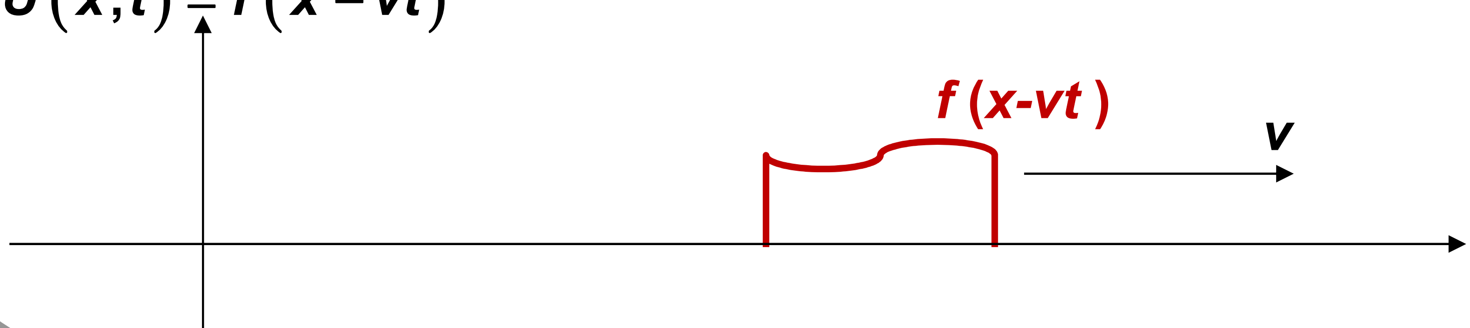
Any function in which position *x* and time *t* come in the form of *x* ± *vt* , e.g. *f*(*x*-*vt*) , will satisfy the wave equation

$$U(x,t) = f(x - vt)$$

$$U(x, t = 0) = f(x)$$



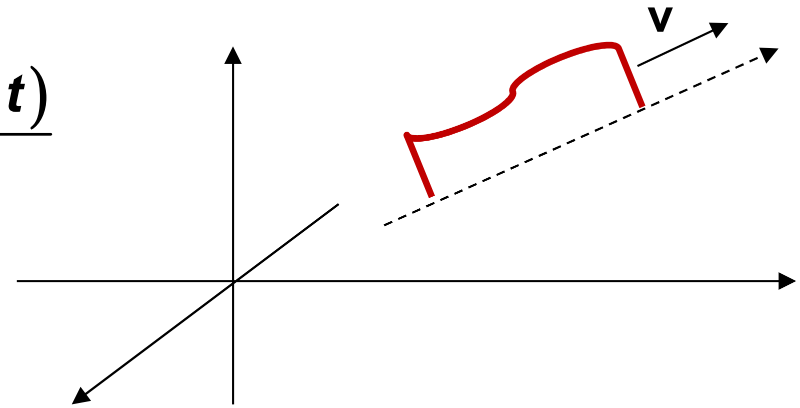
$$U(x,t) = f(x - vt)$$



3D Wave Equation

Most common linear waves are described by a differential equation that is second order in space and second order in time:

$$\nabla^2 U(x,y,z,t) = \frac{1}{v^2}\frac{\partial^2 U(x,y,z,t)}{\partial t^2}$$



Reality of the wave amplitude, time reversal symmetry of basic physics, and isotropy of space, ensure that the above equation is very general

In case you have forgotten your vector calculus:

$$\nabla^2 U(x,y,z,t) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) U(x,y,z,t) \longrightarrow \text{Laplacian}$$

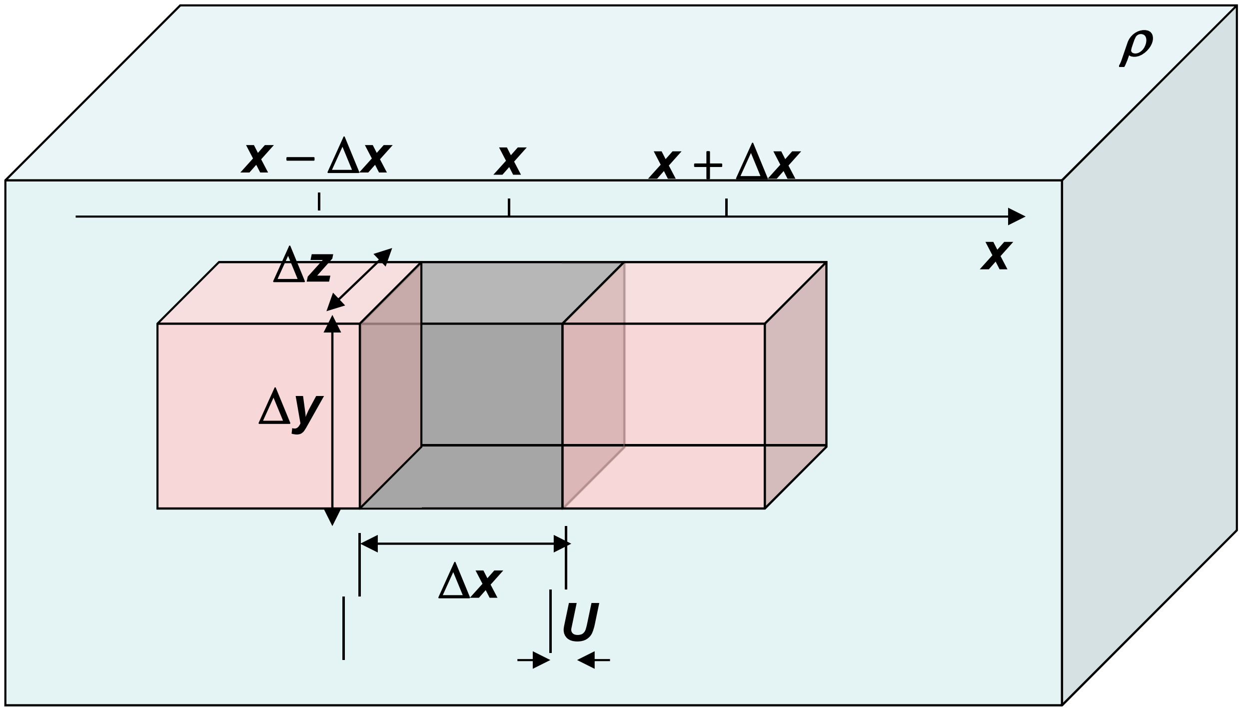
$$\nabla U(x,y,z,t) = \left(e_x \frac{\partial}{\partial x} + e_y \frac{\partial}{\partial y} + e_z \frac{\partial}{\partial z}\right) U(x,y,z,t) \longrightarrow \text{Gradient}$$

In 1D:

$$\frac{\partial^2 U(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 U(x,t)}{\partial t^2}$$

Example: Sound Waves in Solids

Consider a volume Δx Δy Δz inside a solid material of density ρ :

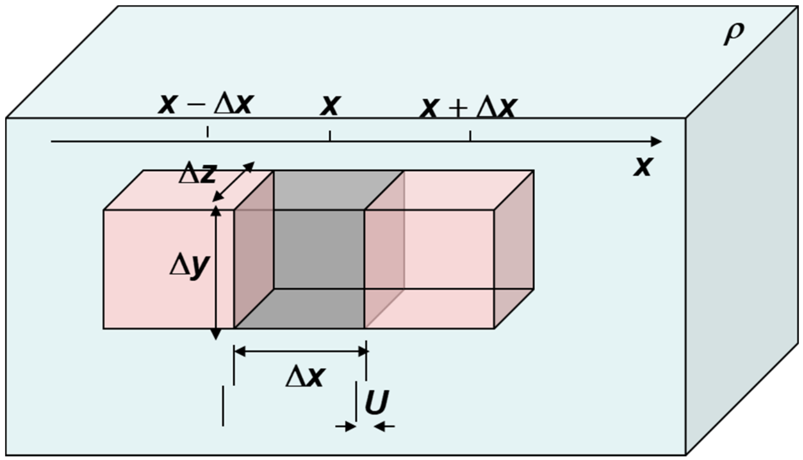


As a result of a wave passing through the solid medium in the x-direction, the indicated volume is displaced by an amount U from its equilibrium position

U is a function of position and time:

$$U(x, y, z, t) \quad \text{or} \quad U(\vec{r}, t) \qquad \left\{ \vec{r} = x e_x + y e_y + z e_z \right.$$

Sound Waves in Solids



Newton's second law states that:

$$F = ma$$

We want to apply the second law to the volume indicated

Velocity of the volume: $$\frac{\partial U(x,y,z,t)}{\partial t}$$

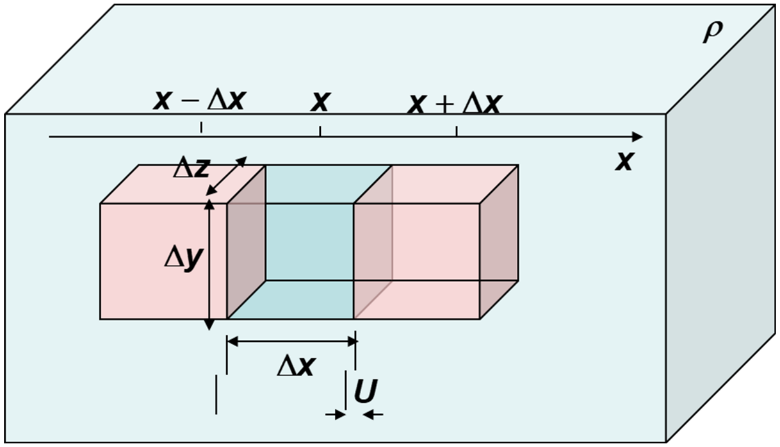
Acceleration of the volume: $$\frac{\partial^2 U(x,y,z,t)}{\partial t^2}$$

Therefore:

$$ma = (\rho \Delta x \Delta y \Delta z)\frac{\partial^2 U(x,y,z,t)}{\partial t^2}$$

Next, we need to find the force acting on this volume!

Sound Waves in Solids



Force in an elastic medium is proportional to the amount of stretching or strain (Hooke's Law)

What if take the force to be proportional to *U* ??

Will not work – because what if the volumes before and after are also displaced by the same *U* !

We write the force *F* as the difference of the forces on the front and back facets:

$$F = K\Delta y\Delta z\left[\frac{U(x+\Delta x)-U(x)}{\Delta x}\right] - K\Delta y\Delta z\left[\frac{U(x)-U(x-\Delta x)}{\Delta x}\right]$$

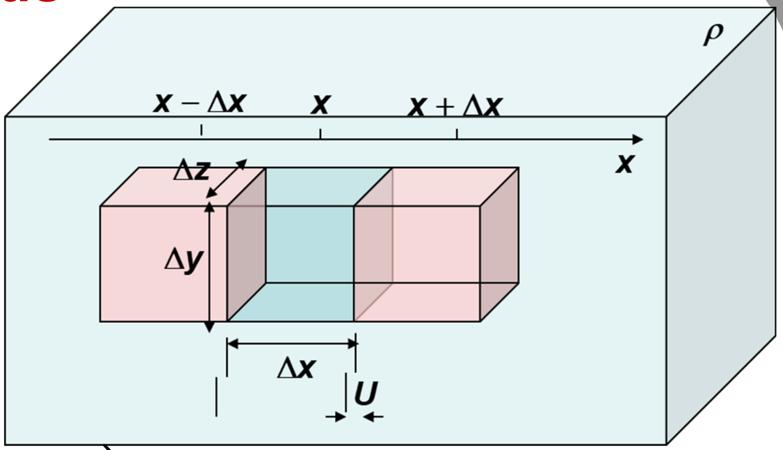
K is an elastic constant of the material with units of N/m^2

$$F = K\Delta y\Delta z\Delta x\left\{\frac{\left[\frac{U(x+\Delta x)-U(x)}{\Delta x}\right]-\left[\frac{U(x)-U(x-\Delta x)}{\Delta x}\right]}{\Delta x}\right\}$$

$$= K\Delta y\Delta z\Delta x\frac{\partial^2 U(x,y,z,t)}{\partial x^2}$$

Sound Waves in Solids

So Newton's second law then becomes:



$$F = ma$$

$$\Rightarrow K\Delta y\Delta z\Delta x\frac{\partial^2 U(x,y,z,t)}{\partial x^2} = (\rho\Delta x\Delta y\Delta z)\frac{\partial^2 U(x,y,z,t)}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 U(x,y,z,t)}{\partial x^2} = \left(\frac{\rho}{K}\right)\frac{\partial^2 U(x,y,z,t)}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 U(x,y,z,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 U(x,y,z,t)}{\partial t^2}$$

This is a wave equation!
Its solutions describe acoustic waves propagating in the material

The wave velocity v is:

$$v = \sqrt{\frac{K}{\rho}}$$

More generally:

$$\boxed{\nabla^2 U(x,y,z,t) = \frac{1}{v^2}\frac{\partial^2 U(x,y,z,t)}{\partial t^2}}$$

Example: Electromagnetic Waves

Maxwell's equation for electromagnetism also allow for electromagnetic waves

$$\nabla \times \vec{E} = -\frac{\partial \mu_o \vec{H}}{\partial t}$$ Faraday's Law

$$\nabla \times \vec{H} = \frac{\partial \varepsilon_o \vec{E}}{\partial t}$$ Ampere's Law

Coupled equations in Free-space

Start by taking the curl of the first equation on both sides and then use the second equation on the LHS:

$$\nabla \times \left(\nabla \times \vec{E}\right) = -\nabla \times \left(\frac{\partial \mu_o \vec{H}}{\partial t}\right) = -\frac{\partial \mu_o \nabla \times \vec{H}}{\partial t} = -\mu_o \varepsilon_o \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\Rightarrow \quad \nabla \times \left(\nabla \times \vec{E}\right) = -\mu_o \varepsilon_o \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\Rightarrow \quad \nabla \times \left(\nabla \times \vec{E}\right) = -\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} \longrightarrow c = \frac{1}{\sqrt{\varepsilon_o \mu_o}} \approx 3 \times 10^8 \text{ m/s}$$

$$\Rightarrow \quad \nabla\left(\underbrace{\nabla . \vec{E}}_{0}\right) - \nabla^2 \vec{E} = -\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

Divergence of the electric field is zero in the absence of charges

$$\Rightarrow \quad \boxed{\nabla^2 \vec{E}(x,y,z,t) = \frac{1}{c^2} \frac{\partial^2 \vec{E}(x,y,z,t)}{\partial t^2}}$$

Equation for a wave traveling at the speed *c* (speed of light)

Basic Wave Motion - I

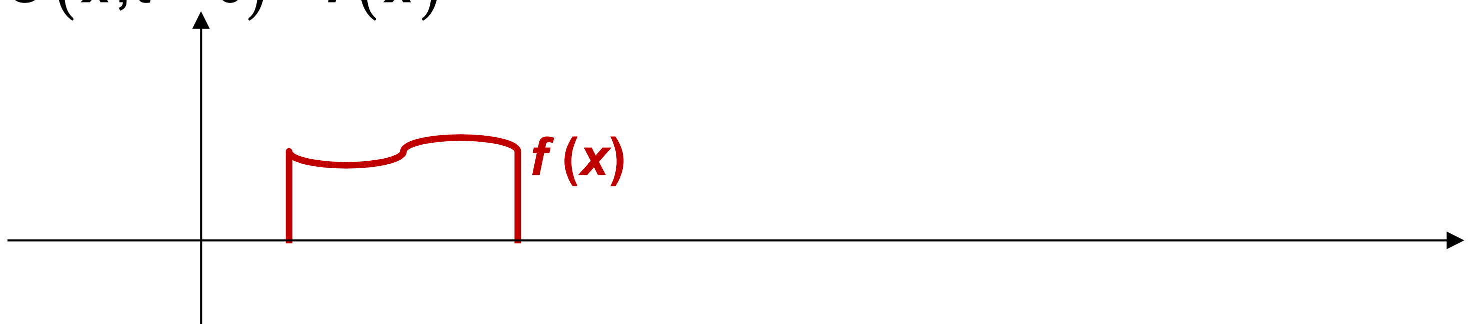
Consider a simple 1-D wave equation:

$$\frac{\partial^2 U(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 U(x,t)}{\partial t^2}$$

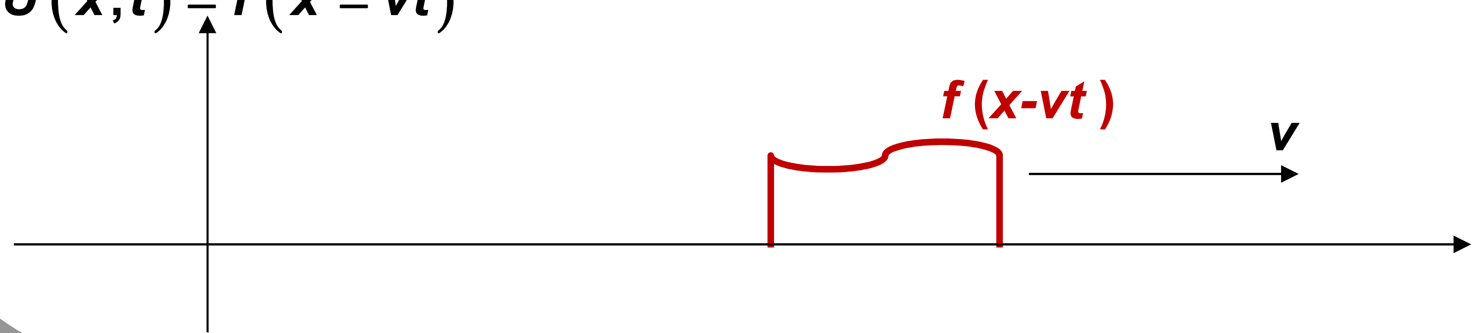
Any function in which position *x* and time *t* come in the form of *x* ± *vt* , e.g. *f*(*x*-*vt*) , will satisfy the wave equation

$$U(x,t) = f(x - vt)$$

$$U(x,t=0) = f(x)$$



$$U(x,t) = f(x = vt)$$



Basic Wave Motion - II

Consider a simple 1-D wave equation:

$$\frac{\partial^2 U(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 U(x,t)}{\partial t^2}$$

Any function in which position x and time t come in the form of $x \pm vt$, e.g. $f(x-vt)$, will satisfy the wave equation

The most commonly used solutions are sinusoids, for example:

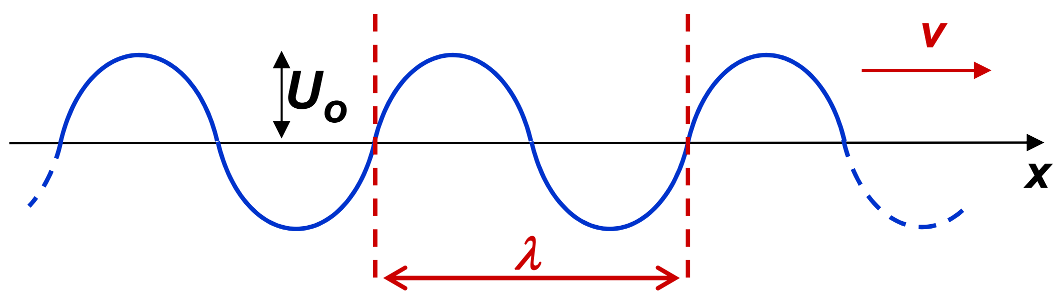
$$U(x,t) = U_o \cos\left(\frac{2\pi}{\lambda}(x - vt)\right)$$

This solution represents a wave that:

i) **Has wavelength λ**

ii) **Is moving in the +x-direction**

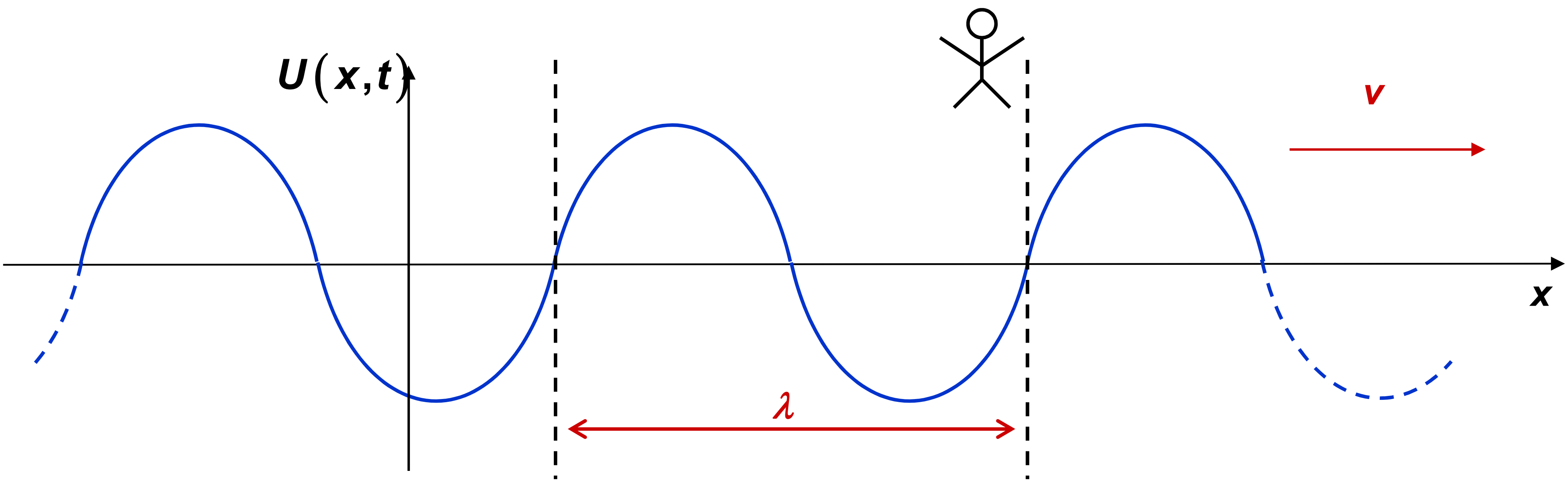
iii) **The velocity of the wave is v**



Basic Wave Motion - III

$$\frac{\partial^2 U(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 U(x,t)}{\partial t^2} \quad \Longrightarrow \quad U(x,t) = U_o \cos\left(\frac{2\pi}{\lambda}(x - v\,t)\right)$$

Consider this wave moving in the +*x*-direction:
The wave travels a distance equal to one wavelength in one time period *T*



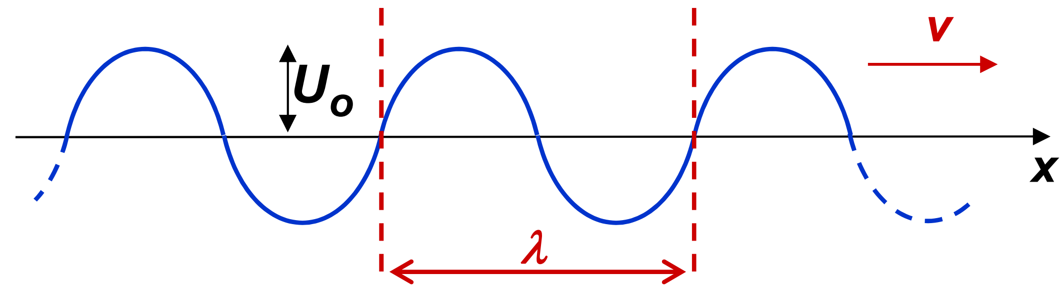
v = velocity of wave propagation
λ = wavelength of the wave
f = frequency of the wave
T = period = $1/f$

$$v = \text{velocity} = \frac{\text{distance}}{\text{time}} = \frac{\lambda}{T} = f\lambda$$

Basic relation for wave motion: $\boxed{f\,\lambda = v}$

Basic Wave Motion - IV

$$\frac{\partial^2 U(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 U(x,t)}{\partial t^2} \longrightarrow U(x,t) = U_o \cos\left(\frac{2\pi}{\lambda}(x - v\,t)\right)$$



The sinusoidal solution,

$$U = U_o \cos\left(\frac{2\pi}{\lambda}(x - v\,t)\right)$$

can also be written as:

$$U = U_o \cos\left(\frac{2\pi}{\lambda}x - \frac{2\pi v}{\lambda}t\right)$$

Define:

$$k = \frac{2\pi}{\lambda} \quad \text{and} \quad \omega = \frac{2\pi v}{\lambda} = 2\pi f$$

(wavevector = k) (radial freq = ω)

To get:

$$U = U_o \cos(kx - \omega t)$$

Basic dispersion relation for wave motion:

$$\boxed{f\,\lambda = v} \longrightarrow \boxed{2\pi f = \frac{2\pi}{\lambda}v} \longrightarrow \boxed{\omega = kv}$$

Basic Wave Motion: Dispersion

$$\frac{\partial^2 U(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 U(x,t)}{\partial t^2} \longrightarrow U = U_o \cos(kx - \omega t)$$

Take the solution: $U = U_o \cos(kx - \omega t)$

And plug it into the wave equation:

$$-k^2 U_o \cos(kx - \omega t) = -\frac{\omega^2}{v^2} U_o \cos(kx - \omega t)$$

$$\Rightarrow k^2 = \frac{\omega^2}{v^2}$$

$$\Rightarrow \omega = kv \longrightarrow$$

This is the dispersion relation of the wave!

Dispersion relation of a wave is the relationship between the frequency ω and the wavevector k

The dispersion is linear in this case – meaning the frequency and the wavevector are linearly related

Basic Wave Motion: Complex Notation

$$\frac{\partial^2 U(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 U(x,t)}{\partial t^2} \longrightarrow U = U_o \cos(kx - \omega t)$$

We can also write the solution as a complex exponential – for CONVENIENCE only:

$$U(x,t) = U_o \cos(kx - \omega t) = U_o \operatorname{Re}\left\{e^{ikx}e^{-i\omega t}\right\}$$

With a little abuse of notation one often writes the solution simply as:

$$U(x,t) = U_o e^{ikx} e^{-i\omega t}$$

But it is understood that the real part is supposed to be taken since $U(x,t)$ represents a real physical quantity (physical quantities are always real – not imaginary)

Some Useful Formulas

Consider the exponential function:

$$e^{ikx}$$

Then note the following relations:

$$\frac{\partial}{\partial x} e^{ikx} = ik\, e^{ikx}$$

$$\frac{\partial^2}{\partial x^2} e^{ikx} = -k^2\, e^{ikx}$$

$$\frac{\partial}{\partial x}\left[e^{ikx} f(x)\right] = ik\, e^{ikx} f(x) + e^{ikx}\,\frac{\partial f(x)}{\partial x} = e^{ikx}\left[\frac{\partial}{\partial x} + ik\right] f(x)$$

Some Useful Formulas

Now consider the following exponential function in 3D:

$$e^{i\left[k_x x + k_y y + k_z z\right]} = e^{i\vec{k}.\vec{r}}$$

$$\left\{ \begin{array}{l} \vec{k} = k_x e_x + k_y e_y + k_z e_z \\ \vec{r} = x e_x + y e_y + z e_z \longrightarrow \text{Position vector} \end{array} \right.$$

Then note the following:

$$\nabla e^{i\vec{k}.\vec{r}} = \left(e_x \frac{\partial}{\partial x} + e_y \frac{\partial}{\partial y} + e_z \frac{\partial}{\partial z} \right) e^{i\vec{k}.\vec{r}} = i\left(k_x e_x + k_y e_y + k_z e_z\right) e^{i\vec{k}.\vec{r}} = i\vec{k}\, e^{i\vec{k}.\vec{r}}$$

$$\nabla^2 e^{i\vec{k}.\vec{r}} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) e^{i\vec{k}.\vec{r}} = -\left(k_x^2 + k_y^2 + k_z^2\right) e^{i\vec{k}.\vec{r}} = -\vec{k}.\vec{k}\, e^{i\vec{k}.\vec{r}} = -k^2\, e^{i\vec{k}.\vec{r}}$$

$$\nabla\left[e^{i\vec{k}.\vec{r}} f\left(\vec{r}\right) \right] = e^{i\vec{k}.\vec{r}} \left[\nabla + i\vec{k} \right] f\left(\vec{r}\right)$$

Basic Wave Motion: Waves in 3D and Wave Dispersion

$$\nabla^2 U(x,y,z,t) = \frac{1}{v^2}\frac{\partial^2 U(x,y,z,t)}{\partial t^2}$$

$$\Rightarrow \quad \nabla^2 U(x,y,z,t) = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] U(x,y,z,t)$$

Solution in 3D for a wave moving in the direction of wavevector $\vec{k}$ is:

$$U(x,y,z,t) = U(\vec{r},t) = U_o \cos\left(\vec{k}.\vec{r} - \omega t\right)$$

$$= U_o \operatorname{Re}\left\{e^{i\vec{k}.\vec{r}} e^{-i\omega t}\right\}$$

$$\left\{\begin{array}{l} \vec{k} = k_x e_x + k_y e_y + k_z e_z \\ \Rightarrow \quad \vec{k}.\vec{k} = k^2 = k_x^2 + k_y^2 + k_z^2 \\ \vec{r} = x e_x + y e_y + z e_z \end{array}\right.$$

$$\nabla^2 U = \frac{1}{v^2}\frac{\partial^2 U}{\partial t^2}$$

$$\Rightarrow \quad -\vec{k}.\vec{k}\left[U_o \cos\left(\vec{k}.\vec{r} - \omega t\right)\right] = -\frac{\omega^2}{v^2}\left[U_o \cos\left(\vec{k}.\vec{r} - \omega t\right)\right]$$

$$\Rightarrow \quad \omega^2 = \left(\vec{k}.\vec{k}\right) v^2 = k^2 v^2$$

$$\Rightarrow \quad \omega = k\,v \longrightarrow$$

The solution can only be correct if: $\omega = k\,v$

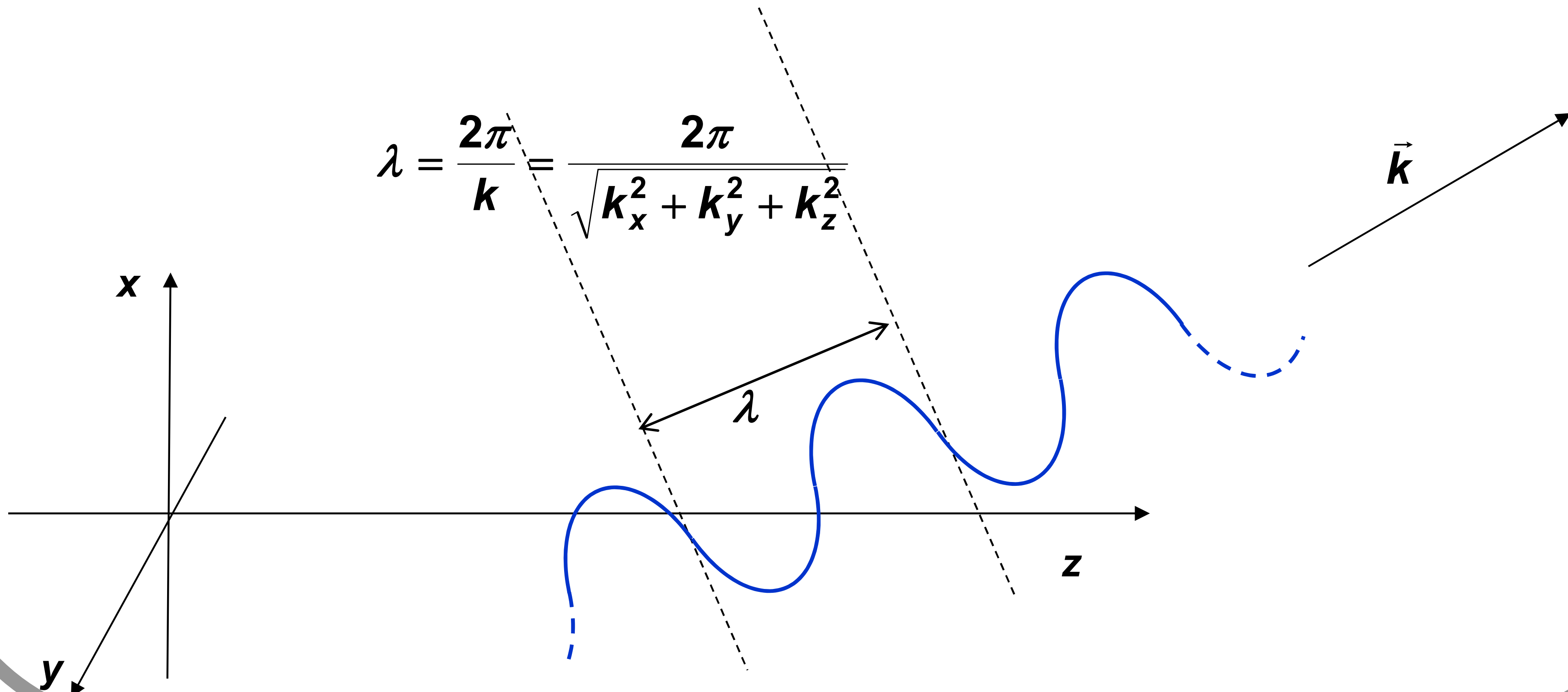
Basic Wave Motion: Waves in 3D

$$U(x,y,z,t) = U(\vec{r},t) = U_o \cos\left(\vec{k}.\vec{r} - \omega t\right)$$

$$= U_o \operatorname{Re}\left\{ e^{i\vec{k}.\vec{r}} e^{-i\omega t} \right\}$$

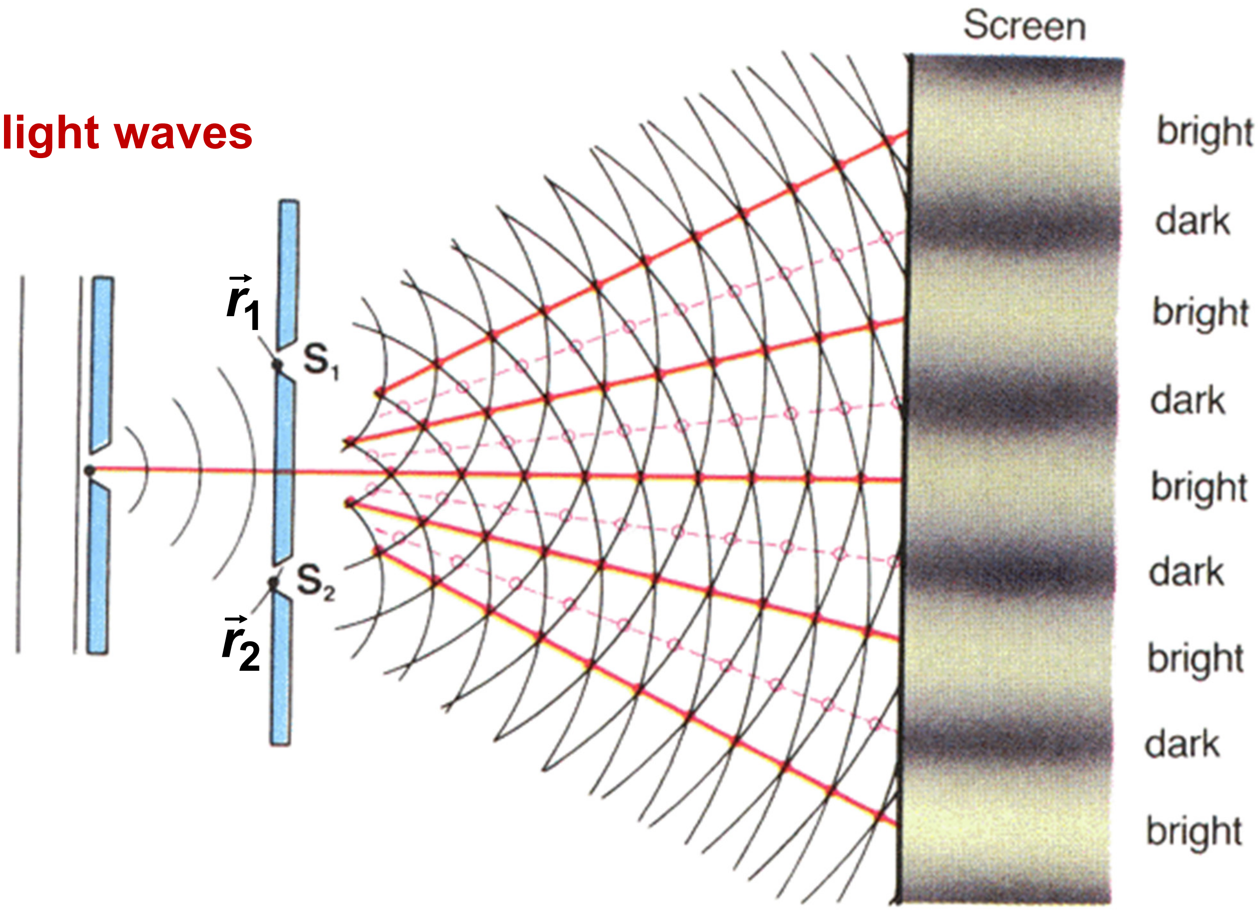
$$\left\{ \begin{array}{l} \vec{k} = k_x e_x + k_y e_y + k_z e_z \\ \Rightarrow \quad \vec{k}.\vec{k} = k^2 = k_x^2 + k_y^2 + k_z^2 \\ \vec{r} = x e_x + y e_y + z e_z \end{array} \right.$$

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{\sqrt{k_x^2 + k_y^2 + k_z^2}}$$



Basic Wave Phenomena: Interference

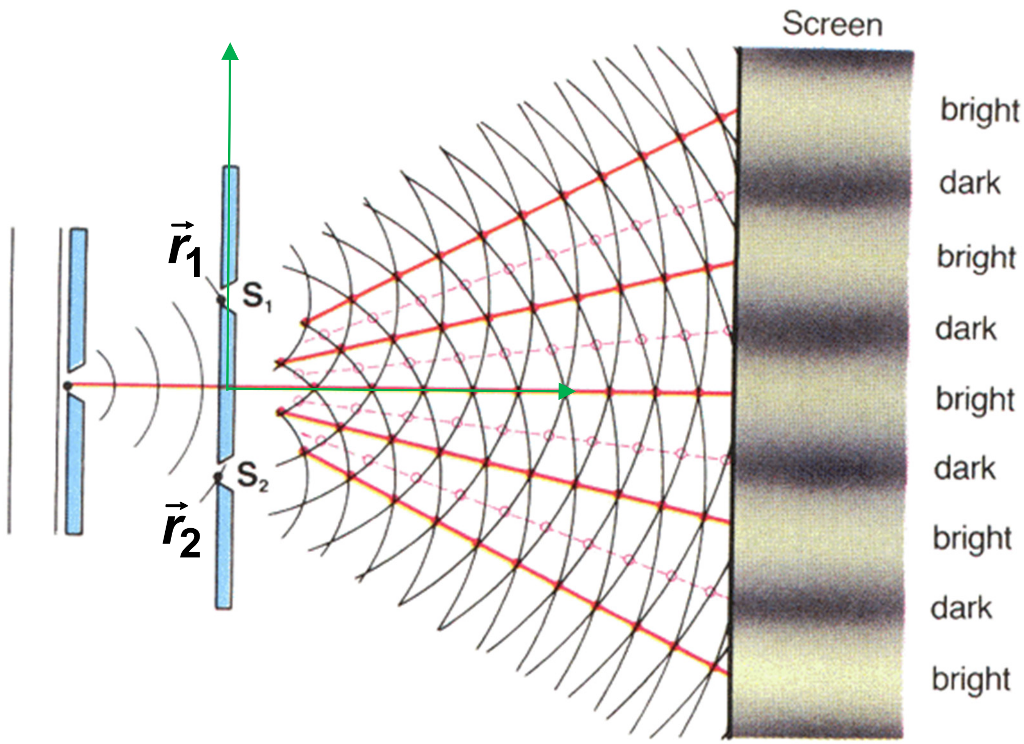
Interference of light waves



Waves interfere!

Points where the waves add in phase, $U(\vec{r},t)$ has the largest swing (anti-nodes)
Points where the waves add out of phase, $U(\vec{r},t)$ has no swing (node)

Basic Wave Phenomena: Interference



$$U(\vec{r},t) = U_o \,\mathrm{Re}\left\{ \frac{e^{i\vec{k}.(\vec{r}-\vec{r}_1)}}{|\vec{r}-\vec{r}_1|} e^{-i\omega t} + \frac{e^{i\vec{k}.(\vec{r}-\vec{r}_2)}}{|\vec{r}-\vec{r}_2|} e^{-i\omega t} \right\} \approx \frac{U_o}{r} \mathrm{Re}\left\{ e^{i\vec{k}.(\vec{r}-\vec{r}_1)} e^{-i\omega t} + e^{i\vec{k}.(\vec{r}-\vec{r}_2)} e^{-i\omega t} \right\}$$

$$|U(\vec{r},t)|^2 \approx \frac{U_o^2}{r^2} \left| e^{i\vec{k}.(\vec{r}-\vec{r}_1)} e^{-i\omega t} + e^{i\vec{k}.(\vec{r}-\vec{r}_2)} e^{-i\omega t} \right|^2$$

$$= \frac{U_o^2}{r^2} \left| e^{i\vec{k}.(\vec{r}-\vec{r}_1)} + e^{i\vec{k}.(\vec{r}-\vec{r}_2)} \right|^2 = 2U_o^2 \left[1 + \cos\left(\vec{k}.(\vec{r}_1 - \vec{r}_2) \right) \right]$$

Wave Equation from Wave Dispersion: A Reverse Engineering Example

Supposing someone tells you that he has a special kind of a medium in which the wave looks like:

$$U(x,t) = U_o e^{ikx} e^{-i\omega t}$$

And that the relation between the frequency and the wavevector (or the relationship between the frequency and the wavelength) of the wave are experimentally found to be:

$$\omega = ak^2 = a\left(\frac{2\pi}{\lambda}\right)^2$$

Question: Find the wave equation!

I know that:

$$\frac{\partial}{\partial x} U(x,t) = \frac{\partial}{\partial x} U_o e^{ikx} e^{-i\omega t} = ikU(x,t)$$

$$\frac{\partial}{\partial t} U(x,t) = \frac{\partial}{\partial t} U_o e^{ikx} e^{-i\omega t} = -i\omega U(x,t)$$

So to get $\omega = ak^2$ the wave equation must look like (verify!!) :

$$i\frac{\partial}{\partial t} U(x,t) = -a^2 \frac{\partial^2}{\partial x^2} U(x,t)$$

First order in time and second order in space

Where in Space is the Wave?

When we write a wave solution of the form:

$$U(x,t) = U_o e^{ikx} e^{-i\omega t}$$

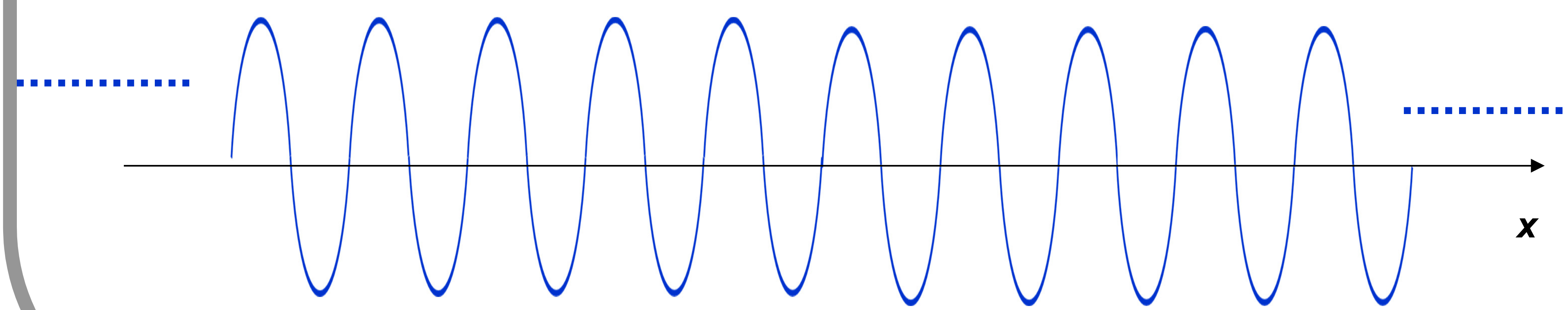
Or if you prefer: $U(x,t) = U_o \operatorname{Re}\left\{e^{ikx} e^{-i\omega t}\right\}$

The wave is spread over All space

It is not localized in an particular location

Clearly this does not correspond to waves that you see in your daily lives

How do we localize waves?



Superposition Principle for Waves

Most simple wave equations, such as:

$$\frac{\partial^2 U(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 U(x,t)}{\partial t^2}$$

are linear differential equations. This means that a superposition (i.e a sum) of functions that satisfy the wave equation will also satisfy the wave equation!

Example:

The function $e^{ik_1 x}e^{-i\omega_1 t}$ satisfies the above wave equation for $\omega_1 = k_1 v$

And the function $e^{ik_2 x}e^{-i\omega_2 t}$ also satisfies the above wave equation for $\omega_2 = k_2 v$

Then the superspoition function (with two arbitrary constants A_1 and A_2):

$$A_1 e^{ik_1 x}e^{-i\omega_1 t} + A_2 e^{ik_2 x}e^{-i\omega_2 t}$$

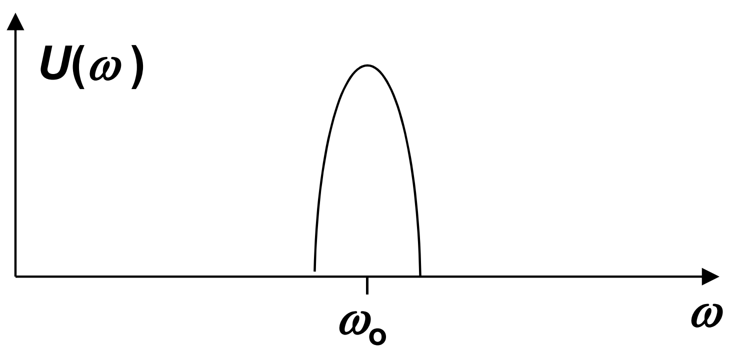
will also satisfy the wave equation

Wave Packets and Group Velocity

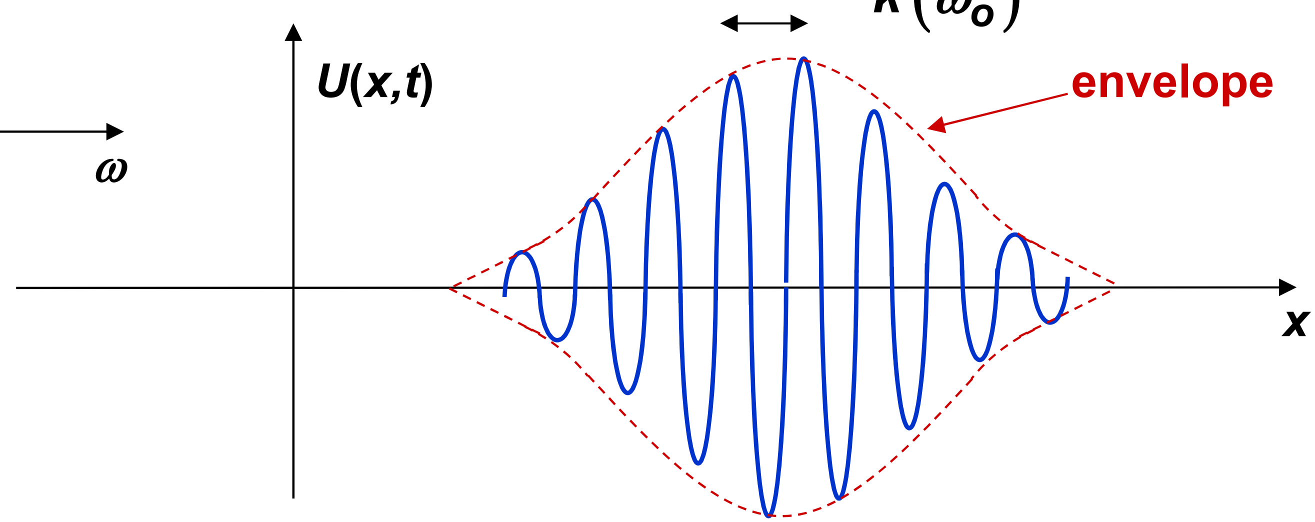
One can always superpose waves of different frequencies to create a localized wave packets of any shape

$$U(x,t) = \mathrm{Re}\left[\int_0^{\infty} \frac{d\omega}{2\pi} U(\omega) e^{ik(\omega)x} e^{-i\omega t}\right]$$

This looks like a Fourier Transform



$$\lambda_o \approx \frac{2\pi}{k(\omega_o)}$$



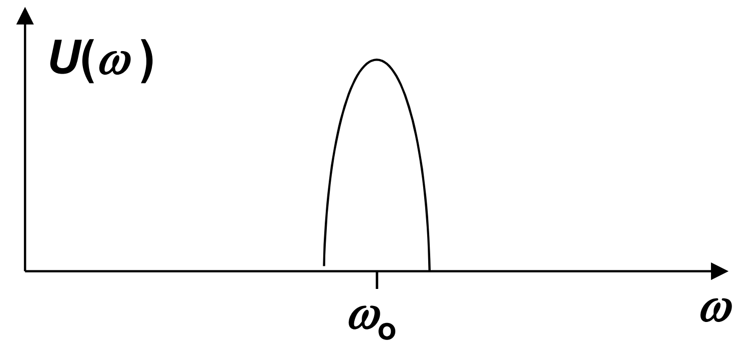
Group velocity = velocity at which the wave packet moves =>

$$\frac{1}{v_g} = \left.\frac{\partial k(\omega)}{\partial \omega}\right|_{\omega=\omega_o}$$

Wave Packets and Group Velocity

Taylor expand:

$$k(\omega) = k(\omega_o) + \left.\frac{\partial k}{\partial \omega}\right|_{\omega=\omega_o} (\omega - \omega_o)$$

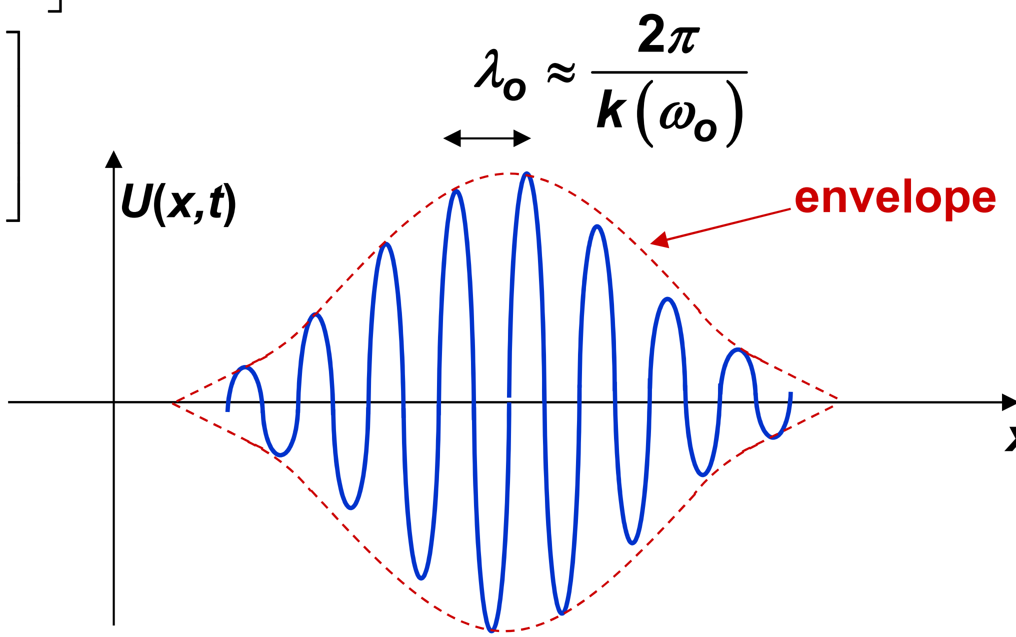


To get:

$$U(x,t) = \mathrm{Re}\left[\int_0^\infty \frac{d\omega}{2\pi} U(\omega) e^{i\left[k(\omega_o) + \left.\frac{\partial k}{\partial \omega}\right|_{\omega=\omega_o}(\omega-\omega_o)\right]x} e^{-i\omega t}\right]$$

$$= \mathrm{Re}\left[e^{ik(\omega_o)x - i\omega_o t} \int_0^\infty \frac{d\omega}{2\pi} U(\omega) e^{i(x - v_g t)\left(\frac{\omega-\omega_o}{v_g}\right)}\right]$$

$$= \mathrm{Re}\left[e^{ik(\omega_o)x - i\omega_o t} U_{evelope}(x - v_g t)\right]$$



Group velocity = velocity at which the wave packet moves:

$$\frac{1}{v_g} = \left.\frac{\partial k(\omega)}{\partial \omega}\right|_{\omega=\omega_o}$$