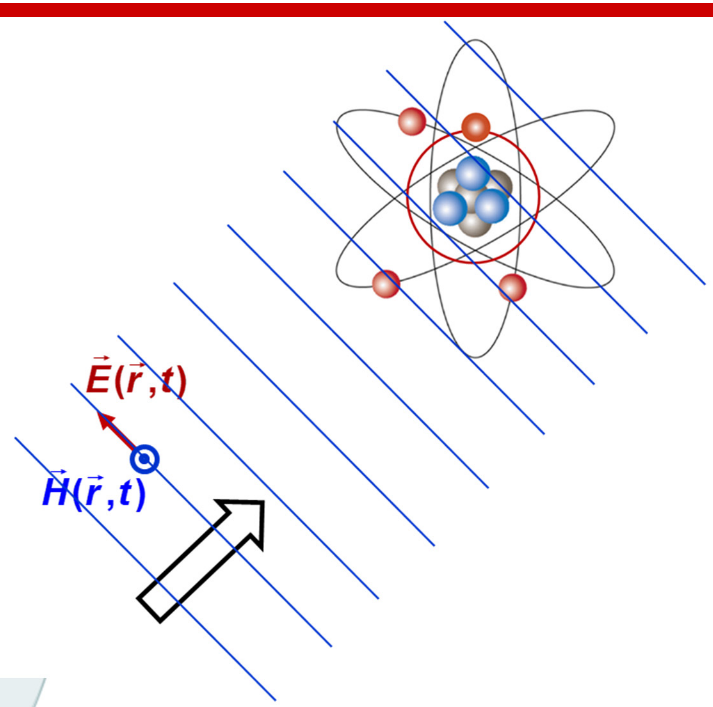
Lecture 18

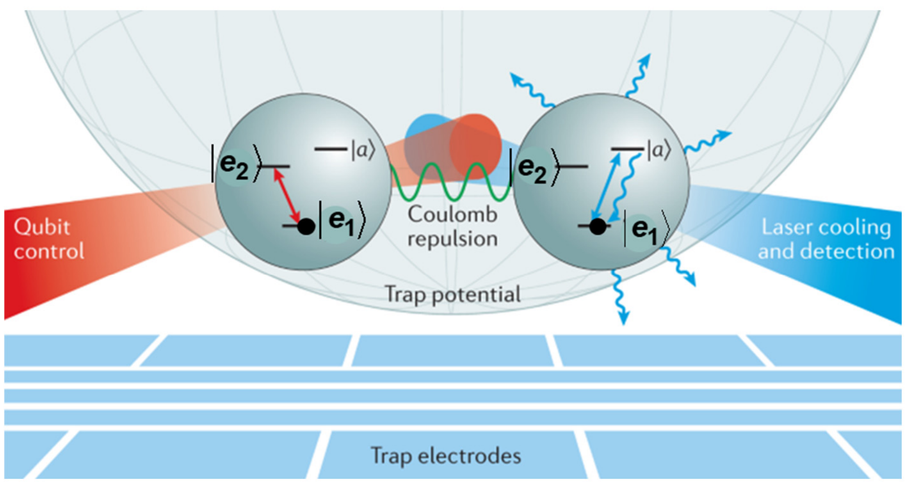
Light-Matter Interaction and Optical Transitions - I

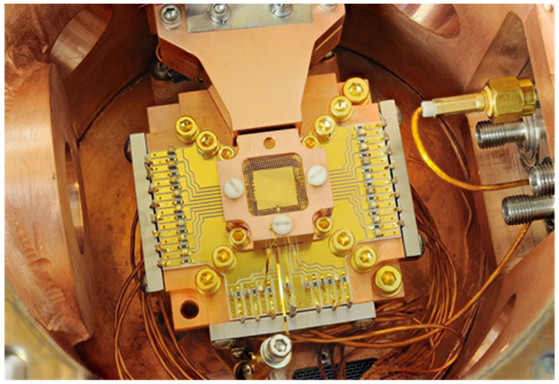
In this lecture you will learn:

Part I:
- Classical electrodynamics
- Gauge transformations
- Light-matter interaction Hamiltonian

Part II:
- Electric-dipole Hamiltonian
- Transformation to a spin Hamiltonian
- Rabi oscillations







An ion-trap chip, NIST (2011)

Light-Matter Interaction

- **Part I (for the self-reading of graduate students)**

  Classical and quantum physics of charged particles
  Derivation of the electric-dipole Hamiltonian for light-matter interaction

- **Part II**

  Interaction of a TLS (an electron in an atom) with light

Part I

Quantum Commutation Relations: A Recap

In quantum mechanics we have for a particle:

$$\left[\hat{r}_k , \hat{p}_j\right] = i\hbar\delta_{kj}$$

This implies:

$$\hat{\vec{p}} = m\hat{\vec{v}} \quad \Longleftrightarrow \quad \frac{\hbar}{i}\nabla \longrightarrow$$

Kinetic momentum of the particle

$$\langle \vec{r} \,|\, \hat{\vec{p}} \,|\, \psi(t)\rangle = \frac{\hbar}{i}\nabla\psi(\vec{r},t)$$

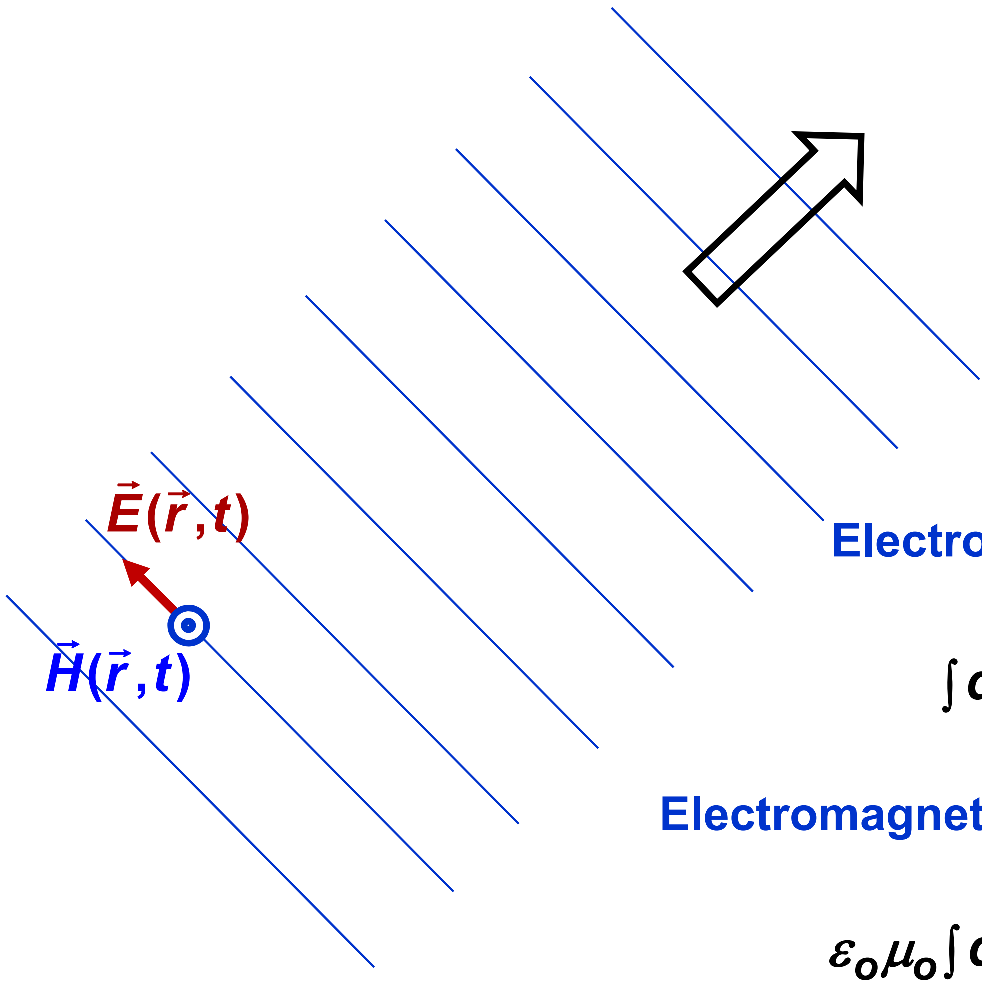
Kinetic energy of the particle in free-space:

$$\hat{H} = \frac{\hat{\vec{p}}.\hat{\vec{p}}}{2m} = \frac{\hat{p}^2}{2m}$$

But what if the particle is charged? Do the above hold?

Classical Electrodynamics (Nothing to do with Quantum Physics)

Consider an electromagnetic wave travelling in free space:



Electromagnetic wave energy is:

$$\int d^3\vec{r}\left\{\frac{1}{2}\varepsilon_o\vec{E}(\vec{r},t).\vec{E}(\vec{r},t)+\frac{1}{2}\mu_o\vec{H}(\vec{r},t).\vec{H}(\vec{r},t)\right\}$$

Electromagnetic wave momentum is:

$$\varepsilon_o\mu_o\int d^3\vec{r}\;\vec{E}\left(\vec{r},t\right)\times\vec{H}\left(\vec{r},t\right)$$

Classical Electrodynamics

The electric and magnetic fields can be represented by scalar and vector potentials:

$$\vec{E}(\vec{r},t) = -\frac{\partial \vec{A}(\vec{r},t)}{\partial t} - \nabla \phi(\vec{r},t)$$

$$\vec{H}(\vec{r},t) = \frac{1}{\mu_o} \nabla \times \vec{A}(\vec{r},t)$$

The vector and scalar potential description is redundant (we are representing 3 degrees of freedom with 4 degrees of freedom)

This means the description of electric and magnetic fields in terms of vector and scalar potentials cannot be unique!

Gauge Transformations and Non-Uniqueness of Electromagnetic Potentials

Suppose one had figured out the vector and scalar potentials such that:

$$\vec{E}(\vec{r},t) = -\frac{\partial \vec{A}(\vec{r},t)}{\partial t} - \nabla \phi(\vec{r},t)$$

$$\vec{H}(\vec{r},t) = \frac{1}{\mu_o} \nabla \times \vec{A}(\vec{r},t)$$

Suppose, amid calculations, one decides to change the vector and scalar potentials as follows:

$$\vec{A}_{new}(\vec{r},t) = \vec{A}(\vec{r},t) + \nabla F(\vec{r},t)$$

$$\phi_{new}(\vec{r},t) = \phi(\vec{r},t) - \frac{\partial}{\partial t} F(\vec{r},t)$$

$F(\vec{r},t)$ is any scalar function

One would still get the same physical electric and magnetic fields:

$$\vec{E}(\vec{r},t) = -\frac{\partial \vec{A}_{new}(\vec{r},t)}{\partial t} - \nabla \phi_{new}(\vec{r},t)$$

$$\vec{H}(\vec{r},t) = \frac{1}{\mu_o} \nabla \times \vec{A}_{new}(\vec{r},t)$$

Scalar and vector potentials are not unique (they depend on the choice of gauge)!
But physical measurable quantities should not depend on the choice of gauge

Classical Electrodynamics and Gauge Choice

The vector and scalar potential description is redundant (we are representing 3 degrees of freedom with 4 degrees of freedom)

We can tie up the extra degree of freedom by a assuming an arbitrary relation between $\vec{A}(\vec{r},t)$ and $\phi(\vec{r},t)$ called the gauge condition:

Coulomb Gauge:

$$\nabla.\vec{A}(\vec{r},t) = 0$$

We will choose this gauge

Lorentz Gauge:

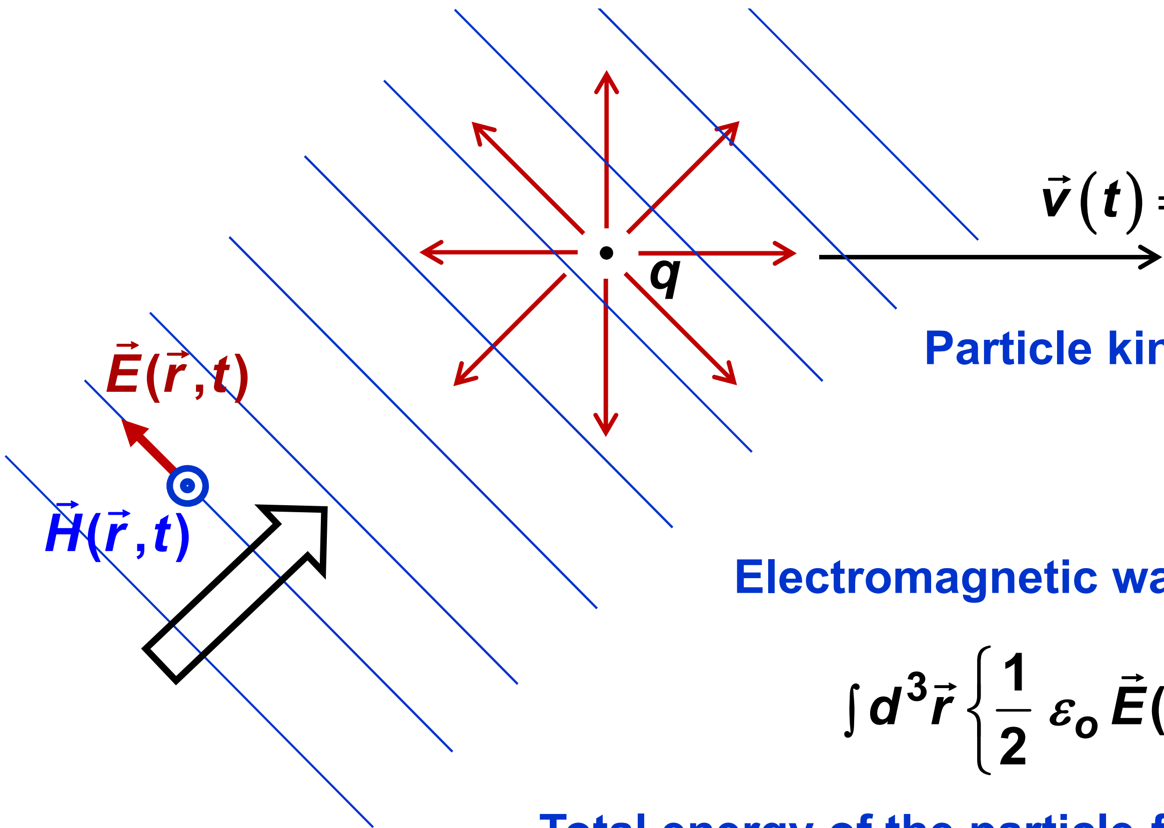
$$\nabla.\vec{A}(\vec{r},t) + \frac{1}{c^2}\frac{\partial\phi(\vec{r},t)}{\partial t} = 0$$

Another Possible Gauge Choice:

$$\phi(\vec{r},t) = 0$$

Classical Electrodynamics

Now consider a charged particle (charge = *q*) in an electromagnetic wave:



$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt}$$

Particle kinetic energy is:

$$\frac{1}{2} m\vec{v}(t).\vec{v}(t)$$

Electromagnetic wave energy is:

$$\int d^3\vec{r} \left\{ \frac{1}{2}\, \varepsilon_o \vec{E}(\vec{r},t).\vec{E}(\vec{r},t) + \frac{1}{2} \mu_o \vec{H}(\vec{r},t).\vec{H}(\vec{r},t) \right\}$$

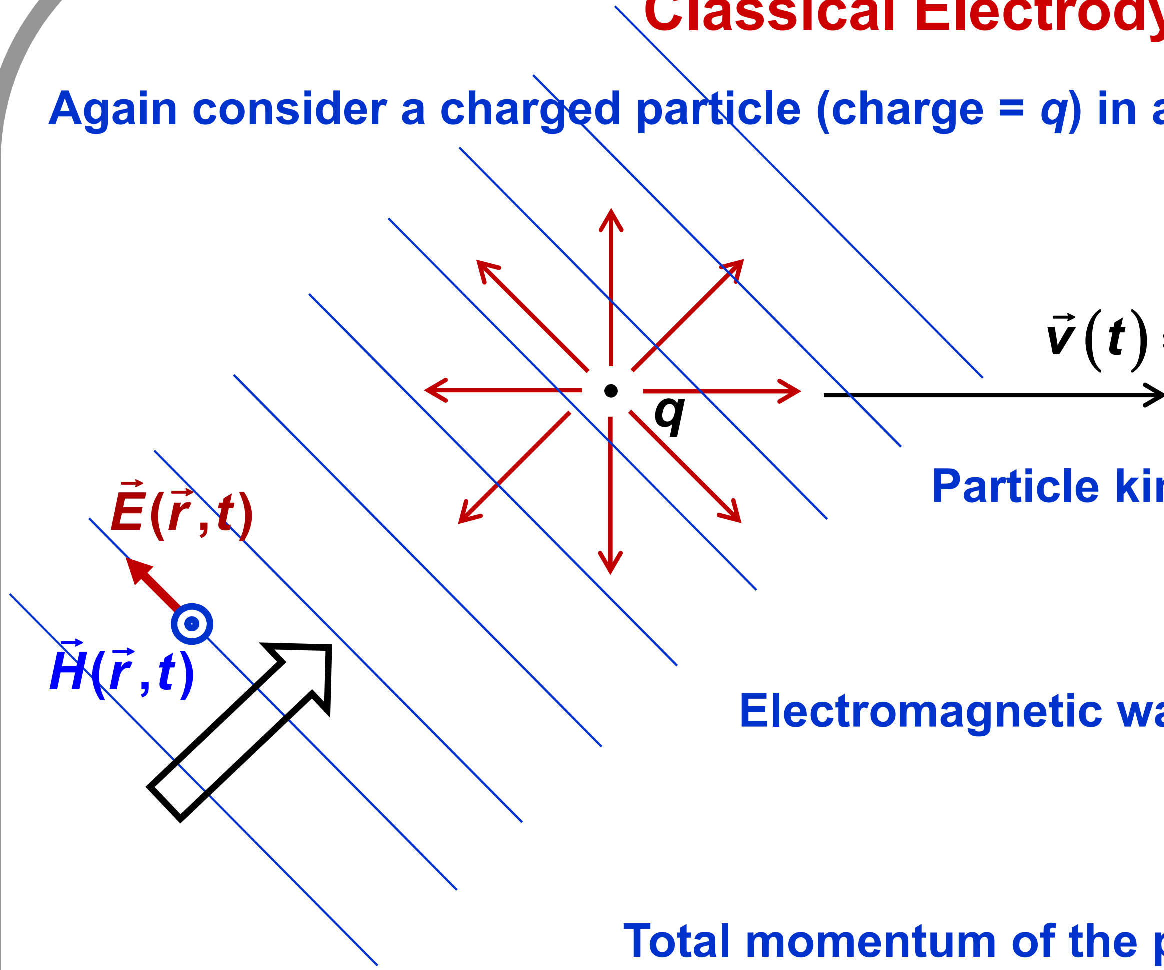
Total energy of the particle-field system is:

$$E_{total} = \frac{1}{2} m\vec{v}(t).\vec{v}(t) + q\phi\left(\vec{r}(t),t\right) + \int d^3\vec{r} \left\{ \frac{1}{2}\, \varepsilon_o \vec{E}(\vec{r},t).\vec{E}(\vec{r},t) + \frac{1}{2} \mu_o \vec{H}(\vec{r},t).\vec{H}(\vec{r},t) \right\}$$

Easy peasy !!

Classical Electrodynamics

Again consider a charged particle (charge = q) in an electromagnetic wave:



$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt}$$

Particle kinetic momentum is:

$$m\vec{v}(t) = m\frac{d\vec{r}(t)}{dt}$$

Electromagnetic wave momentum is:

$$\varepsilon_o\mu_o \int d^3\vec{r}\ \vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t)$$

Total momentum of the particle-field system is:

$$\vec{p}_{total}(t) = \underbrace{m\frac{d\vec{r}(t)}{dt}}_{\text{Kinetic momentum of the particle}} + \underbrace{q\vec{A}(\vec{r}(t),t)}_{} + \underbrace{\varepsilon_o\mu_o \int d^3\vec{r}\ \vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t)}_{\text{Electromagnetic momentum of the wave}}$$

Classical Electrodynamics

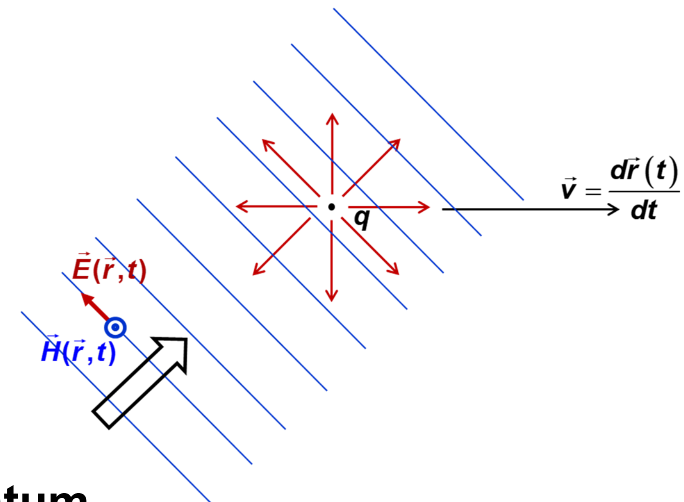
$$\vec{p}_{total}(t) = \underbrace{m\frac{d\vec{r}(t)}{dt}}_{\text{Kinetic momentum of the particle}} + \underbrace{q\vec{A}\big(\vec{r}(t),t\big)}_{} + \underbrace{\varepsilon_o\mu_o\int d^3\vec{r}\;\vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t)}_{\text{Electromagnetic momentum of the wave}}$$

Define a canonical momentum of the particle as:

$$\vec{p}(t) = m\vec{v}(t) + q\vec{A}\big(\vec{r}(t),t\big)$$

$$\Rightarrow m\vec{v}(t) = \vec{p}(t) - q\vec{A}\big(\vec{r}(t),t\big)$$

Kinetic momentum



Particle kinetic energy is:

$$\frac{1}{2}m\vec{v}(t).\vec{v}(t) = \frac{\left[\vec{p}(t) - q\vec{A}\big(\vec{r}(t),t\big)\right]^2}{2m}$$

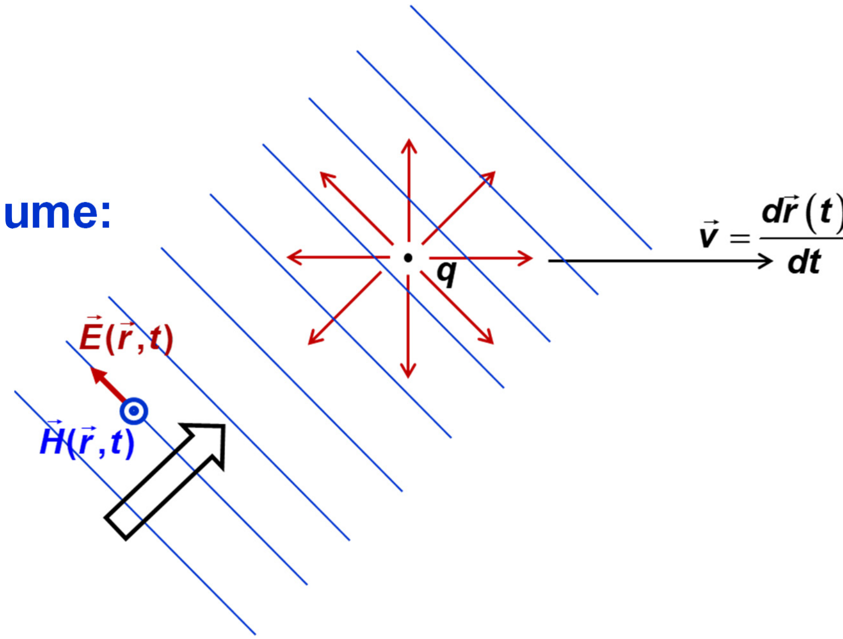
Total particle energy (kinetic + potential) is:

$$H = \frac{1}{2}m\vec{v}(t).\vec{v}(t) + q\phi\big(\vec{r}(t),t\big) = \frac{\left[\vec{p}(t) - q\vec{A}\big(\vec{r}(t),t\big)\right]^2}{2m} + q\phi\big(\vec{r}(t),t\big)$$

Quantum Commutation Relations

We have:

$$\vec{p}(t) = m\vec{v}(t) + q\vec{A}(\vec{r}(t),t)$$



In imposing commutation relations, should we assume:

$$\left[\hat{r}_k, m\hat{v}_j\right] = i\hbar\delta_{kj}$$

Kinetic momentum

Or, should we assume:

$$\left[\hat{r}_k, \hat{p}_j\right] = i\hbar\delta_{kj}$$

Canonical momentum

Turns out only the latter works!

$$\left[\hat{r}_k, \hat{p}_j\right] = i\hbar\delta_{kj} \qquad \left[\hat{p}_k, \hat{p}_j\right] = 0 \qquad \left[\hat{r}_k, \hat{r}_j\right] = 0$$

$$\Rightarrow \langle\vec{r}\,|\hat{\vec{p}}|\psi(t)\rangle = \frac{\hbar}{i}\nabla\psi(\vec{r},t)$$

$$\left[m\hat{v}_k, m\hat{v}_j\right] = i\hbar q\varepsilon_{kjs}\mu_o H_s(\hat{r},t)$$

$$\varepsilon_{123} = 1 \qquad (\text{fully antisymmetric})$$

e.g.: $\varepsilon_{123} = 1, \varepsilon_{132} = -1, \varepsilon_{312} = 1, \ldots\ldots$

Levi-Civita symbol

Time-Dependent Quantum Hamiltonian

Total classical particle energy (kinetic + potential) is:

$$H = \frac{1}{2} m\vec{v}(t).\vec{v}(t) + q\phi\left(\vec{r}(t),t\right) = \frac{\left[\vec{p}(t) - q\vec{A}\left(\vec{r}(t),t\right)\right]^2}{2m} + q\phi\left(\vec{r}(t),t\right)$$

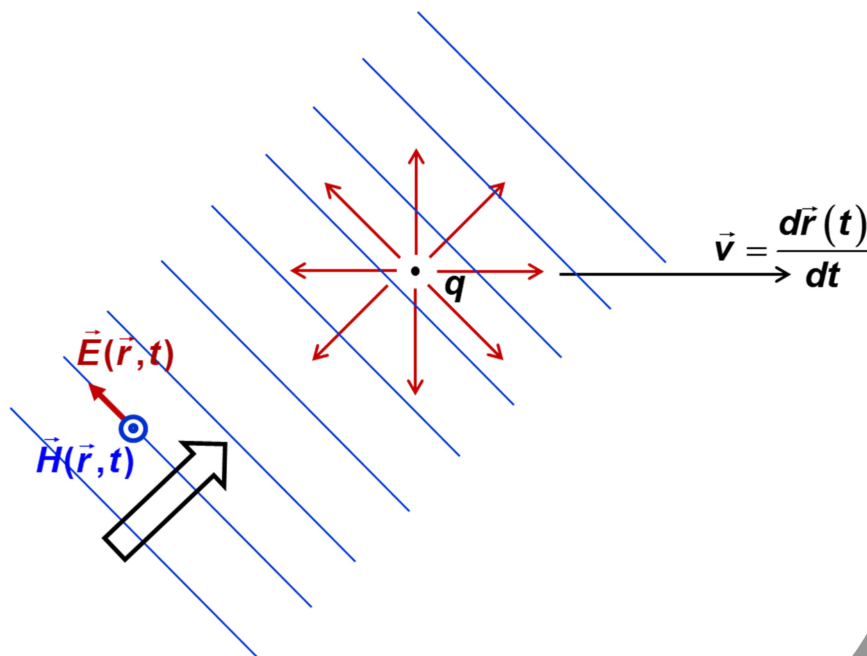
The quantum Hamiltonian operator becomes:

$$\hat{H}(t) = \frac{1}{2} m\hat{\vec{v}}.\hat{\vec{v}} + q\phi\left(\hat{\vec{r}},t\right) = \frac{\left[\hat{\vec{p}} - q\vec{A}\left(\hat{\vec{r}},t\right)\right]^2}{2m} + q\phi\left(\hat{\vec{r}},t\right)$$

The Hamiltonian operator is <u>time-dependent!</u>

We will now need to solve the time-dependent Schrödinger equation:

$$i\hbar \frac{\partial \left|\psi(t)\right\rangle}{\partial t} = \hat{H}(t)\left|\psi(t)\right\rangle$$



Time-Dependent Schrödinger Equation

We have:

$$i\hbar \frac{\partial |\psi(t)\rangle}{\partial t} = \hat{H}(t)|\psi(t)\rangle$$

Where:

$$\hat{H}(t) = \frac{\left[\hat{\vec{p}} - q\vec{A}\left(\hat{\vec{r}},t\right)\right]^2}{2m} + q\phi\left(\hat{\vec{r}},t\right)$$

$$\left\{ \begin{array}{l} \left[\hat{r}_k, \hat{p}_j\right] = i\hbar\delta_{kj} \\ \Rightarrow \langle \vec{r}|\hat{\vec{p}}|\psi(t)\rangle = \frac{\hbar}{i}\nabla\psi(\vec{r},t) \end{array} \right.$$

It follows that:

$$i\hbar \frac{\partial \langle \vec{r}|\psi(t)\rangle}{\partial t} = \langle \vec{r}|\hat{H}(t)|\psi(t)\rangle$$

$$\Rightarrow i\hbar \frac{\partial \psi(\vec{r},t)}{\partial t} = \langle \vec{r}|\frac{\left[\hat{\vec{p}} - q\vec{A}\left(\hat{\vec{r}},t\right)\right]^2}{2m} + q\phi\left(\hat{\vec{r}},t\right)|\psi(t)\rangle$$

$$\Rightarrow i\hbar \frac{\partial \psi(\vec{r},t)}{\partial t} = \frac{\left[\frac{\hbar}{i}\nabla - q\vec{A}(\vec{r},t)\right]^2}{2m}\psi(\vec{r},t) + q\phi\left(\hat{\vec{r}},t\right)\psi(\vec{r},t)$$

But wait a minute …… the vector and scalar potentials are not unique and are gauge-dependent !!

So how can the above Schrodinger equation be universally correct ??

Quantum States and Gauge Choice

$$i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t}=\frac{\left[\frac{\hbar}{i}\nabla-q\vec{A}(\vec{r},t)\right]^2}{2m}\psi(\vec{r},t)+q\phi(\hat{\vec{r}},t)\psi(\vec{r},t)$$

Suppose we make a gauge transformation (in our heads):

$$\vec{A}_{new}(\vec{r},t)=\vec{A}(\vec{r},t)+\nabla F(\vec{r},t)$$

$$\phi_{new}(\vec{r},t)=\phi(\vec{r},t)-\frac{\partial}{\partial t}F(\vec{r},t)$$

$F(\vec{r},t)$ is a scalar function

And if we also assume that under this gauge transformation:

$$\psi_{new}(\vec{r},t)=e^{i\frac{q}{\hbar}F(\vec{r},t)}\psi(\vec{r},t)$$

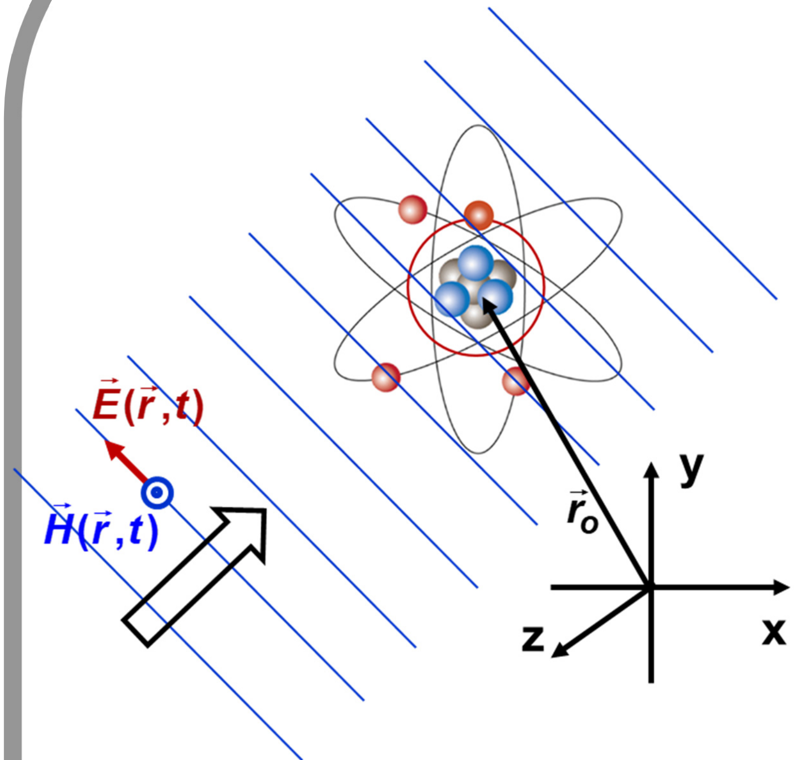
$$|\psi_{new}(t)\rangle=e^{i\frac{q}{\hbar}F(\hat{\vec{r}},t)}|\psi(t)\rangle$$

Then we get (after substituting the above in the Schrodinger equation at the top):

$$i\hbar\frac{\partial\psi_{new}(\vec{r},t)}{\partial t}=\frac{\left[\frac{\hbar}{i}\nabla-q\vec{A}_{new}(\vec{r},t)\right]^2}{2m}\psi_{new}(\vec{r},t)+q\phi_{new}(\hat{\vec{r}},t)\psi_{new}(\vec{r},t)$$

- Quantum states and quantum wavefunctions are gauge dependent !
- But the form of the Schrodinger equation is gauge independent !
- The probability density $|\psi(\vec{r},t)|^2$ is also gauge independent !

Particle in a Potential Well (Atom) Interacting with Radiation



$$i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t}=\frac{\left[\frac{\hbar}{i}\nabla-q\vec{A}(\vec{r},t)\right]^2}{2m}\psi(\vec{r},t)$$

$$+\underbrace{q\phi\left(\hat{\vec{r}},t\right)}_{\text{Scalar potential of the wave}}\psi(\vec{r},t)+\underbrace{V(\vec{r})}_{\text{Atomic potential well}}\psi(\vec{r},t)$$

Coulomb Gauge: $\nabla.\vec{A}(\vec{r},t)=0$

$$\vec{E}(\vec{r},t)=-\frac{\partial\vec{A}(\vec{r},t)}{\partial t}-\nabla\phi(\vec{r},t)$$

$$\nabla.\vec{E}(\vec{r},t)=0\Rightarrow-\nabla^2\phi(\vec{r},t)=0$$

$$\Rightarrow\phi(\vec{r},t)=0$$

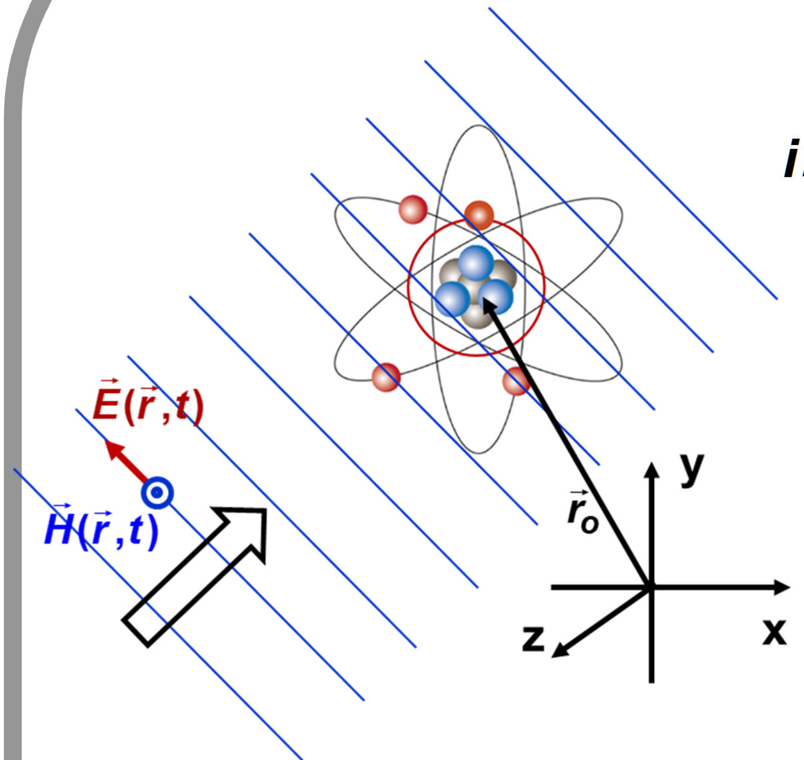
So we have:

$$\vec{E}(\vec{r},t)=-\frac{\partial\vec{A}(\vec{r},t)}{\partial t}$$

$$\vec{H}(\vec{r},t)=\frac{1}{\mu_o}\nabla\times\vec{A}(\vec{r},t)$$

$$i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t}=\frac{\left[\frac{\hbar}{i}\nabla-q\vec{A}(\vec{r},t)\right]^2}{2m}\psi(\vec{r},t)+V(\vec{r})\psi(\vec{r},t)$$

Particle in a Potential Well (Atom) Interacting with Radiation



$$i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t}=\frac{\left[\frac{\hbar}{i}\nabla-q\vec{A}(\vec{r},t)\right]^2}{2m}\psi(\vec{r},t)+V(\vec{r})\psi(\vec{r},t)$$

Now:

$$\vec{A}(\vec{r},t)=A_o\cos\left(\vec{k}.\vec{r}-\omega t\right)$$

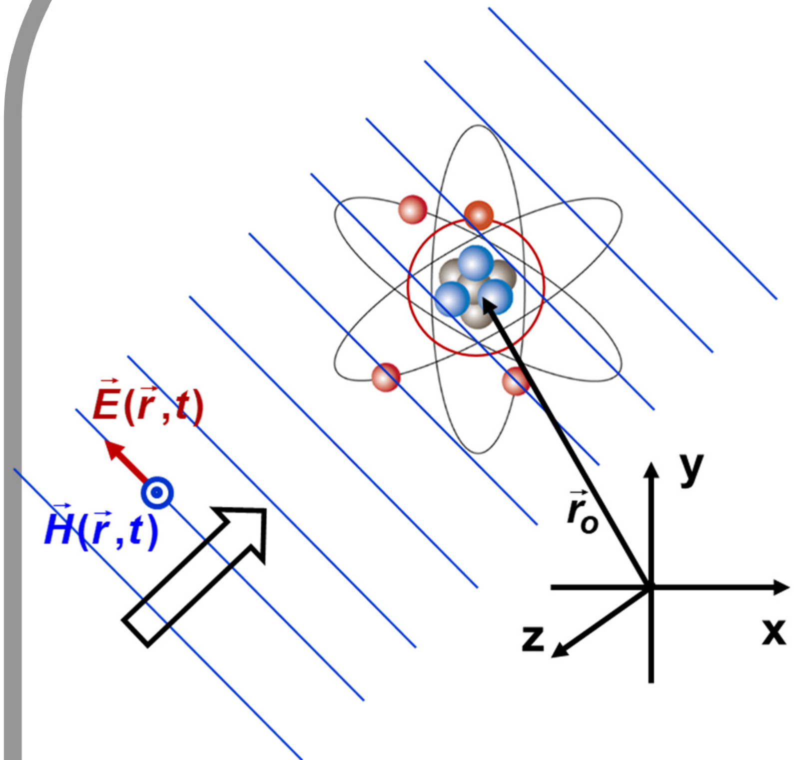
Assume that the potential well (atom) is located at $\vec{r}=\vec{r}_o$ then at the location of the atom:

$$\vec{A}(\vec{r},t)\approx\vec{A}(\vec{r}_o,t)=A_o\cos\left(\vec{k}.\vec{r}_o-\omega t\right)$$

Then we can approximate the Schrodinger equation as (because the field is pretty much uniform in space as far as the atom is concerned):

$$i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t}=\frac{\left[\frac{\hbar}{i}\nabla-q\vec{A}(\vec{r}_o,t)\right]^2}{2m}\psi(\vec{r},t)+V(\vec{r})\psi(\vec{r},t)$$

Particle in a Potential Well (Atom) Interacting with Radiation



$$i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t}=\frac{\left[\frac{\hbar}{i}\nabla-q\vec{A}(\vec{r}_o,t)\right]^2}{2m}\psi(\vec{r},t)+V(\vec{r})\psi(\vec{r},t)$$

Now do a gauge transformation:

$$\vec{A}_{new}(\vec{r},t)=\vec{A}(\vec{r},t)+\nabla F(\vec{r},t)$$

$$\phi_{new}(\vec{r},t)=\cancelto{0}{\phi(\vec{r},t)}-\frac{\partial}{\partial t}F(\vec{r},t)$$

$$\psi_{new}(\vec{r},t)=e^{i\frac{q}{\hbar}F(\vec{r},t)}\psi(\vec{r},t)$$

Choose:

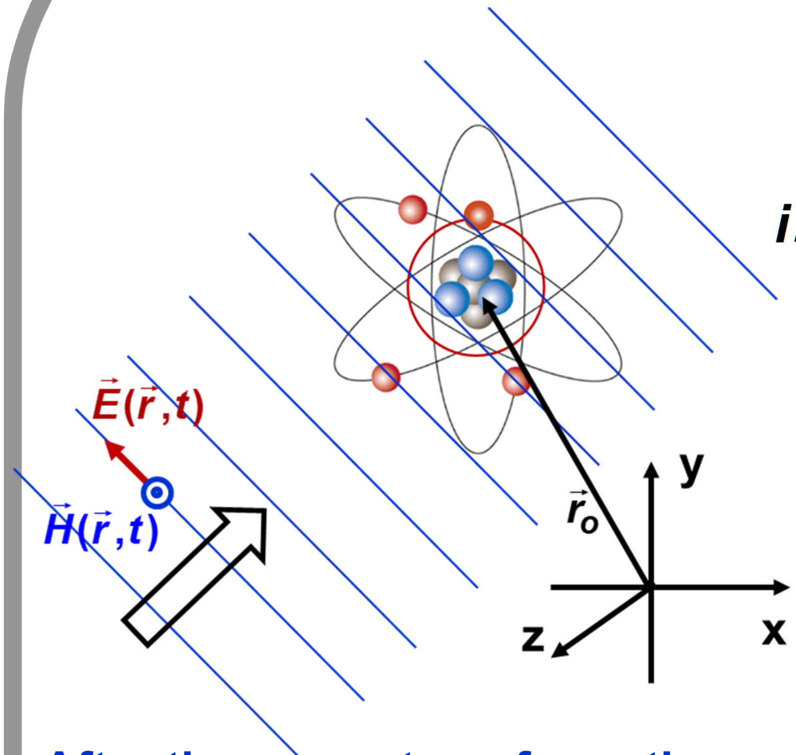
$$F(\vec{r},t)=-\vec{A}(\vec{r}_o,t).\,\vec{r}\qquad\Rightarrow\qquad\nabla F(\vec{r},t)=-\vec{A}(\vec{r}_o,t)$$

Which gives:

$$\vec{A}_{new}(\vec{r}_o,t)=0$$

$$\phi_{new}(\vec{r}_o,t)=\cancelto{0}{\phi(\vec{r}_o,t)}+\frac{\partial}{\partial t}\vec{A}(\vec{r}_o,t).\,\vec{r}=-\vec{E}(\vec{r}_o,t).\,\vec{r}$$

Particle in a Potential Well (Atom) Interacting with Radiation



$$i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t}=\frac{\left[\frac{\hbar}{i}\nabla-q\vec{A}(\vec{r}_o,t)\right]^2}{2m}\psi(\vec{r},t)+V(\vec{r})\psi(\vec{r},t)$$

After the gauge transformation we get:

$$i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t}=\frac{\left[\frac{\hbar}{i}\nabla\right]^2}{2m}\psi(\vec{r},t)+\left[V(\vec{r})-q\vec{E}(\vec{r}_o,t).\vec{r}\right]\psi(\vec{r},t)$$

The particle Hamiltonian is effectively:

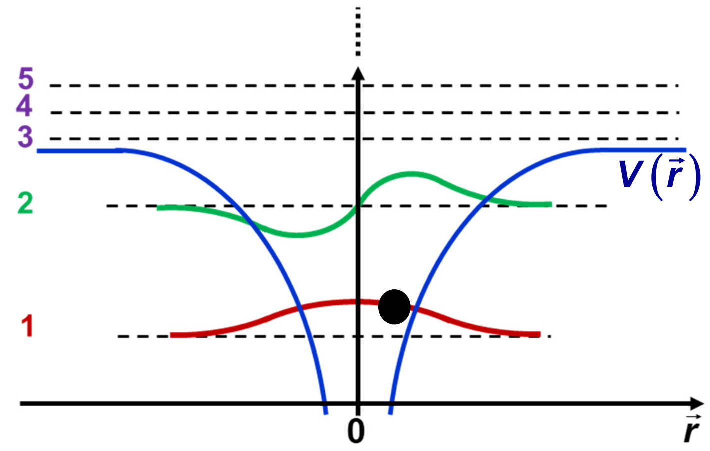
$$\boxed{\hat{H}(t)=\frac{\hat{p}^2}{2m}+V(\hat{r})-q\vec{E}(\vec{r}_o,t).\vec{r}}$$

Part II

Particle in a Potential Well (an Atom)

Suppose the Hamiltonian of a confined particle inside some potential is:

$$\hat{H}_o = \frac{\hat{p}^2}{2m} + V(\vec{r})$$



Let the energy eigenstates be defined as:

$$\hat{H}_o \left| e_j \right\rangle = E_j \left| e_j \right\rangle$$

$$\left\{ \begin{array}{l} \sum_j \left| e_j \right\rangle \left\langle e_j \right| = \hat{1} \\ \left\langle e_j | e_k \right\rangle = \delta_{jk} \end{array} \right.$$

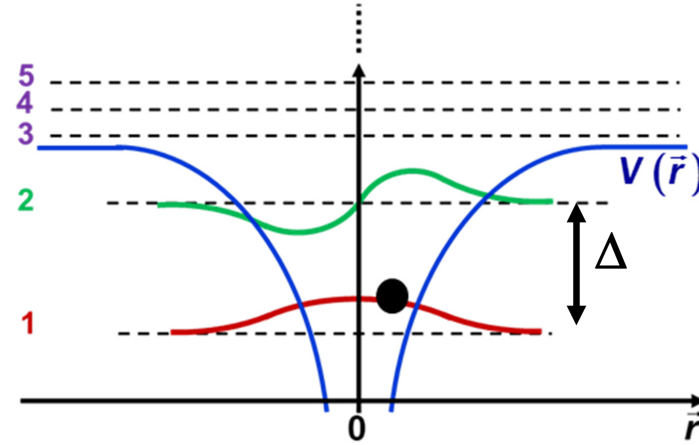
We can write the Hamiltonian as:

$$\hat{H}_o = \hat{1}\hat{H}_o\hat{1}$$

$$= \sum_{j,k} \left| e_j \right\rangle \left\langle e_j \right| \hat{H}_o \left| e_k \right\rangle \left\langle e_k \right| = \sum_k E_k \left| e_k \right\rangle \left\langle e_k \right|$$

Particle in a Potential Well: A Two Level System (TLS)

Assume the Hilbert space is restricted to only the two lowest two energy states of $\hat{H}_o$:



$$|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2| = \hat{1} \longrightarrow$$

New approximate completeness
Two Level System (TLS) approximation

The Hamiltonian becomes:

$$\hat{H}_o|e_1\rangle = E_1|e_1\rangle$$

$$\hat{H}_o|e_2\rangle = E_2|e_2\rangle$$

$$\hat{H}_o = \hat{1}\hat{H}_o\hat{1} = E_1|e_1\rangle\langle e_1| + E_2|e_2\rangle\langle e_2|$$

Make the following mapping:

$$|e_1\rangle \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad |e_2\rangle \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\hat{H}_o = E_1|e_1\rangle\langle e_1| + E_2|e_2\rangle\langle e_2|$$

$$= \begin{bmatrix} E_2 & 0 \\ 0 & E_1 \end{bmatrix} \qquad \left\{ \Delta = E_2 - E_1 \right.$$

$$= \frac{E_1 + E_2}{2} + \begin{bmatrix} \Delta/2 & 0 \\ 0 & -\Delta/2 \end{bmatrix}$$

$$= \frac{E_1 + E_2}{2} + \frac{\Delta}{2}\hat{\sigma}_z$$

Same as the Hamiltonian of a spin 1/2 in a DC magnetic field !!!

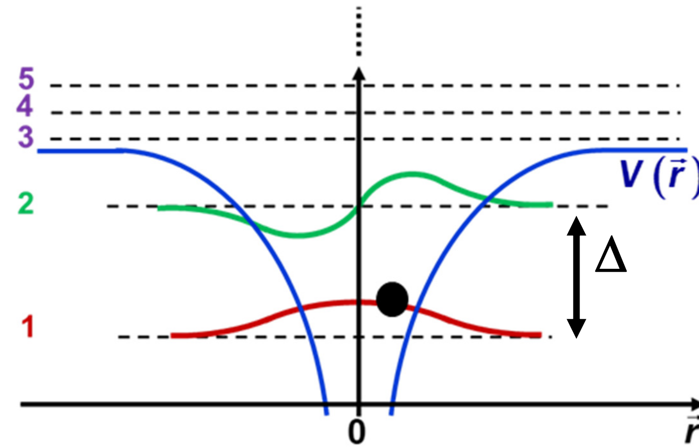
A Trapped Ion Qubit

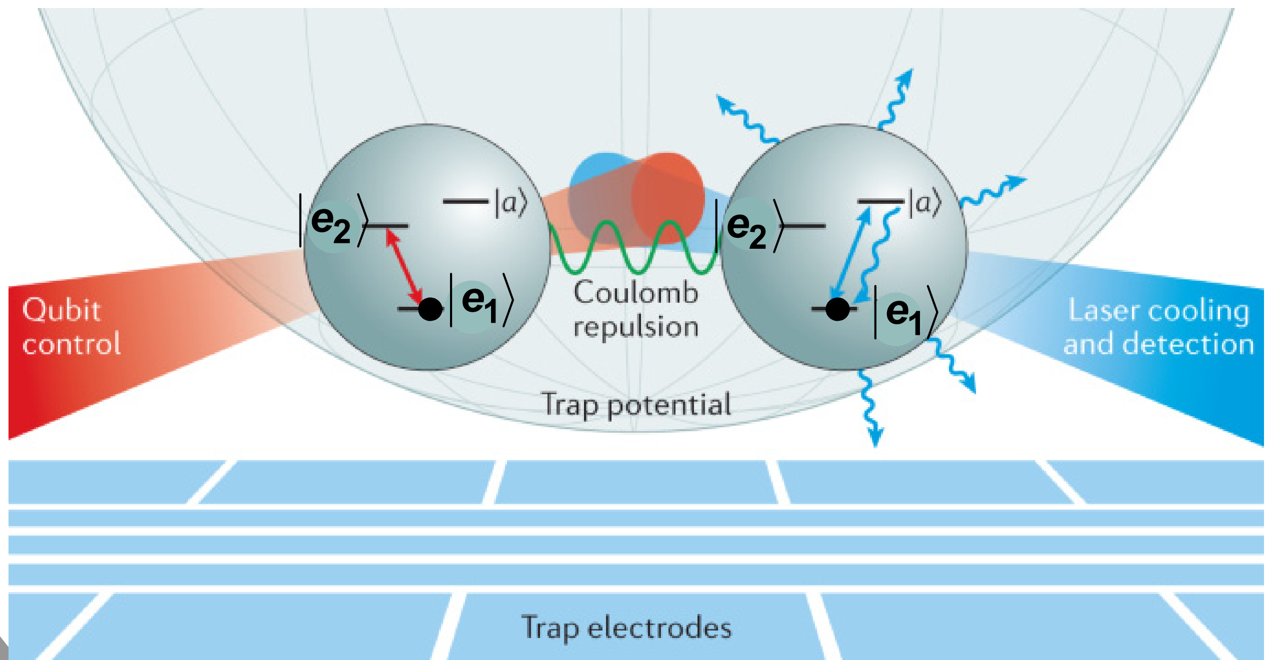
Make the following mapping:

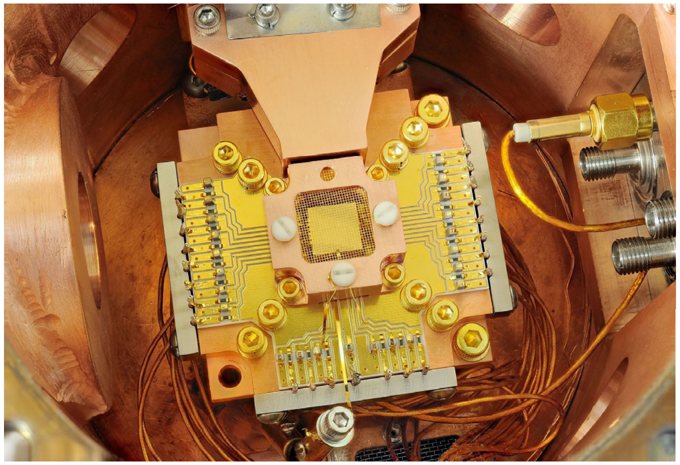
$$|e_1\rangle \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow |1\rangle \quad \text{and} \quad |e_2\rangle \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow |0\rangle$$

Logical "1" qubit

Logical "0" qubit







An ion-trap chip, NIST (2011)

A Detour on Linear Algebra

Consider the Hamiltonian $\hat{H}_o$ whose eigenvalues and eigenstates have been found:

$$\hat{H}_o \left| e_j \right\rangle = E_j \left| e_j \right\rangle$$

$$\left\{ \begin{array}{l} \sum_j \left| e_j \right\rangle \left\langle e_j \right| = \hat{1} \\ \left\langle e_j \middle| e_k \right\rangle = \delta_{jk} \end{array} \right.$$

Now consider what happens when we add another term to the original Hamiltonian to get a new Hamiltonian:

$$\hat{H} = \hat{H}_o + \hat{O}$$

We can write the new Hamiltonian as:

$$\begin{aligned} \hat{H} &= \hat{1}\hat{H}_o\hat{1} + \hat{1}\hat{O}\hat{1} \\ &= \sum_{j,k} \left| e_j \right\rangle \left\langle e_j \right| \hat{H}_o \left| e_k \right\rangle \left\langle e_k \right| + \sum_{j,k} \left| e_j \right\rangle \left\langle e_j \right| \hat{O} \left| e_k \right\rangle \left\langle e_k \right| \\ &= \sum_k E_k \left| e_k \right\rangle \left\langle e_k \right| + \sum_{j,k} O_{jk} \left| e_j \right\rangle \left\langle e_k \right| \end{aligned}$$

Particle in a Potential Well Interacting with Light

The classical expression for the potential energy of a charged particle (charge *q*) in an electromagnetic field is (see Part I):

$$-q\vec{E}(t).\vec{r}$$

Electric field at the location of the particle

$$\vec{r} = xe_x + ye_y + ze_z$$

$$\vec{E}(t) = \hat{n}E_o\cos(\omega t)$$

$$\vec{E}(t).\vec{r} = E_o(\hat{n}.\vec{r})\cos(\omega t)$$

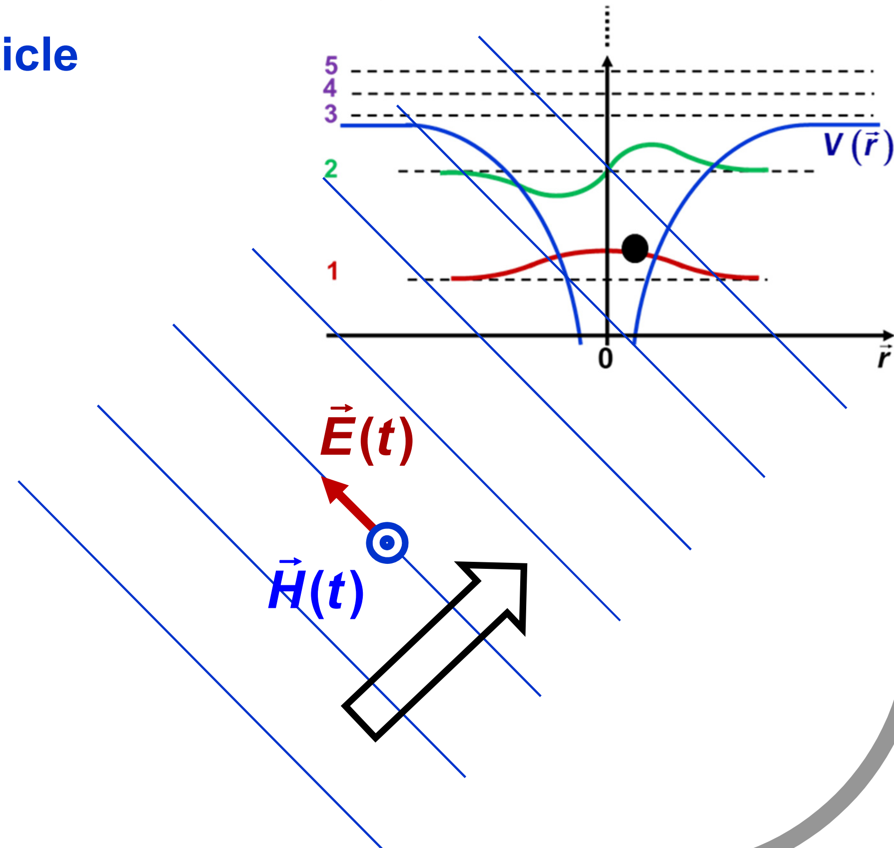
$\hat{n}$ is a unit vector in the direction of the electric field polarization

We add this energy to the Hamiltonian of the particle confined in a 1D potential:

$$\hat{H}(t) = \frac{\hat{p}^2}{2m} + V(\hat{\vec{r}}) - q\vec{E}(t).\hat{\vec{r}}$$

$$= \frac{\hat{p}^2}{2m} + V(\hat{\vec{r}}) - qE_o(\hat{n}.\hat{\vec{r}})\cos(\omega t)$$

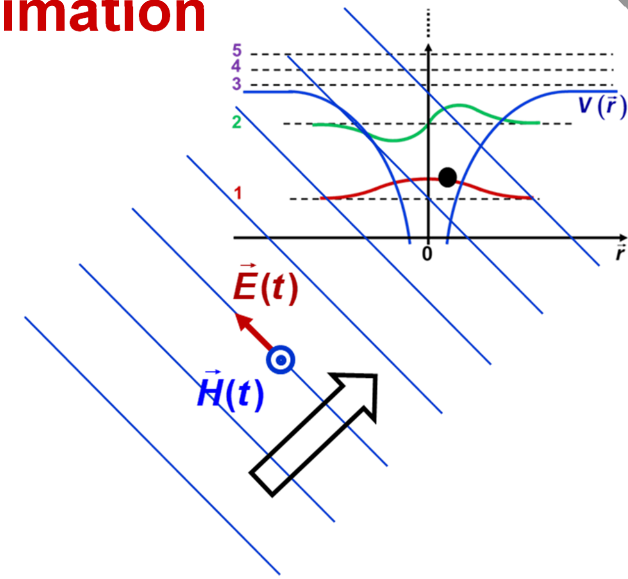
$$= \hat{H}_o + \hat{H}_i(t)$$



Hamiltonian becomes time-dependent !!!

Two Level System (TLS) Approximation

$$\hat{H}(t) = \hat{H}_o + \hat{H}_i(t) = \frac{\hat{p}^2}{2m} + V\left(\hat{\vec{r}}\right) - qE_o\left(\hat{n}.\hat{\vec{r}}\right)\cos(\omega t)$$



Assume a Hilbert space consisting of only the two lowest energy states of $\hat{H}_o$:

$$\hat{H}_o|e_1\rangle = E_1|e_1\rangle$$

$$\hat{H}_o|e_2\rangle = E_2|e_2\rangle$$

$$|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2| = \hat{1} \longrightarrow$$ Approximate completeness
Two Level System (TLS) approximation

Hamiltonian becomes:

$$\hat{H}(t) = \hat{H}_o - qE_o\left(\hat{n}.\hat{\vec{r}}\right)\cos(\omega t)$$

$$= \hat{1}\hat{H}_o\hat{1} - \hat{1}qE_o\left(\hat{n}.\hat{\vec{r}}\right)\cos(\omega t)\hat{1}$$

$$= E_1|e_1\rangle\langle e_1| + E_2|e_2\rangle\langle e_2|$$

$$-qE_o\cos(\omega t)\begin{bmatrix} \cancelto{0}{\langle e_1|\hat{n}.\hat{\vec{r}}|e_1\rangle}|e_1\rangle\langle e_1| + \cancelto{0}{\langle e_2|\hat{n}.\hat{\vec{r}}|e_2\rangle}|e_2\rangle\langle e_2| \\ +\langle e_1|\hat{n}.\hat{\vec{r}}|e_2\rangle|e_1\rangle\langle e_2| + \langle e_2|\hat{n}.\hat{\vec{r}}|e_1\rangle|e_2\rangle\langle e_1| \end{bmatrix}$$

Dipole Matrix Elements

Hamiltonian becomes:

$$\hat{H}(t) = E_1|e_1\rangle\langle e_1| + E_2|e_2\rangle\langle e_2| - qE_o\cos(\omega t)\Big[\underbrace{\langle e_1|\hat{n}.\hat{\vec{r}}|e_2\rangle}_{d_{12}}|e_1\rangle\langle e_2| + \underbrace{\langle e_2|\hat{n}.\hat{\vec{r}}|e_1\rangle}_{d_{21}}|e_2\rangle\langle e_1|\Big]$$

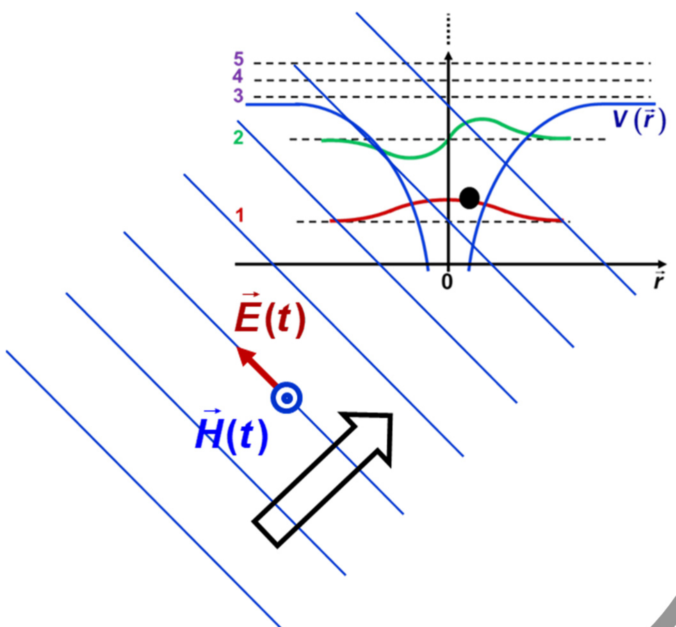
Dipole matrix element:

$$d_{12} = \langle e_1|\hat{n}.\hat{\vec{r}}|e_2\rangle = \int d^3\vec{r}\ \phi_1^*(\vec{r})\,\hat{n}.\vec{r}\,\phi_2(\vec{r}) = \int d^3\vec{r}\ \phi_1^*(\vec{r})\big(n_x x + n_y y + n_z z\big)\phi_2(\vec{r})$$

$$d_{21} = d_{12}^* = \langle e_2|\hat{n}.\hat{\vec{r}}|e_1\rangle$$

Assume:

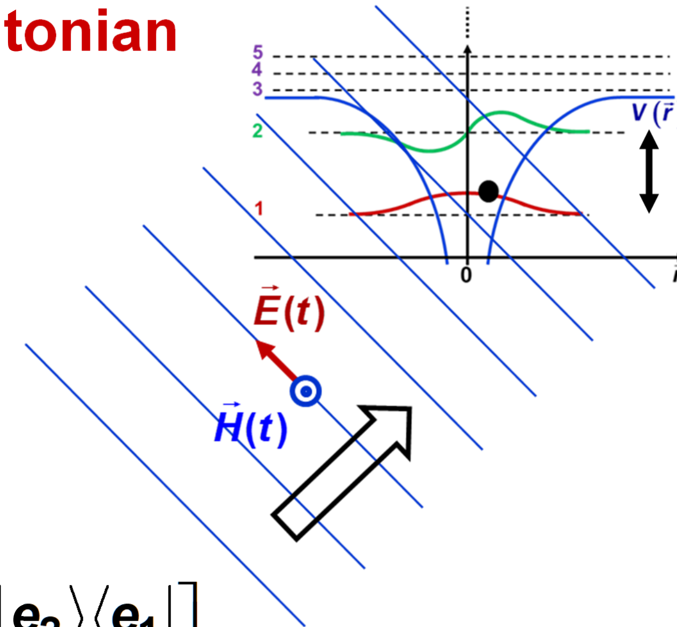
$$d_{12} = d_{21} = d$$



Hamiltonian becomes:

$$\hat{H}(t) = E_1|e_1\rangle\langle e_1| + E_2|e_2\rangle\langle e_2| - qE_o d\cos(\omega t)\big[|e_1\rangle\langle e_2| + |e_2\rangle\langle e_1|\big]$$

Mapping to a Spin Hamiltonian



Make the following mapping:

$$|e_1\rangle \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad |e_2\rangle \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The Hamiltonian becomes:

$$\hat{H}(t) = E_1|e_1\rangle\langle e_1| + E_2|e_2\rangle\langle e_2| - qE_o d\cos(\omega t)\big[|e_1\rangle\langle e_2| + |e_2\rangle\langle e_1|\big]$$

$$= \begin{bmatrix} E_2 & 0 \\ 0 & E_1 \end{bmatrix} - qE_o d\cos(\omega t)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \frac{E_1+E_2}{2} + \frac{\Delta}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - qE_o d\cos(\omega t)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \underbrace{\frac{E_1+E_2}{2}}_{\text{offset}} + \frac{\Delta}{2}\hat{\sigma}_z - qE_o d\cos(\omega t)\hat{\sigma}_x$$

$$\sim \frac{\Delta}{2}\hat{\sigma}_z + \kappa\cos(\omega t)\hat{\sigma}_x$$

Spin Hamiltonian !!!

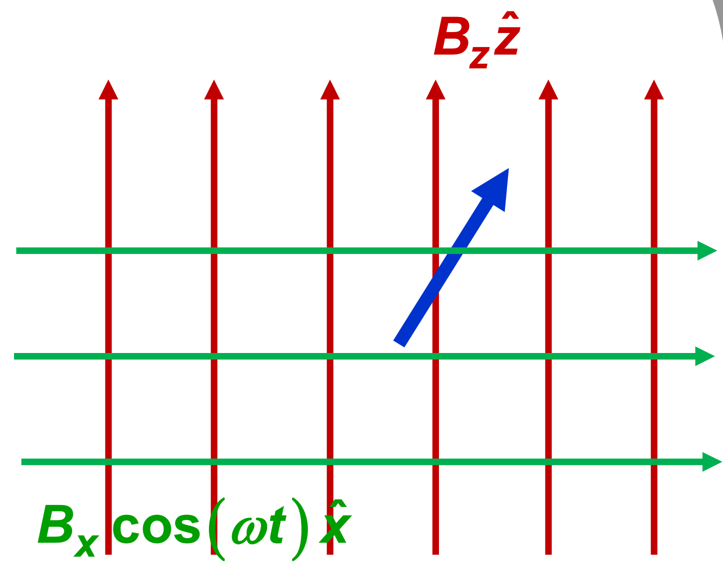
For a 1D potential well:

$$\Delta = E_2 - E_1$$

$$\kappa = -qE_o d$$

The Hamiltonian, and therefore the physics, is that of spin 1/2 in a DC z-directed magnetic field and an AC x-directed magnetic field!!!!!

A Single Spin ½ Qubit in DC and AC Magnetic Fields

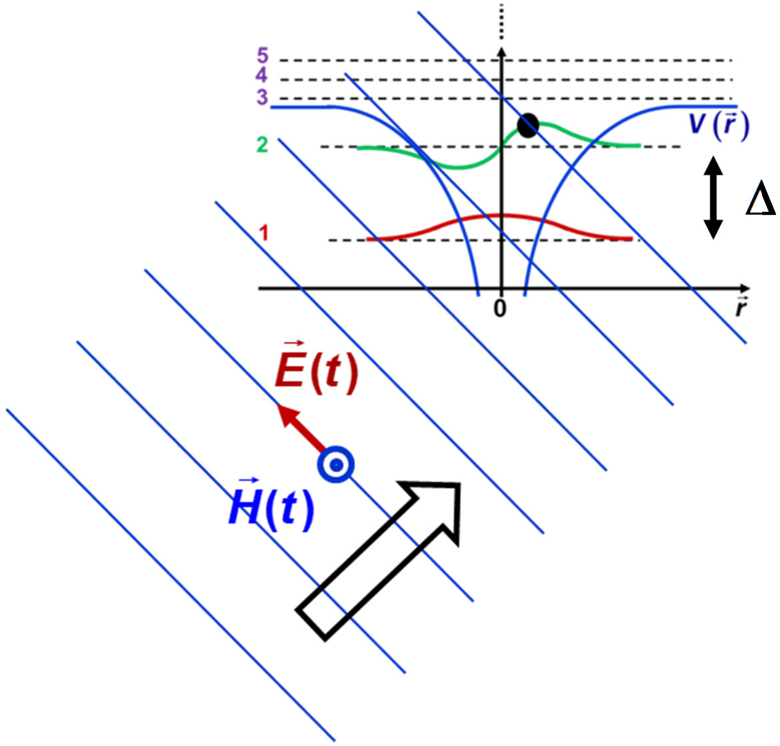


The Hamiltonian was:

$$\hat{H}(t) = \mu_B B_z \hat{\sigma}_z + \mu_B B_x \cos(\omega t)\,\hat{\sigma}_x$$
$$= \frac{\Delta}{2}\hat{\sigma}_z + \kappa \cos(\omega t)\,\hat{\sigma}_x$$

Rabi Oscillations and the Rabi Frequency

$$\hat{H}(t) = \frac{\Delta}{2}\hat{\sigma}_z - qE_o d\cos(\omega t)\hat{\sigma}_x$$
$$= \frac{\Delta}{2}\hat{\sigma}_z + \kappa\cos(\omega t)\hat{\sigma}_x$$



Assume zero detuning:

$$\Delta = E_2 - E_1 = \hbar\omega$$

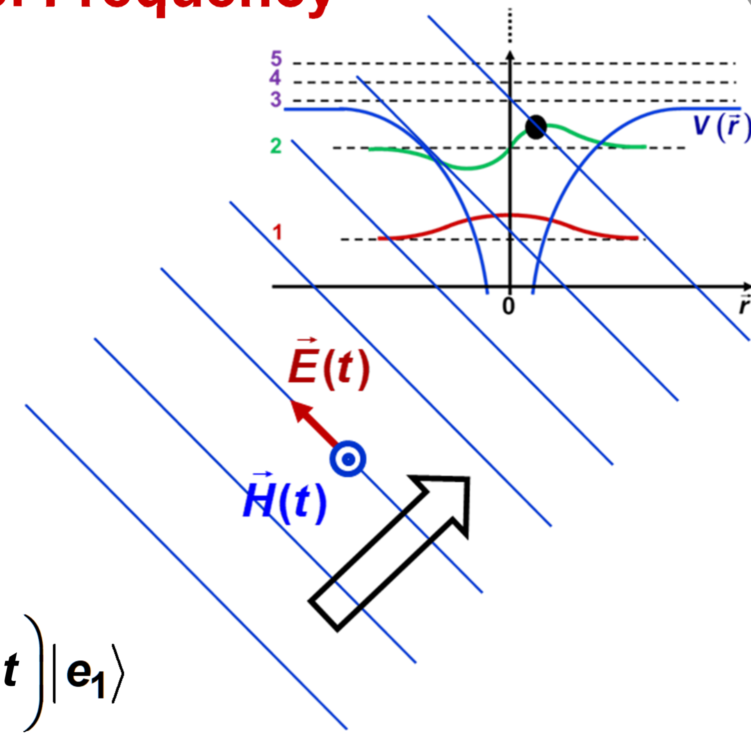
Need to solve:

$$i\hbar\frac{\partial|\psi(t)\rangle}{\partial t} = \hat{H}(t)|\psi(t)\rangle$$

Subject to the initial condition:

$$|\psi(t=0)\rangle = |e_2\rangle = \begin{bmatrix}1\\0\end{bmatrix}$$

Rabi Oscillations and the Rabi Frequency



Need to solve:

$$i\hbar\frac{\partial|\psi(t)\rangle}{\partial t}=\hat{H}(t)|\psi(t)\rangle$$

Subject to the initial condition:

$$|\psi(t=0)\rangle=|e_2\rangle=\begin{bmatrix}1\\0\end{bmatrix}$$

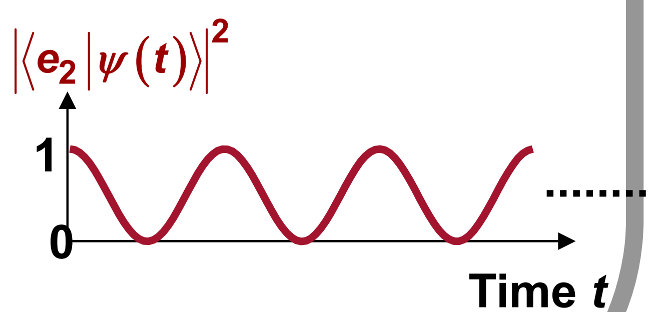
Solution is (from lecture 15, see also the Appendix):

$$|\psi(t)\rangle=e^{-i\frac{\omega}{2}t}\cos\left(\frac{\kappa}{2\hbar}t\right)|e_2\rangle-ie^{+i\frac{\omega}{2}t}\sin\left(\frac{\kappa}{2\hbar}t\right)|e_1\rangle$$

$$=e^{-i\frac{\omega}{2}t}\cos\left(\frac{qE_od}{2\hbar}t\right)|e_2\rangle+ie^{+i\frac{\omega}{2}t}\sin\left(\frac{qE_od}{2\hbar}t\right)|e_1\rangle$$

Probability of finding the electron in the initial upper state at later time *t*:

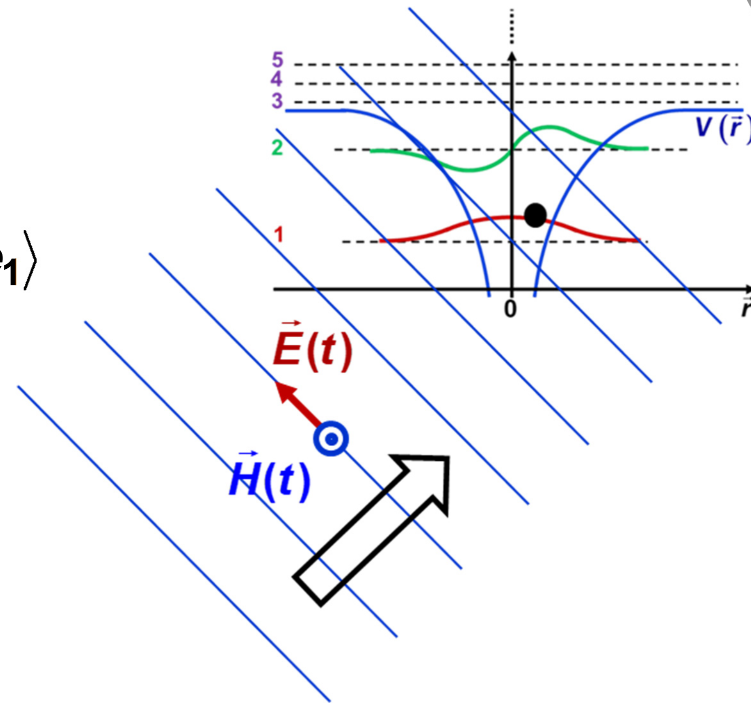
$$|\langle e_2|\psi(t)\rangle|^2=\cos^2\left(\frac{qE_od}{2\hbar}t\right)=\frac{1}{2}+\frac{1}{2}\cos\left(\frac{qE_od}{\hbar}t\right)$$



Probability is oscillating at the frequency:

$$\Omega=\frac{|qE_od|}{\hbar}\longrightarrow\text{Rabi frequency}$$

Rabi Oscillations and the Rabi Frequency



Solution is:

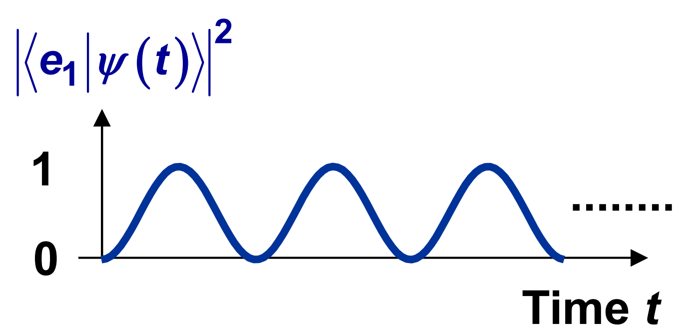
$$\left|\psi(t)\right\rangle = e^{-i\frac{\omega}{2}t}\cos\left(\frac{qE_od}{2\hbar}t\right)\left|e_2\right\rangle + ie^{+i\frac{\omega}{2}t}\sin\left(\frac{qE_od}{2\hbar}t\right)\left|e_1\right\rangle$$

Probability of finding the electron in the lower state at later time *t*:

$$\left|\left\langle e_1\middle|\psi(t)\right\rangle\right|^2 = \sin^2\left(\frac{qE_od}{2\hbar}t\right) = \frac{1}{2} - \frac{1}{2}\cos\left(\frac{qE_od}{\hbar}t\right)$$

Probability is oscillating at the frequency:

$$\Omega = \frac{\left|qE_od\right|}{\hbar} \longrightarrow \text{Rabi frequency}$$



Rabi Oscillations and the Rabi Frequency

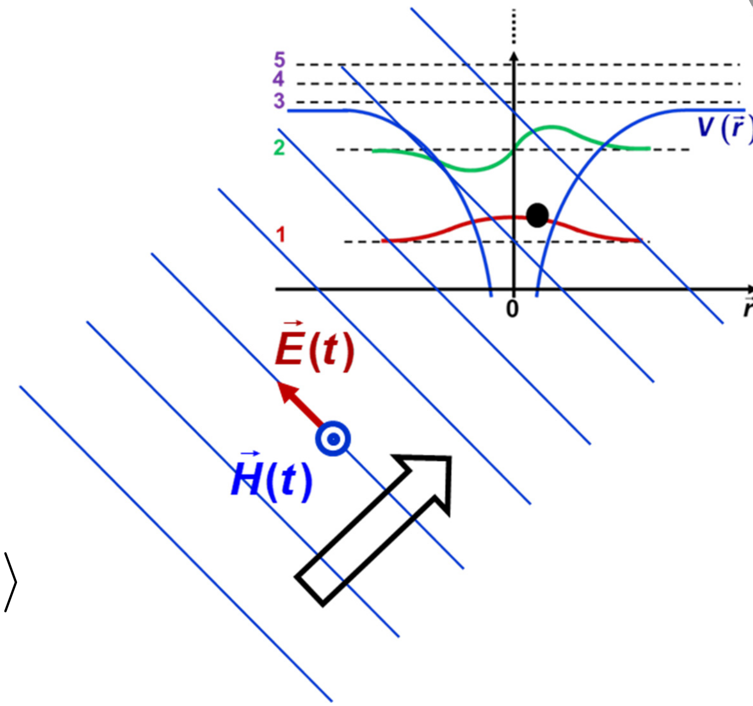
Initial condition:

$$\left|\psi(t=0)\right\rangle = \left|e_2\right\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

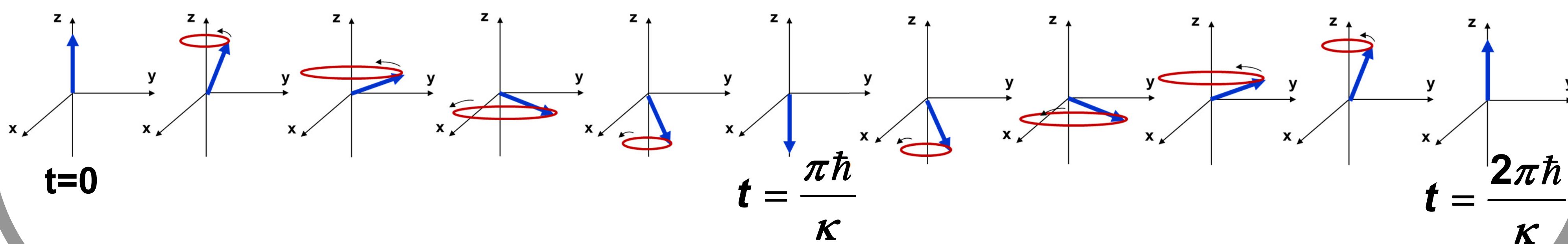
Solution is (from lecture 15, see also the Appendix):

$$\left|\psi(t)\right\rangle = e^{-i\frac{\omega}{2}t}\cos\left(\frac{\kappa}{2\hbar}t\right)\left|e_2\right\rangle - ie^{+i\frac{\omega}{2}t}\sin\left(\frac{\kappa}{2\hbar}t\right)\left|e_1\right\rangle$$

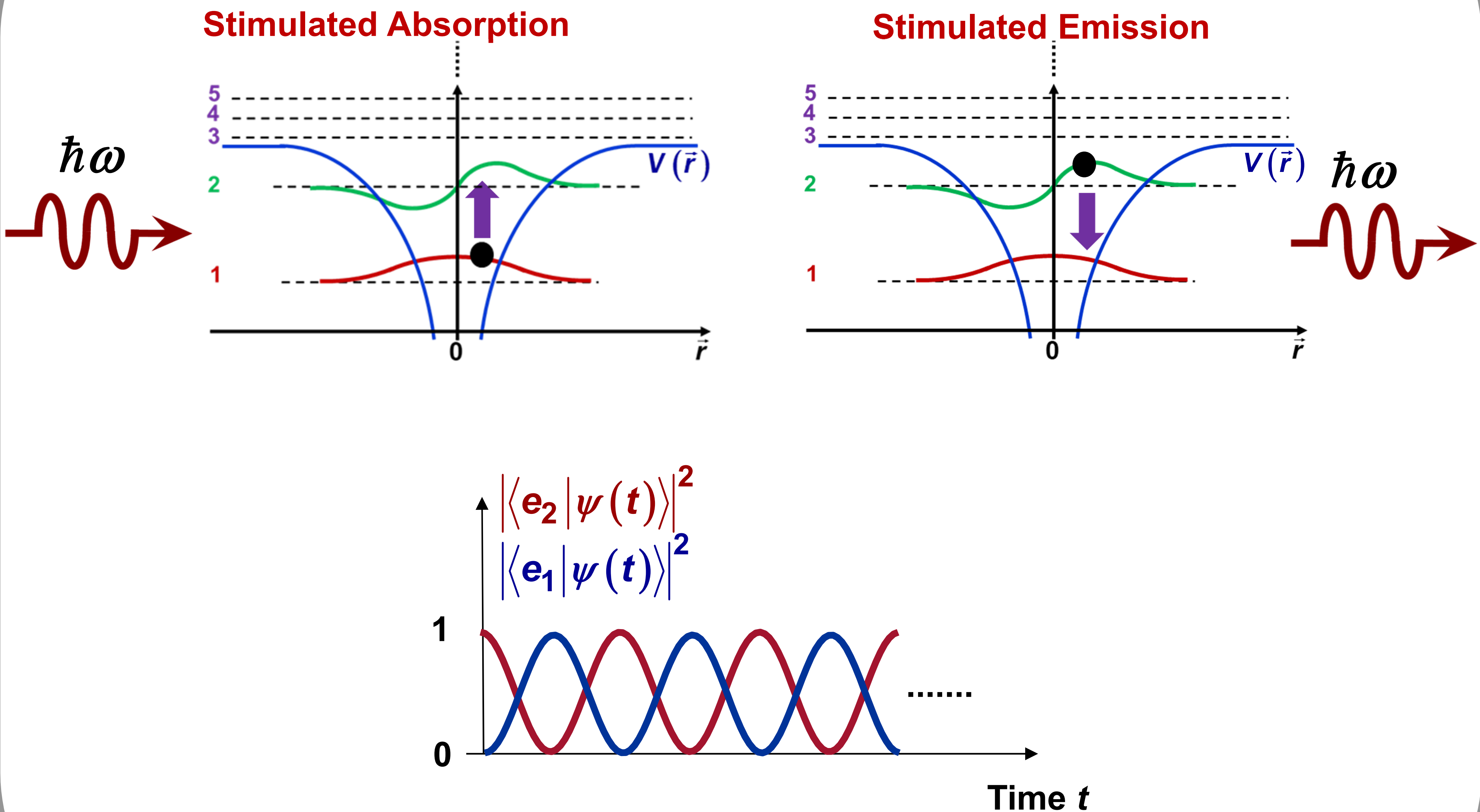
$$= e^{-i\frac{\omega}{2}t}\cos\left(\frac{qE_o d}{2\hbar}t\right)\left|e_2\right\rangle + ie^{+i\frac{\omega}{2}t}\sin\left(\frac{qE_o d}{2\hbar}t\right)\left|e_1\right\rangle$$



If you want to visualize the dynamics of the quantum state:

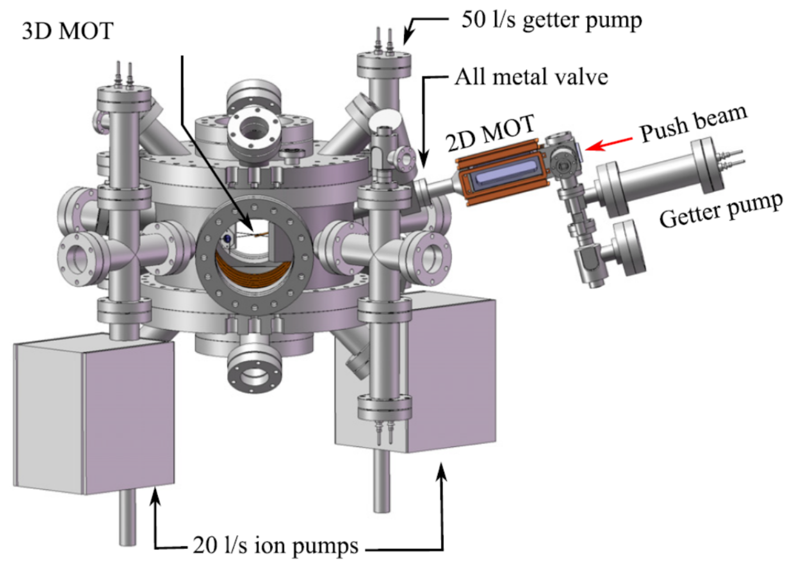


Rabi Oscillations: An Interpretation

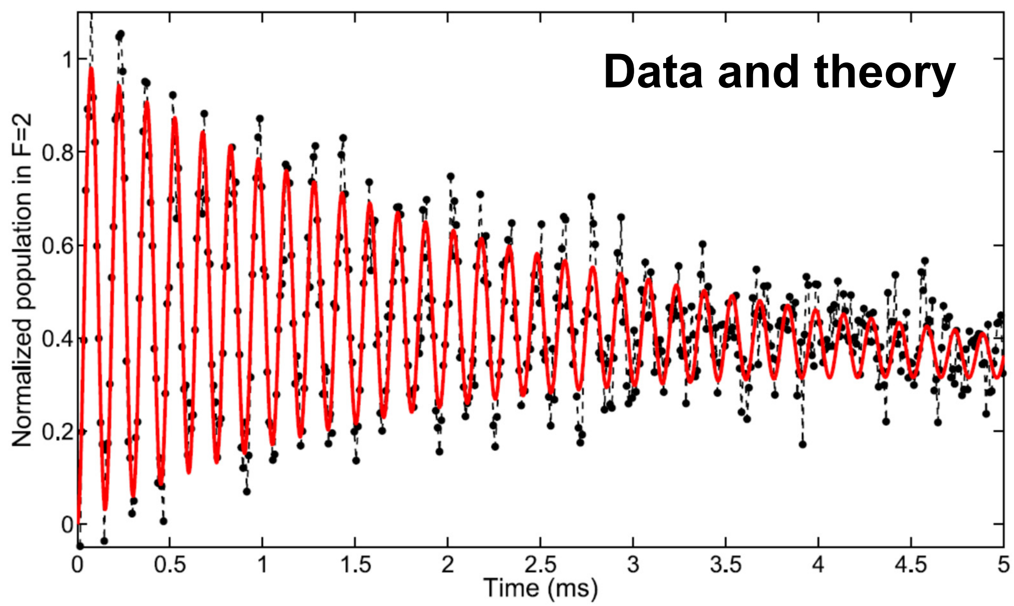


Rabi Oscillations in Cold Rubidium Atoms

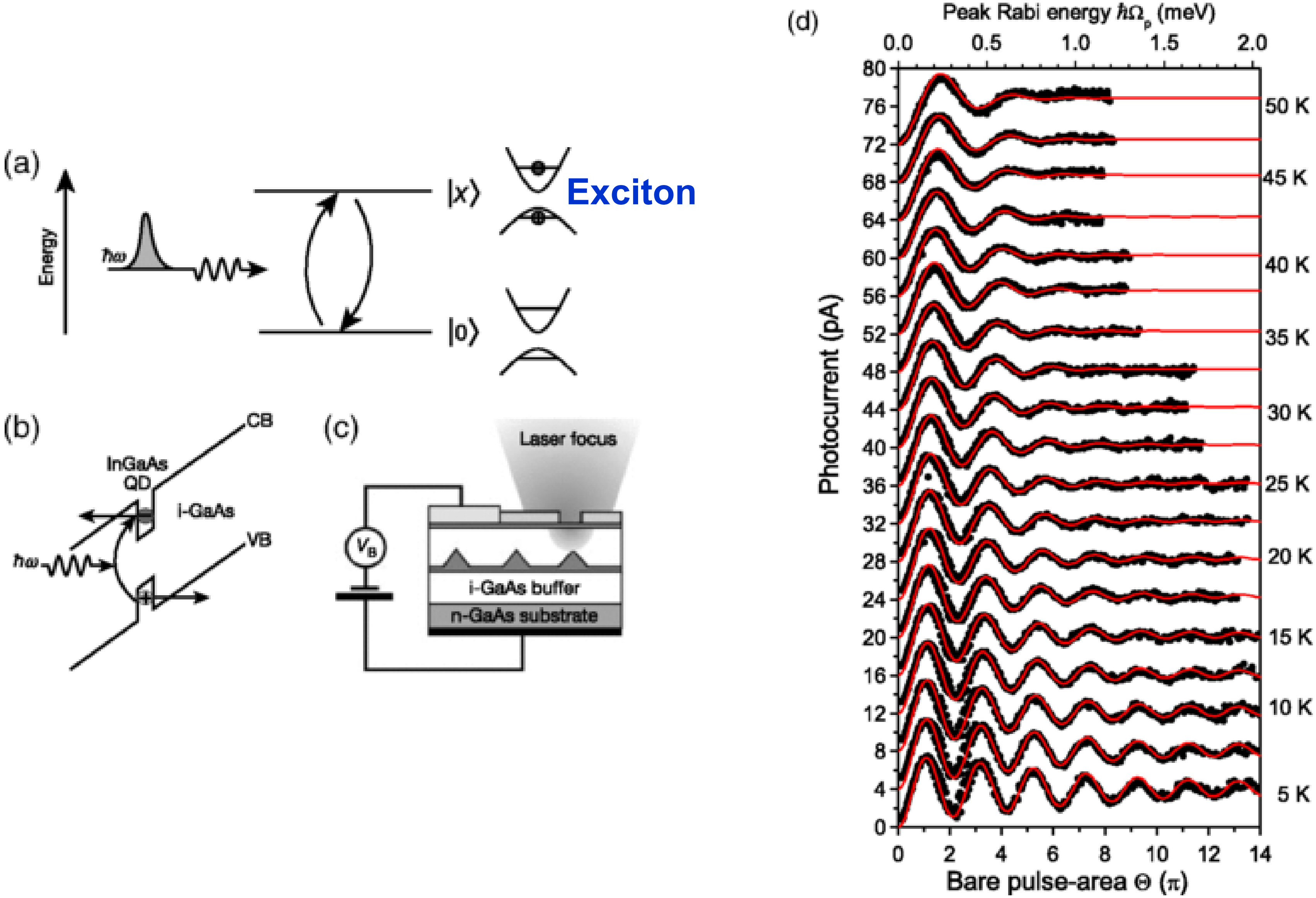
Driven Rabi oscillation in Rubidium atoms



New Journal of Physics, 13 065021 (2011)

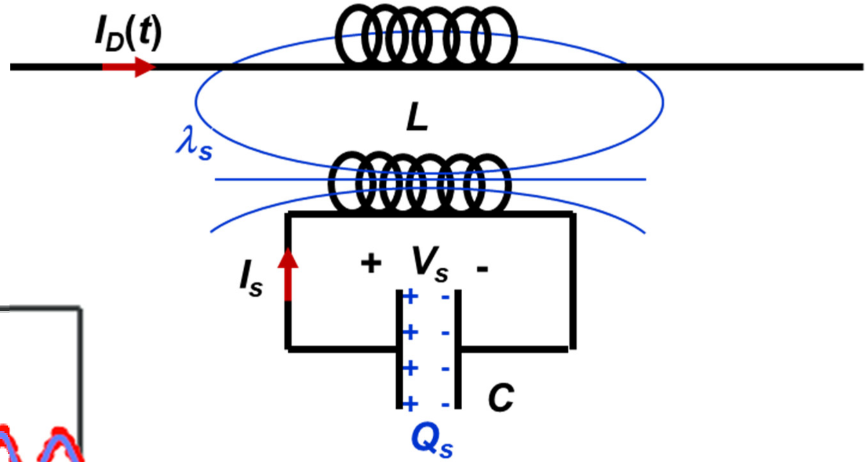


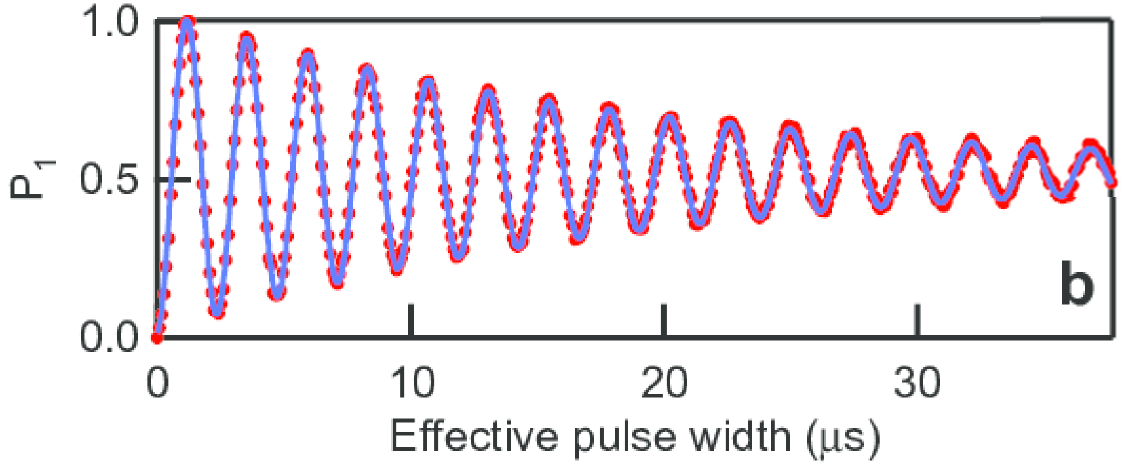
Rabi Oscillations in Semiconductor Quantum Dots



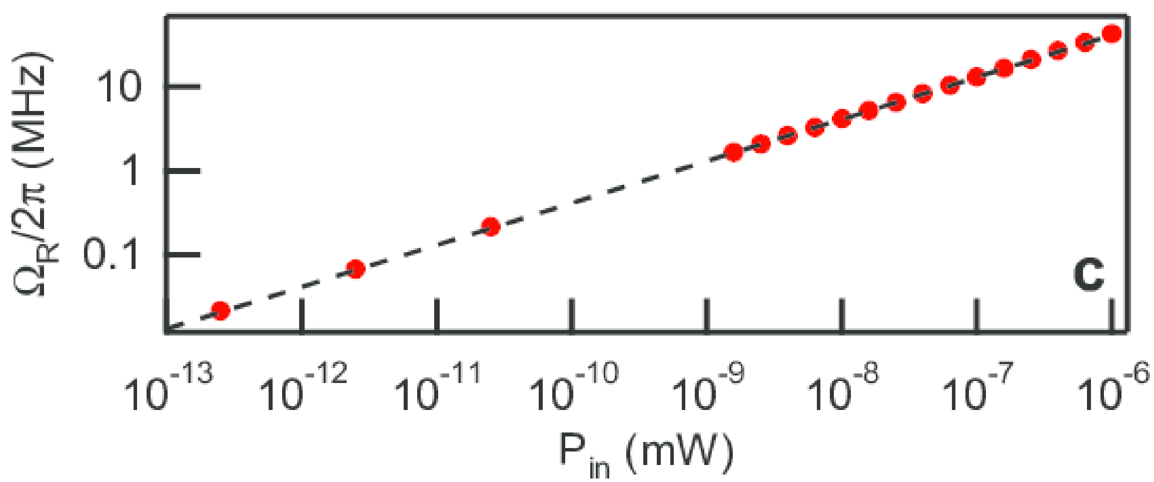
Rev. Mod. Phys., 87, 347 (2015)

Rabi Oscillations in Superconducting Qubits



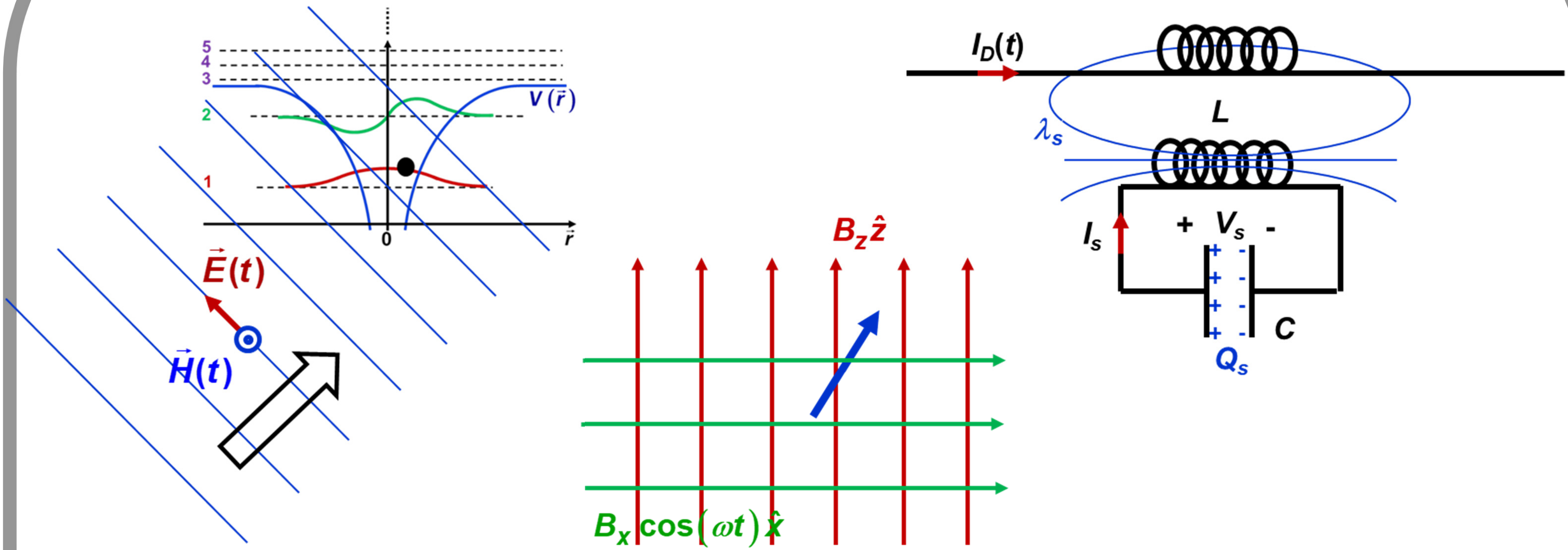


Phys. Rev. Lett. **107**, 240501 **(2011)**



$$\Omega_R \propto I_{DO} \propto \sqrt{P_{in}}$$

The Spin Hamiltonians of Two Level Systems



Note that all these three TLS have a (relevant) Hilbert space of dimension 2 and a Hamiltonian of the same general form:

$$\hat{H}(t) = A + B\hat{\sigma}_z + C\cos(\omega t)\hat{\sigma}_x$$

Appendix: Solution Technique

Suppose we make the following mapping:

$$|e_1\rangle \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad |e_2\rangle \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

And suppose under this mapping:

$$\hat{H}(t) = \frac{\Delta}{2}\hat{\sigma}_z + \kappa \cos(\omega t)\hat{\sigma}_x$$

Need to solve:

$$i\hbar \frac{\partial |\psi(t)\rangle}{\partial t} = \hat{H}(t)|\psi(t)\rangle$$

With the initial condition:

$$|\psi(t=0)\rangle = \alpha|e_1\rangle + \beta|e_2\rangle = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$$

Appendix: Solution Technique

Need to solve:

$$i\hbar\frac{\partial\left|\psi(t)\right\rangle}{\partial t}=\hat{H}(t)\left|\psi(t)\right\rangle$$

Assume:

Time development of states without interaction

$$\left|\psi(t)\right\rangle=a(t)e^{i\frac{\Delta}{2\hbar}t}\left|e_1\right\rangle+b(t)e^{-i\frac{\Delta}{2\hbar}t}\left|e_2\right\rangle=\begin{bmatrix}b(t)e^{-i\frac{\Delta}{2\hbar}t}\\a(t)e^{+i\frac{\Delta}{2\hbar}t}\end{bmatrix}$$

LHS:

$$i\hbar\frac{\partial\left|\psi(t)\right\rangle}{\partial t}=i\hbar\frac{\partial}{\partial t}\begin{bmatrix}b(t)e^{-i\frac{\Delta}{2\hbar}t}\\a(t)e^{i\frac{\Delta}{2\hbar}t}\end{bmatrix}=\begin{bmatrix}e^{-i\frac{\Delta}{2\hbar}t}i\hbar\frac{\partial b(t)}{\partial t}\\e^{i\frac{\Delta}{2\hbar}t}i\hbar\frac{\partial a(t)}{\partial t}\end{bmatrix}+\frac{\Delta}{2}\hat{\sigma}_z\begin{bmatrix}b(t)e^{-i\frac{\Delta}{2\hbar}t}\\a(t)e^{i\frac{\Delta}{2\hbar}t}\end{bmatrix}$$

RHS:

$$\hat{H}(t)\left|\psi(t)\right\rangle=\frac{\Delta}{2}\hat{\sigma}_z\begin{bmatrix}b(t)e^{-i\frac{\Delta}{2\hbar}t}\\a(t)e^{i\frac{\Delta}{2\hbar}t}\end{bmatrix}+\kappa\cos(\omega t)\begin{bmatrix}a(t)e^{i\frac{\Delta}{2\hbar}t}\\b(t)e^{-i\frac{\Delta}{2\hbar}t}\end{bmatrix}$$

Appendix: Solution Technique

$$i\hbar\frac{\partial|\psi(t)\rangle}{\partial t}=\hat{H}(t)|\psi(t)\rangle$$

$$\Rightarrow\begin{bmatrix} e^{-i\frac{\Delta}{2\hbar}t}\,i\hbar\dfrac{\partial b(t)}{\partial t} \\ e^{i\frac{\Delta}{2\hbar}t}\,i\hbar\dfrac{\partial a(t)}{\partial t}\end{bmatrix}=\kappa\cos(\omega t)\begin{bmatrix} a(t)e^{i\frac{\Delta}{2\hbar}t} \\ b(t)e^{-i\frac{\Delta}{2\hbar}t}\end{bmatrix}$$

$$\Rightarrow i\hbar\frac{\partial}{\partial t}\begin{bmatrix} b(t) \\ a(t)\end{bmatrix}=\kappa\begin{bmatrix} a(t)e^{i\frac{\Delta}{\hbar}t}\cos(\omega t) \\ b(t)e^{-i\frac{\Delta}{\hbar}t}\cos(\omega t)\end{bmatrix}$$

Assume zero detuning: $\Delta=\hbar\omega$ **and using the rotating wave approximation:**

$$i\hbar\frac{\partial}{\partial t}\begin{bmatrix} b(t) \\ a(t)\end{bmatrix}=\frac{\kappa}{2}\begin{bmatrix} a(t) \\ b(t)\end{bmatrix}\xrightarrow{\text{Solution}}\begin{bmatrix} b(t) \\ a(t)\end{bmatrix}=\begin{bmatrix} A \\ C\end{bmatrix}\cos\left(\frac{\kappa}{2\hbar}t\right)+\begin{bmatrix} B \\ D\end{bmatrix}\sin\left(\frac{\kappa}{2\hbar}t\right)$$

Appendix: Solution Technique

$$i\hbar\frac{\partial}{\partial t}\begin{bmatrix} b(t) \\ a(t) \end{bmatrix} = \frac{\kappa}{2}\begin{bmatrix} a(t) \\ b(t) \end{bmatrix} \xrightarrow{\text{Solution}} \begin{bmatrix} b(t) \\ a(t) \end{bmatrix} = \begin{bmatrix} A \\ C \end{bmatrix}\cos\left(\frac{\kappa}{2\hbar}t\right) + \begin{bmatrix} B \\ D \end{bmatrix}\sin\left(\frac{\kappa}{2\hbar}t\right)$$

Initial conditions:

$$\left.\begin{bmatrix} b(t) \\ a(t) \end{bmatrix}\right|_{t=0} = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$$

$$\left. i\hbar\frac{\partial}{\partial t}\begin{bmatrix} b(t) \\ a(t) \end{bmatrix}\right|_{t=0} = \frac{\kappa}{2}\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Final solution:

$$\begin{bmatrix} b(t) \\ a(t) \end{bmatrix} = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}\cos\left(\frac{\kappa}{2\hbar}t\right) - i\begin{bmatrix} \alpha \\ \beta \end{bmatrix}\sin\left(\frac{\kappa}{2\hbar}t\right)$$

$$\Rightarrow |\psi(t)\rangle = \begin{bmatrix} b(t)e^{-i\frac{\Delta}{2\hbar}t} \\ a(t)e^{+i\frac{\Delta}{2\hbar}t} \end{bmatrix} = \begin{bmatrix} \beta e^{-i\frac{\Delta}{2\hbar}t} \\ \alpha e^{+i\frac{\Delta}{2\hbar}t} \end{bmatrix}\cos\left(\frac{\kappa}{2\hbar}t\right) - i\begin{bmatrix} \alpha e^{-i\frac{\Delta}{2\hbar}t} \\ \beta e^{+i\frac{\Delta}{2\hbar}t} \end{bmatrix}\sin\left(\frac{\kappa}{2\hbar}t\right)$$