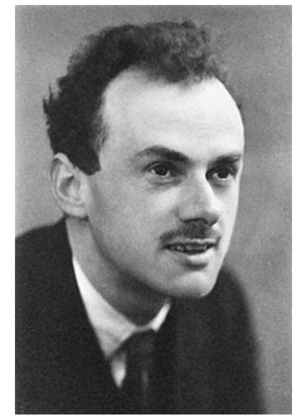
# Lecture15

# The Electron Spin and the Spin Qubit



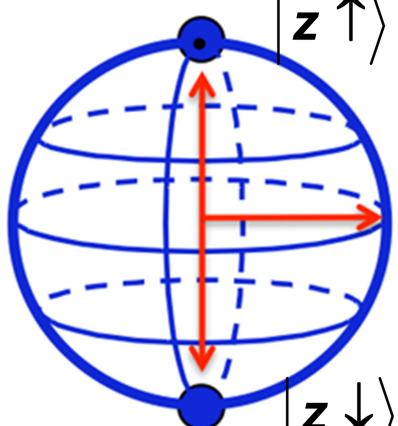
Paul Adrien Maurice Dirac
(1902 – 1984)
Nobel Prize 1933

**In this lecture you will learn:**

- Quantum bits (or qubits) vs classical bits
- Dealing with quantum states that have no wavefunctions
- Spin 1/2 in quantum physics
- Spin 1/2 and quantum two level systems
- Spin 1/2 in DC and AC magnetic fields


Classical Bit

$$|z\uparrow\rangle = |0\rangle$$



$$|x\uparrow\rangle = \frac{|z\uparrow\rangle + |z\downarrow\rangle}{\sqrt{2}} = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

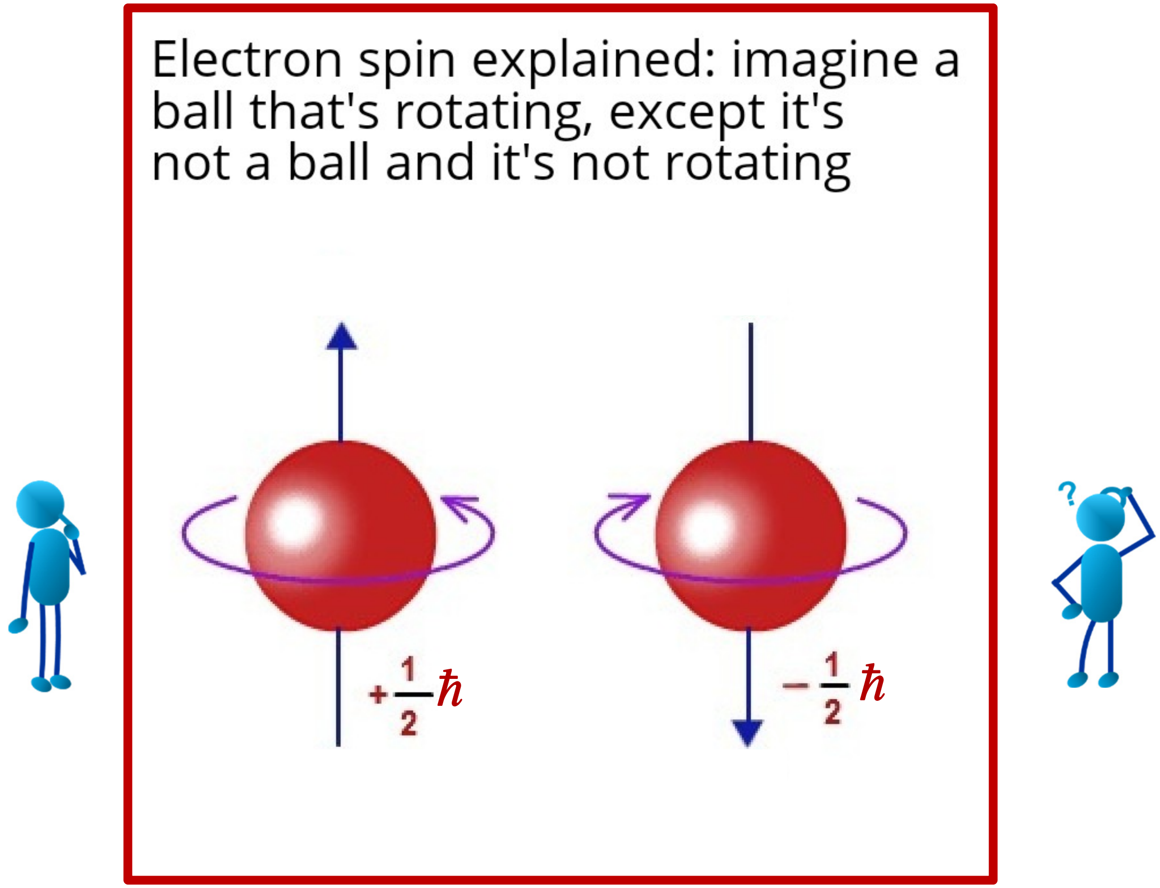
$$|z\downarrow\rangle = |1\rangle$$

Qubit

# Electron Spin Explained (…… NOT)

**Electrons (like most other elementary particles) possesses an internal degree of freedom called "spin"**

**We don't know what spin really is ………but ……… we know it is associated with an internal angular momentum of some sort …………**




**The spin (or, more precisely, the spin angular momentum) is an observable**

# Electron Spin Angular Momentum

The electron spin angular momentum operator is a vector operator (just like the momentum operator or the angular momentum operator):

$$\hat{\vec{S}} = \hat{S}_x e_x + \hat{S}_y e_y + \hat{S}_z e_z$$

By convention, the x, y, and z components of the spin angular momentum are written as:

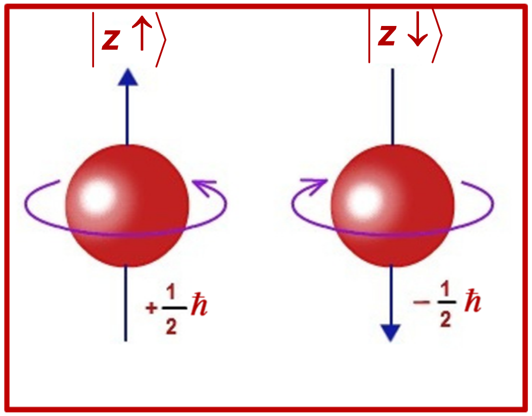
$$\hat{S}_x = \frac{\hbar}{2}\hat{\sigma}_x \qquad \hat{S}_y = \frac{\hbar}{2}\hat{\sigma}_y \qquad \hat{S}_z = \frac{\hbar}{2}\hat{\sigma}_z$$

Pauli Spin Operators

Angular momentum has units of $\hbar$ (i.e. energy-second) and so the eigenvalues of the Pauli operators, whatever they might turn out to be, will be dimensionless

# The Hilbert Space of Electron Spin States

**Experimental fact:** electron spin angular momentum, measured along any axis, say z-axis, can take two values:




$$+\frac{\hbar}{2} \quad \text{or} \quad -\frac{\hbar}{2}$$

up or down

**Implication:** the z-component of the spin angular momentum operator $\hat{S}_z$ has only two eigenvalues

$$+\frac{\hbar}{2} \quad \text{and} \quad -\frac{\hbar}{2}$$

The corresponding eigenstates of $\hat{S}_z$ corresponding to the up and down spins along z-axis are:

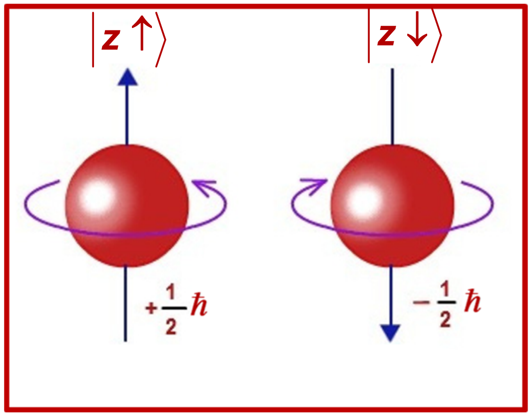
$$|z\uparrow\rangle \quad \Longleftrightarrow \quad \hat{S}_z|z\uparrow\rangle = +\frac{\hbar}{2}|z\uparrow\rangle$$

$$|z\downarrow\rangle \quad \Longleftrightarrow \quad \hat{S}_z|z\downarrow\rangle = -\frac{\hbar}{2}|z\downarrow\rangle$$

# The Operator for Spin Angular Momentum: z-axis

The operator for spin angular momentum measured along z-axis is:

$$\hat{S}_z = \frac{\hbar}{2}\hat{\sigma}_z$$




And its eigenvectors must be the two spin states:

$$\hat{S}_z|z\uparrow\rangle = +\frac{\hbar}{2}|z\uparrow\rangle \qquad \hat{S}_z|z\downarrow\rangle = -\frac{\hbar}{2}|z\downarrow\rangle$$

$$\Rightarrow \frac{\hbar}{2}\hat{\sigma}_z|z\uparrow\rangle = +\frac{\hbar}{2}|z\uparrow\rangle \qquad \Rightarrow \frac{\hbar}{2}\hat{\sigma}_z|z\downarrow\rangle = -\frac{\hbar}{2}|z\downarrow\rangle$$

$$\Rightarrow \hat{\sigma}_z|z\uparrow\rangle = +|z\uparrow\rangle \qquad \Rightarrow \hat{\sigma}_z|z\downarrow\rangle = -|z\downarrow\rangle$$

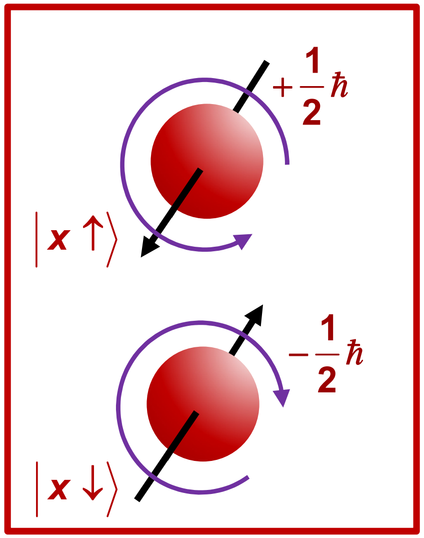
So now the measured values of the electron spin angular momentum's z-component can be $+\hbar/2$ or $-\hbar/2$ as observed in experiments

The eigenvalues of $\hat{\sigma}_z$ must then be $+1$ and $-1$

# The Operator for Spin Angular Momentum: x-axis

The operator for spin angular momentum measured along x-axis is:

$$\hat{S}_x = \frac{\hbar}{2}\hat{\sigma}_x$$




And its eigenvectors must be the two spin states:

$$\hat{S}_x|x\uparrow\rangle = +\frac{\hbar}{2}|x\uparrow\rangle \qquad \hat{S}_x|x\downarrow\rangle = -\frac{\hbar}{2}|x\downarrow\rangle$$

$$\Rightarrow \frac{\hbar}{2}\hat{\sigma}_x|x\uparrow\rangle = +\frac{\hbar}{2}|x\uparrow\rangle \qquad \Rightarrow \frac{\hbar}{2}\hat{\sigma}_x|x\downarrow\rangle = -\frac{\hbar}{2}|x\downarrow\rangle$$

$$\Rightarrow \hat{\sigma}_x|x\uparrow\rangle = +|x\uparrow\rangle \qquad \Rightarrow \hat{\sigma}_x|x\downarrow\rangle = -|x\downarrow\rangle$$

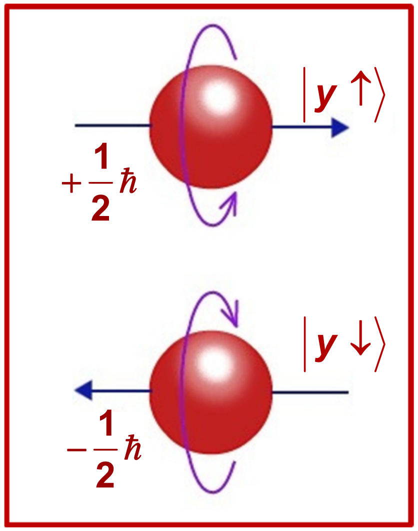
So now the measured values of the electron spin angular momentum's x-component can be $+\hbar/2$ or $-\hbar/2$ as observed in experiments

The eigenvalues of $\hat{\sigma}_x$ must then be $+1$ and $-1$

# The Operator for Spin Angular Momentum: y-axis

The operator for spin angular momentum measured along y-axis is:

$$\hat{S}_y = \frac{\hbar}{2}\hat{\sigma}_y$$




And its eigenvectors must be the two spin states:

$$\hat{S}_y|y\uparrow\rangle = +\frac{\hbar}{2}|y\uparrow\rangle \qquad \hat{S}_y|y\downarrow\rangle = -\frac{\hbar}{2}|y\downarrow\rangle$$

$$\Rightarrow \frac{\hbar}{2}\hat{\sigma}_y|y\uparrow\rangle = +\frac{\hbar}{2}|y\uparrow\rangle \qquad \Rightarrow \frac{\hbar}{2}\hat{\sigma}_y|y\downarrow\rangle = -\frac{\hbar}{2}|y\downarrow\rangle$$

$$\Rightarrow \hat{\sigma}_y|y\uparrow\rangle = +|y\uparrow\rangle \qquad \Rightarrow \hat{\sigma}_y|y\downarrow\rangle = -|y\downarrow\rangle$$

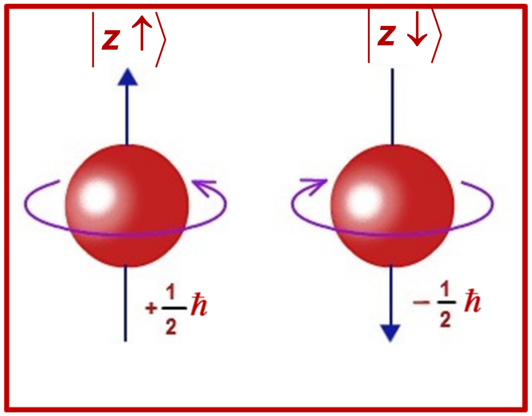
So now the measured values of the electron spin angular momentum's y-component can be $+\hbar/2$ or $-\hbar/2$ as observed in experiments

The eigenvalues of $\hat{\sigma}_y$ must then be $+1$ and $-1$

# The Operator for Spin Angular Momentum

The operator for spin angular momentum measured along **z-axis** is:

$$\hat{S}_z = \frac{\hbar}{2}\hat{\sigma}_z$$




And its **eigenvectors** must be the two spin states:

$$\hat{S}_z|z\uparrow\rangle = +\frac{\hbar}{2}|z\uparrow\rangle \qquad \hat{S}_z|z\downarrow\rangle = -\frac{\hbar}{2}|z\downarrow\rangle$$

$$\Rightarrow \frac{\hbar}{2}\hat{\sigma}_z|z\uparrow\rangle = +\frac{\hbar}{2}|z\uparrow\rangle \qquad \Rightarrow \frac{\hbar}{2}\hat{\sigma}_z|z\downarrow\rangle = -\frac{\hbar}{2}|z\downarrow\rangle$$

$$\Rightarrow \hat{\sigma}_z|z\uparrow\rangle = +|z\uparrow\rangle \qquad \Rightarrow \hat{\sigma}_z|z\downarrow\rangle = -|z\downarrow\rangle$$

So now the measured values of the electron spin angular momentum's z-component can be $+\hbar/2$ or $-\hbar/2$ as observed in experiments

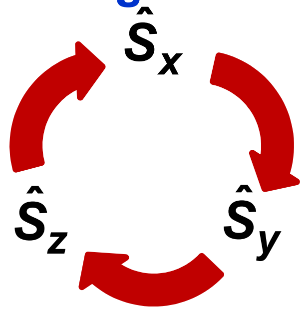
The eigenvalues of $\hat{\sigma}_z$ must then be $+1$ and $-1$

# Spin 1/2 Commutation Relations

$$\hat{\vec{S}} = \hat{S}_x e_x + \hat{S}_y e_y + \hat{S}_z e_z$$

$$\left\{ \begin{aligned} \hat{S}_x &= \frac{\hbar}{2}\hat{\sigma}_x \\ \hat{S}_y &= \frac{\hbar}{2}\hat{\sigma}_y \\ \hat{S}_z &= \frac{\hbar}{2}\hat{\sigma}_z \end{aligned} \right.$$

**The components of the spin angular momentum don't commute:**




$$\left[\hat{S}_x, \hat{S}_y\right] = i\hbar\hat{S}_z$$

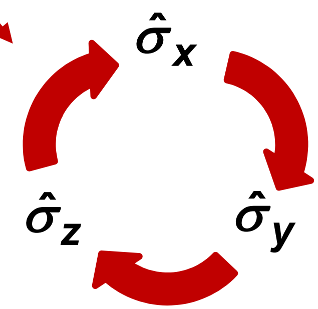
$$\left[\hat{S}_y, \hat{S}_z\right] = i\hbar\hat{S}_x$$

$$\left[\hat{S}_z, \hat{S}_x\right] = i\hbar\hat{S}_y$$

$$\left[\hat{\sigma}_x, \hat{\sigma}_y\right] = 2i\hat{\sigma}_z$$

$$\left[\hat{\sigma}_y, \hat{\sigma}_z\right] = 2i\hat{\sigma}_x$$

$$\left[\hat{\sigma}_z, \hat{\sigma}_x\right] = 2i\hat{\sigma}_y$$




**The above commutation relation express fundamental laws, consistent with the Lorentz invariance of the universe, and cannot be derived from anything more fundamental**

# Spin 1/2 Commutation Relations

$$\left[\hat{S}_x, \hat{S}_y\right] = i\hbar \hat{S}_z$$

$$\left[\hat{S}_y, \hat{S}_z\right] = i\hbar \hat{S}_x$$

$$\left[\hat{S}_z, \hat{S}_x\right] = i\hbar \hat{S}_y$$

**If two operators don't commute, the corresponding observables cannot have a simultaneous reality**

**Example:** Because position and momentum operators don't commute,

$$\left[\hat{x}, \hat{p}\right] = i\hbar$$

Position and momentum observables don't have simultaneous reality; a particle can be in a position eigenstate, or in a momentum eigenstate, but not in both at the same time.

Consequently, either the position eigenkets taken by themselves, or the momentum eigenkets taken by themselves, form a complete set:

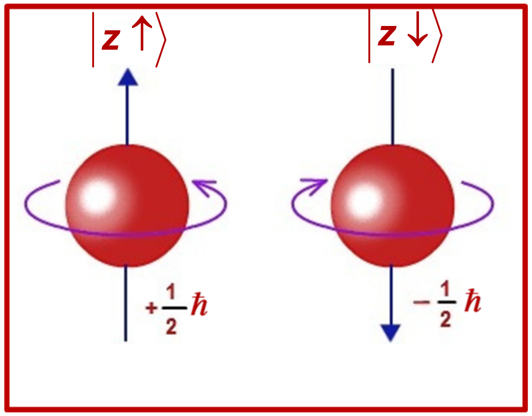
$$\int dx \, |x\rangle\langle x| = \hat{1} \qquad \int \frac{dp}{2\pi} |p\rangle\langle p| = \hat{1}$$

**Lesson:** One doesn't need eigenstates of $\hat{S}_x$ or $\hat{S}_y$ to be included in the completeness relation:

$$|z\uparrow\rangle\langle z\uparrow| + |z\downarrow\rangle\langle z\downarrow| = \hat{1}$$

# The Hilbert Space of Electron Spin States

The operator $\hat{S}_z = \frac{\hbar}{2}\hat{\sigma}_z$ is Hermitian; this means:




**The Hilbert space of electron spin angular momentum is completely spanned by just these two states:**

$$|z\uparrow\rangle\langle z\uparrow| + |z\downarrow\rangle\langle z\downarrow| = \hat{1} \longrightarrow \text{Completeness}$$

**We must also have:**

$$\langle z\uparrow|z\uparrow\rangle = \langle z\downarrow|z\downarrow\rangle = 1 \longrightarrow \text{Normalization}$$

$$\langle z\uparrow|z\downarrow\rangle = 0 \longrightarrow \text{Orthogonality}$$

In general, the spin state of an electron can also be a superposition of the up and down spin states:

$$|\psi\rangle = a|z\uparrow\rangle + b|z\downarrow\rangle$$

$$\Rightarrow \langle\psi|\psi\rangle = |a|^2 + |b|^2 = 1$$

## The Hilbert Space of Electron Spin States

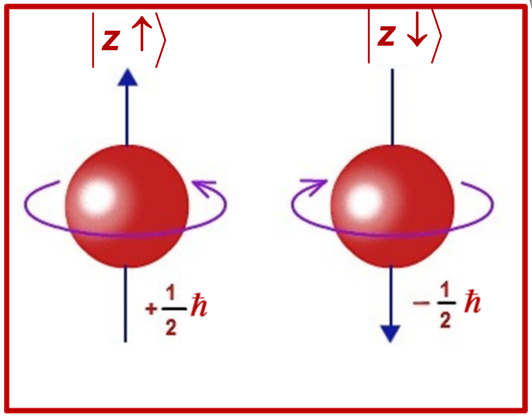
**We can write the spin angular momentum operator as:**

$$\hat{S}_z = \frac{\hbar}{2}\hat{\sigma}_z = \hat{1}\frac{\hbar}{2}\hat{\sigma}_z\hat{1}$$

$$= \left[|z\uparrow\rangle\langle z\uparrow| + |z\downarrow\rangle\langle z\downarrow|\right]\frac{\hbar}{2}\hat{\sigma}_z\left[|z\uparrow\rangle\langle z\uparrow| + |z\downarrow\rangle\langle z\downarrow|\right]$$

$$= \frac{\hbar}{2}|z\uparrow\rangle\langle z\uparrow| - \frac{\hbar}{2}|z\downarrow\rangle\langle z\downarrow|$$

# The Matrix Representation for Spin Angular Momentum




**Start from:**

$$\hat{S}_z = \frac{\hbar}{2}\hat{\sigma}_z = \frac{\hbar}{2}|z\uparrow\rangle\langle z\uparrow| - \frac{\hbar}{2}|z\downarrow\rangle\langle z\downarrow|$$

**Use a representation by the Hilbert space formed by 2x2 matrices and 2x1 column vectors:**

$$|z\uparrow\rangle \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |z\downarrow\rangle \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

**Then since an operator is always diagonal in the representation of its eigenvectors:**

$$\hat{S}_z = \frac{\hbar}{2}\hat{\sigma}_z \rightarrow \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Longrightarrow \hat{\sigma}_z \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

**Example: Spin angular momentum expectation value or mean value:**

$$|\psi\rangle = a|z\uparrow\rangle + b|z\downarrow\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\langle\hat{S}_z\rangle = \langle\psi|\frac{\hbar}{2}\hat{\sigma}_z|\psi\rangle = [a^* \quad b^*]\frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \frac{\hbar}{2}|a|^2 - \frac{\hbar}{2}|b|^2$$

# The Full Hilbert Space of Electrons

Since the spin state of an electron can be $|z\uparrow\rangle$ or $|z\downarrow\rangle$, the complete quantum state of the electron is obtained by "gluing" together its spin degree of freedom with its other degrees (position, momentum, etc) of freedom:

$$|\psi\rangle = |\phi\rangle \otimes |z\uparrow\rangle \qquad \text{or} \qquad |\psi'\rangle = |\phi\rangle \otimes |z\downarrow\rangle$$

So for example, the wavefunction of a spin up electron becomes:

$$\langle x|\psi\rangle = \langle x|\phi\rangle|z\uparrow\rangle = \phi(x)|z\uparrow\rangle$$

Electron with position wavefunction $\phi(x)$ and spin up

Or, for a spin down electron:

$$\langle x|\psi'\rangle = \langle x|\phi\rangle|z\downarrow\rangle = \phi(x)|z\downarrow\rangle$$

Electron with position wavefunction $\phi(x)$ and spin down

The inner products in this Hilbert space work as follows:

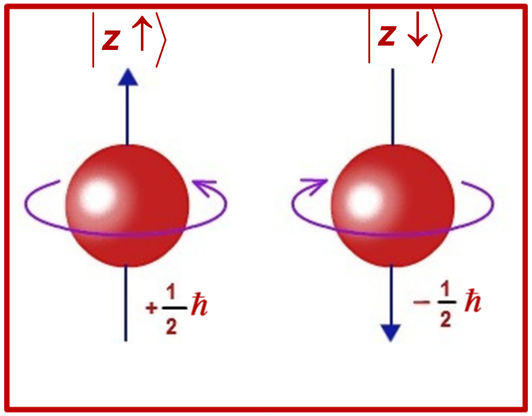
$$|\psi\rangle = |\phi\rangle \otimes |z\uparrow\rangle$$
$$\Rightarrow \langle\psi|\psi\rangle = \langle\phi|\phi\rangle\langle z\uparrow|z\uparrow\rangle = 1$$

and

$$|\psi'\rangle = |\chi\rangle \otimes \frac{1}{\sqrt{2}}\left[|z\uparrow\rangle + |z\downarrow\rangle\right]$$
$$\Rightarrow \langle\psi'|\psi'\rangle = \langle\chi|\chi\rangle\frac{1}{2}\left[\begin{array}{l}\langle z\uparrow|z\uparrow\rangle + \langle z\downarrow|z\downarrow\rangle \\ +\langle z\uparrow|z\downarrow\rangle + \langle z\downarrow|z\uparrow\rangle\end{array}\right] = 1$$

$$\langle\psi'|\psi\rangle = \langle\chi|\phi\rangle\frac{1}{\sqrt{2}}\left[\langle z\uparrow|z\uparrow\rangle + \langle z\downarrow|z\uparrow\rangle\right]$$
$$= \langle\chi|\phi\rangle\frac{1}{\sqrt{2}}$$

# The Nature of Spin 1/2




**Very tempting to take this too literally ……….**

# Spin 1/2 Commutation Relations

**The components of the spin angular momentum don't commute:**

$$\left[\hat{S}_x, \hat{S}_y\right] = i\hbar\hat{S}_z \qquad \left[\hat{\sigma}_x, \hat{\sigma}_y\right] = 2i\hat{\sigma}_z$$

$$\left[\hat{S}_y, \hat{S}_z\right] = i\hbar\hat{S}_x \quad \Longrightarrow \quad \left[\hat{\sigma}_y, \hat{\sigma}_z\right] = 2i\hat{\sigma}_x$$

$$\left[\hat{S}_z, \hat{S}_x\right] = i\hbar\hat{S}_y \qquad \left[\hat{\sigma}_z, \hat{\sigma}_x\right] = 2i\hat{\sigma}_y$$

**In the basis:**

$$|z\uparrow\rangle \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |z\downarrow\rangle \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\left\{ \begin{array}{l} \langle z\uparrow|z\uparrow\rangle = \langle z\downarrow|z\downarrow\rangle = 1 \\ \langle z\uparrow|z\downarrow\rangle = 0 \longrightarrow \text{Orthogonality} \\ |z\uparrow\rangle\langle z\uparrow| + |z\downarrow\rangle\langle z\downarrow| = \hat{1} \longrightarrow \text{Completeness} \end{array} \right.$$

**We get:**

$$\hat{\sigma}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \hat{\sigma}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \hat{\sigma}_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Diagonal

**Pauli Spin Matrices**

# The Operators for Spin Angular Momentum

$$\hat{S}_x = \frac{\hbar}{2}\hat{\sigma}_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \hat{S}_y = \frac{\hbar}{2}\hat{\sigma}_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \hat{S}_z = \frac{\hbar}{2}\hat{\sigma}_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

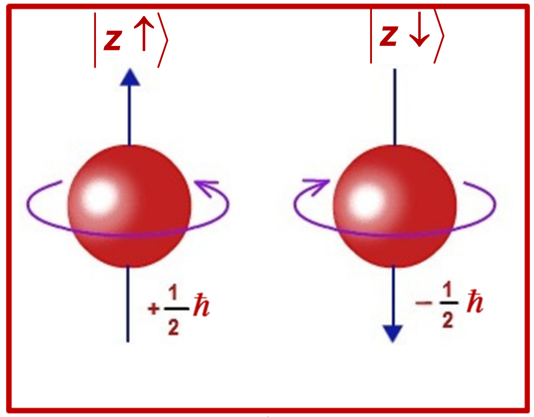
**Pauli Spin Matrices**

**Since we have chosen the eigenstates of the z-component of the spin angular momentum for the matrix representation, the operator for the z-component of the spin angular momentum is diagonal in the chosen representation, but the operators for the x- and y-components are not diagonal !**

$$\hat{S}^2 = \hat{\vec{S}}.\hat{\vec{S}} = \hat{S}_x\hat{S}_x + \hat{S}_y\hat{S}_y + \hat{S}_z\hat{S}_z = \frac{3}{4}\hbar^2\,\hat{1} = \frac{3}{4}\hbar^2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\left[\hat{S}^2, \hat{S}_j\right] = 0 \qquad \{\, j = x, y, z$$

# The Nature of Spin 1/2




**Very tempting to take this too literally ……….**

**Since the x, y, and z components of the spin angular momentum operator do not commute, different components of the spin angular momentum cannot be simultaneously measured with complete accuracy**

# Spin 1/2: Classical vs Quantum Pictures

$$\left[\hat{S}_x, \hat{S}_y\right] = i\hbar\hat{S}_z \qquad \left[\hat{S}_y, \hat{S}_z\right] = i\hbar\hat{S}_x \qquad \left[\hat{S}_z, \hat{S}_x\right] = i\hbar\hat{S}_y$$

**Consider the state:** $|\psi\rangle = |z\uparrow\rangle$

**It is an eigenstate of the z-component of the spin angular momentum:** $\hat{S}_z|z\uparrow\rangle = +\frac{\hbar}{2}|z\uparrow\rangle$

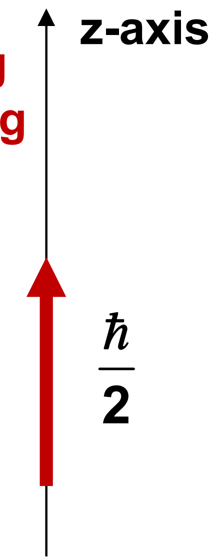
**We get:**

$$\langle\hat{S}_z\rangle = \langle z\uparrow|\frac{\hbar}{2}\hat{\sigma}_z|z\uparrow\rangle = \frac{\hbar}{2}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \frac{\hbar}{2}$$

$$\langle\hat{S}_x\rangle = \langle z\uparrow|\frac{\hbar}{2}\hat{\sigma}_x|z\uparrow\rangle = \frac{\hbar}{2}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = 0$$

$$\langle\hat{S}_y\rangle = \langle z\uparrow|\frac{\hbar}{2}\hat{\sigma}_y|z\uparrow\rangle = \frac{\hbar}{2}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = 0$$

**Tempting but wrong classical picture**



# Spin 1/2: Classical vs Quantum Pictures

**For the state:** $|\psi\rangle = |z\uparrow\rangle$

$$\langle \hat{S}_z \rangle = \frac{\hbar}{2} \qquad \langle \hat{S}_z^2 \rangle = \frac{\hbar^2}{4}$$

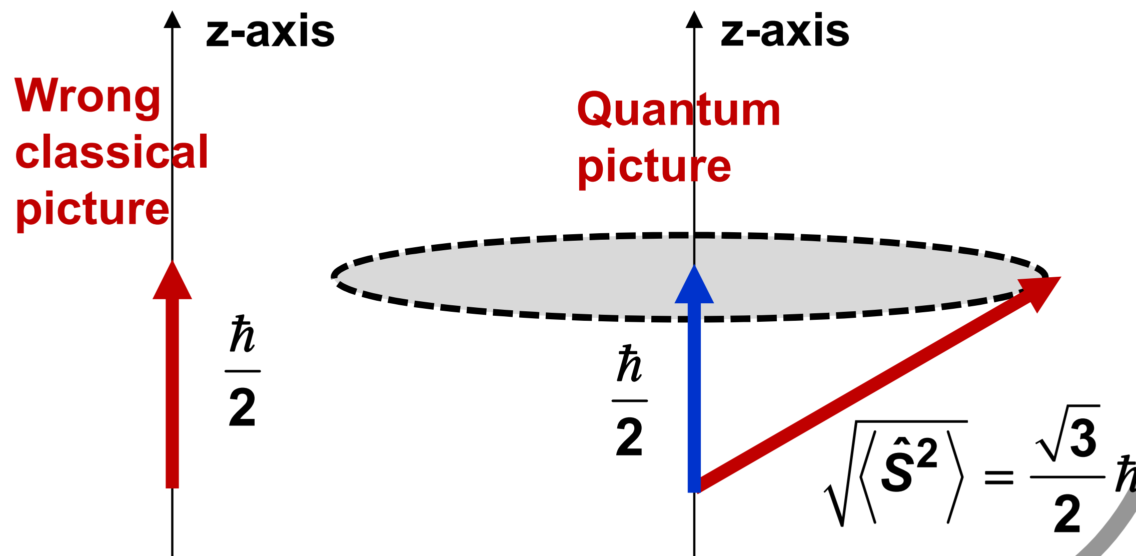
$$\langle \hat{S}_x \rangle = \langle \hat{S}_y \rangle = 0 \qquad \langle \hat{S}_x^2 \rangle = \langle \hat{S}_y^2 \rangle = \frac{\hbar^2}{4}$$

$$\Rightarrow \langle \hat{S}^2 \rangle = \langle \hat{S}_x^2 \rangle + \langle \hat{S}_y^2 \rangle + \langle \hat{S}_z^2 \rangle = \frac{3}{4}\hbar^2$$

$$\langle \Delta\hat{S}_z^2 \rangle = \left\langle \left( \hat{S}_z - \langle \hat{S}_z \rangle \right)^2 \right\rangle = 0$$

$$\langle \Delta\hat{S}_x^2 \rangle = \left\langle \left( \hat{S}_x - \langle \hat{S}_x \rangle \right)^2 \right\rangle = \frac{\hbar^2}{4}$$

$$\langle \Delta\hat{S}_y^2 \rangle = \left\langle \left( \hat{S}_y - \langle \hat{S}_y \rangle \right)^2 \right\rangle = \frac{\hbar^2}{4}$$

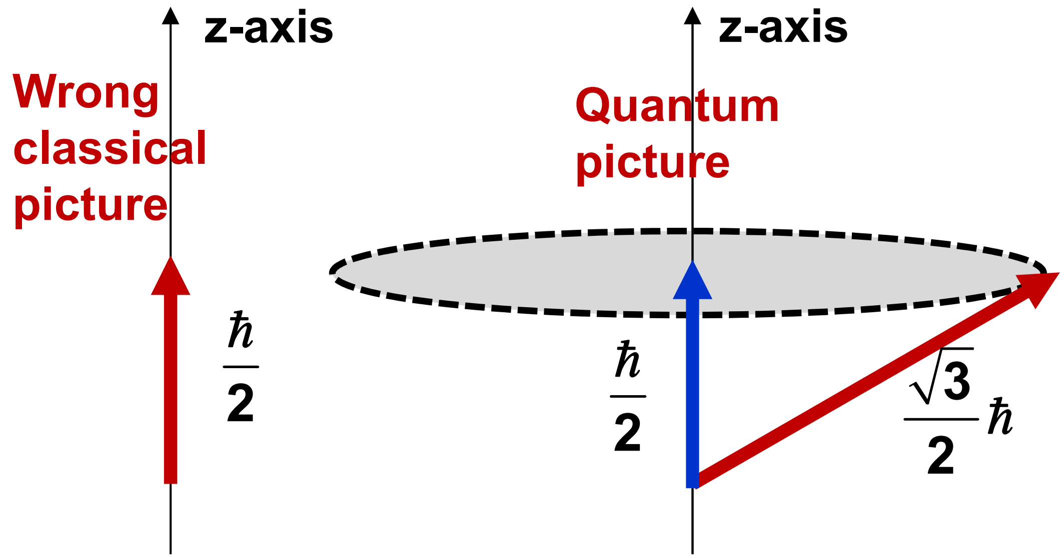


# Spin 1/2: Classical vs Quantum Pictures

$$\left[\hat{S}_x,\hat{S}_y\right]=i\hbar\hat{S}_z$$

$$\left[\hat{S}_y,\hat{S}_z\right]=i\hbar\hat{S}_x$$

$$\left[\hat{S}_z,\hat{S}_x\right]=i\hbar\hat{S}_y$$




**Quantum state is:** $\left|\psi\right\rangle=\left|z\uparrow\right\rangle$

$$\left\langle\hat{S}_x\right\rangle=\left\langle\hat{S}_y\right\rangle=0 \qquad \left\langle\hat{S}_z\right\rangle=\frac{\hbar}{2}$$

$$\left\langle\Delta\hat{S}_x^2\right\rangle\left\langle\Delta\hat{S}_y^2\right\rangle=\frac{\hbar^4}{16}$$

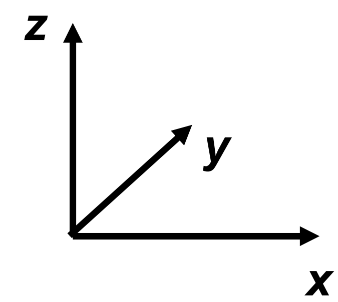
**What does the uncertainty principle say:**

$$\left\langle\Delta\hat{S}_x^2\right\rangle\left\langle\Delta\hat{S}_y^2\right\rangle\geq\frac{\left|\left\langle\left[\hat{S}_x,\hat{S}_y\right]\right\rangle\right|^2}{4}=\frac{\left|\left\langle i\hbar\hat{S}_z\right\rangle\right|^2}{4}=\frac{\hbar^4}{16}$$

**The chosen state satisfies the uncertainty principle with equality**

# The Operators for Spin Angular Momentum

The operator for spin angular momentum component along the x-axis, in the basis defined by the eigenvectors of $\hat{S}_z$, is:




$$\hat{S}_x = \frac{\hbar}{2}\hat{\sigma}_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Eigenvalues and eigenvectors are:

$$\left.\begin{array}{l} \frac{\hbar}{2}\hat{\sigma}_x \left| x\uparrow \right\rangle = +\frac{\hbar}{2}\left| x\uparrow \right\rangle \\ \frac{\hbar}{2}\hat{\sigma}_x \left| x\downarrow \right\rangle = -\frac{\hbar}{2}\left| x\downarrow \right\rangle \end{array}\right\} \quad \begin{array}{l} \left| x\uparrow \right\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{\left| z\uparrow \right\rangle + \left| z\downarrow \right\rangle}{\sqrt{2}} \\ \left| x\downarrow \right\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{\left| z\uparrow \right\rangle - \left| z\downarrow \right\rangle}{\sqrt{2}} \end{array}$$

The operator for spin angular momentum component along the y-axis is:

$$\hat{S}_y = \frac{\hbar}{2}\hat{\sigma}_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Eigenvalues are eigenvectors are:

$$\left.\begin{array}{l} \frac{\hbar}{2}\hat{\sigma}_y \left| y\uparrow \right\rangle = +\frac{\hbar}{2}\left| y\uparrow \right\rangle \\ \frac{\hbar}{2}\hat{\sigma}_y \left| y\downarrow \right\rangle = -\frac{\hbar}{2}\left| y\downarrow \right\rangle \end{array}\right\} \quad \begin{array}{l} \left| y\uparrow \right\rangle = \frac{\left| z\uparrow \right\rangle + i\left| z\downarrow \right\rangle}{\sqrt{2}} \\ \left| y\downarrow \right\rangle = \frac{\left| z\uparrow \right\rangle - i\left| z\downarrow \right\rangle}{\sqrt{2}} \end{array}$$

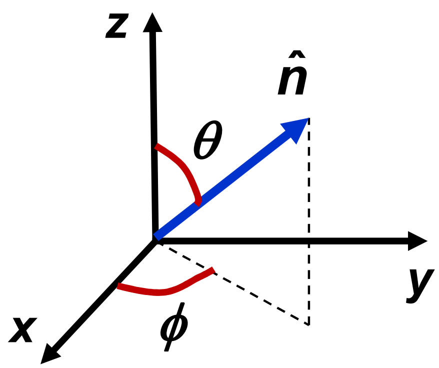
# Spin Eigenstates for a General Direction

A spin 1/2 state pointing in the +z direction is: $|\psi\rangle = |z\uparrow\rangle$

A spin 1/2 state pointing in the +x direction is: $|\psi\rangle = |x\uparrow\rangle = \dfrac{|z\uparrow\rangle + |z\downarrow\rangle}{\sqrt{2}}$

A spin 1/2 state pointing in the +y direction is: $|\psi\rangle = |y\uparrow\rangle = \dfrac{|z\uparrow\rangle + i|z\downarrow\rangle}{\sqrt{2}}$

**Lets generalize:** what if one wants a spin 1/2 state pointing in the direction of unit vector $\hat{n}$ ?




**First, define a spin operator for the direction $\hat{n}$ :**

$$\hat{n} = \sin\theta\cos\phi\, e_x + \sin\theta\sin\phi\, e_y + \cos\theta\, e_z$$

$$\hat{S}_{\hat{n}} = \hat{\vec{S}}.\hat{n} = \left[\hat{S}_x e_x + \hat{S}_y e_y + \hat{S}_z e_z\right].\hat{n} = \frac{\hbar}{2}\begin{bmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{bmatrix}$$

Operator for spin pointing in $\hat{n}$ direction

Next we find the two eigenstates of the operator $\hat{S}_{\hat{n}}$ ............

# Spin Eigenstates for a General Direction

$$\hat{S}_{\hat{n}} = \hat{\vec{S}}.\hat{n} = \left[\hat{S}_x \hat{x} + \hat{S}_y \hat{y} + \hat{S}_z \hat{z}\right].\hat{n} = \frac{\hbar}{2}\begin{bmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{bmatrix}$$

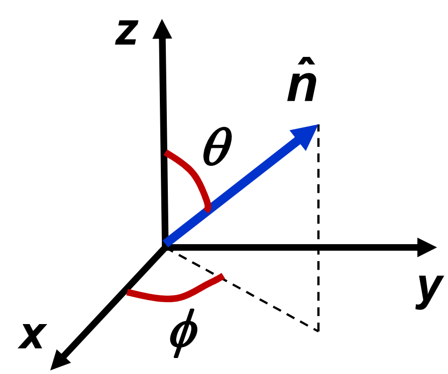
**The two eigenstates and the eigenvalues of $\hat{S}_{\hat{n}}$ are found to be:**

$$|\hat{n}\uparrow\rangle = \cos(\theta/2)e^{-i\phi/2}|z\uparrow\rangle + \sin(\theta/2)e^{i\phi/2}|z\downarrow\rangle$$

$$\Rightarrow \hat{S}_{\hat{n}}|\hat{n}\uparrow\rangle = +\frac{\hbar}{2}|\hat{n}\uparrow\rangle$$

$$|\hat{n}\downarrow\rangle = \sin(\theta/2)e^{-i\phi/2}|z\uparrow\rangle - \cos(\theta/2)e^{i\phi/2}|z\downarrow\rangle$$

$$\Rightarrow \hat{S}_{\hat{n}}|\hat{n}\downarrow\rangle = -\frac{\hbar}{2}|\hat{n}\downarrow\rangle$$



# Classical Larmor Precession of a Magnetic Momemet in a Magnetic Field

**Consider a classical magnetic moment in a DC magnetic field**

Energy = $E = -\vec{m} \cdot \vec{B}$

**Angular momentum of a magnetic moment:**

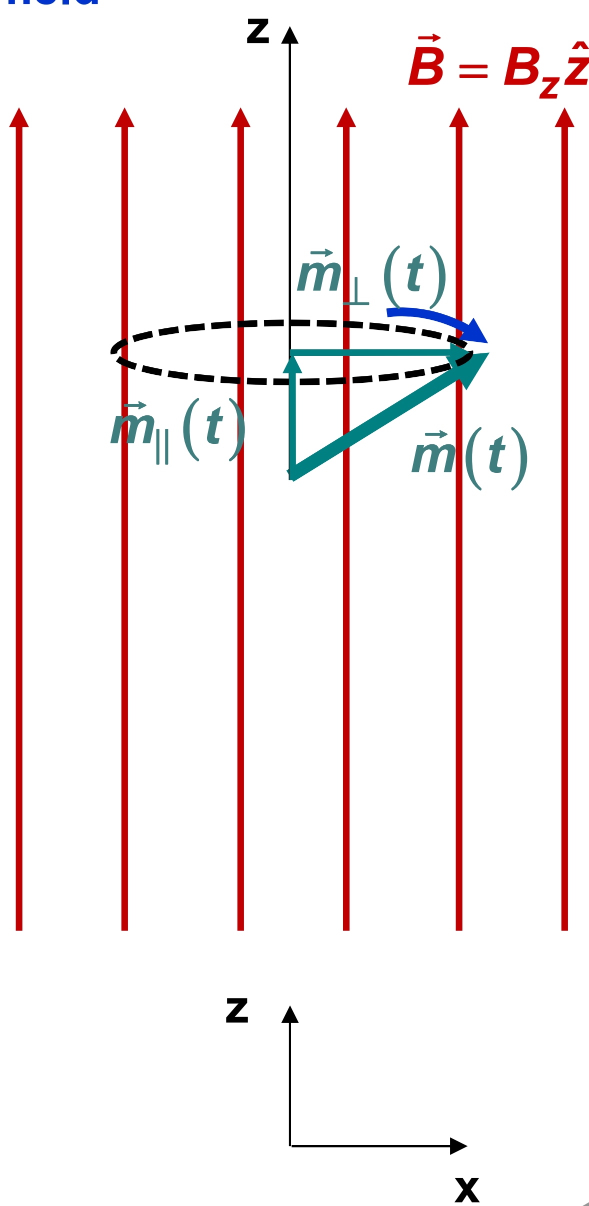
$$\vec{m} = \gamma \vec{L}$$

Gyromagnetic ratio

**Torque and Dynamics:**

$$\frac{d\vec{L}(t)}{dt} = \vec{\tau} = \vec{m} \times \vec{B}$$

$$\Rightarrow \frac{d\vec{m}(t)}{dt} = \gamma \vec{m}(t) \times \vec{B}$$



# Classical Larmor Precession in a Magnetic Field

$$\frac{d\vec{m}(t)}{dt} = \gamma \vec{m}(t) \times \vec{B} \longrightarrow \boxed{\frac{d\vec{m}_{\parallel}(t)}{dt} = 0}$$

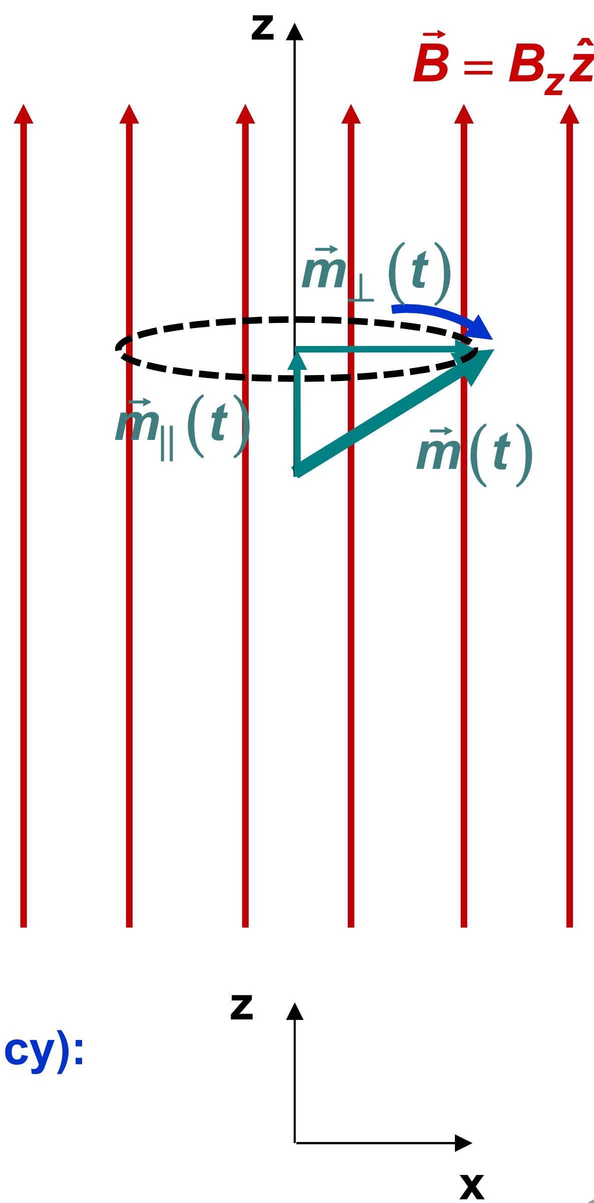
$$\Rightarrow \frac{d^2\vec{m}(t)}{dt^2} = \gamma \frac{d\vec{m}(t)}{dt} \times \vec{B}$$

$$\Rightarrow \frac{d^2\vec{m}(t)}{dt^2} = \gamma^2 \left[\vec{m}(t) \times \vec{B}\right] \times \vec{B}$$

$$\Rightarrow \frac{d^2\vec{m}(t)}{dt^2} = -\left(\gamma^2 \vec{B} . \vec{B}\right) \vec{m}(t) + \gamma^2 \vec{B}\left[\vec{B} . \vec{m}(t)\right]$$

$$\Rightarrow \frac{d^2\vec{m}(t)}{dt^2} + \omega_L^2 \, \vec{m}(t) = \gamma^2 \vec{B}\left[\vec{B} . \vec{m}(t)\right]$$

$$\boxed{\frac{d^2\vec{m}_{\perp}(t)}{dt^2} + \omega_L^2 \, \vec{m} \perp (t) = 0}$$




**Frequency of precession (also called the Larmor frequency):**

$$\omega_L = \left|\gamma B_z\right|$$

# Spin 1/2 Qubit in an External Magnetic Field

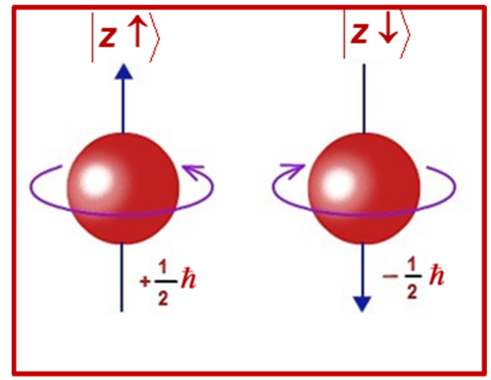
Electron spin angular momentum has an associated **magnetic moment**:

Bohr magneton

$$\mu_B = \frac{e\hbar}{2m}$$

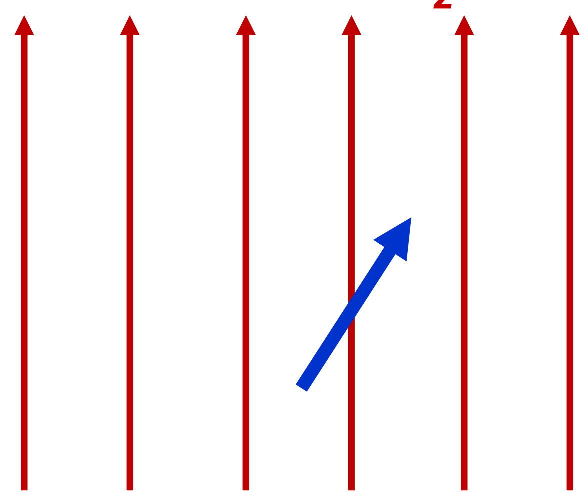
$$\hat{\vec{m}} = \gamma \hat{\vec{S}} = -\frac{e}{m}\hat{\vec{S}} = -\frac{e}{m}\left[\hat{S}_x e_x + \hat{S}_y e_y + \hat{S}_z e_z\right]$$

$$= -\frac{e\hbar}{2m}\left[\hat{\sigma}_x e_x + \hat{\sigma}_y e_y + \hat{\sigma}_z e_z\right] = -\mu_B \hat{\vec{\sigma}}$$




Consider a spin 1/2 particle (electron) in an applied magnetic field:

$$\vec{B} = B_z \hat{z}$$



The classical expression for the energy of a magnetic moment in an external magnetic field is:

$$E = -\vec{m} \,.\, \vec{B}$$

The Hamiltonian of a spin 1/2 in an external magnetic field is:

$$\hat{H} = -\hat{\vec{m}} \,.\, \vec{B} = \mu_B \hat{\vec{\sigma}} \,.\, \vec{B} = \mu_B B_z \hat{\sigma}_z = \mu_B B_z \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

# Spin 1/2 Qubit in an External Magnetic Field: Zeeman Splitting

$$\hat{H} = \mu_B B_z \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

**Eigenstates and eigenvalues of the Hamiltonian:**

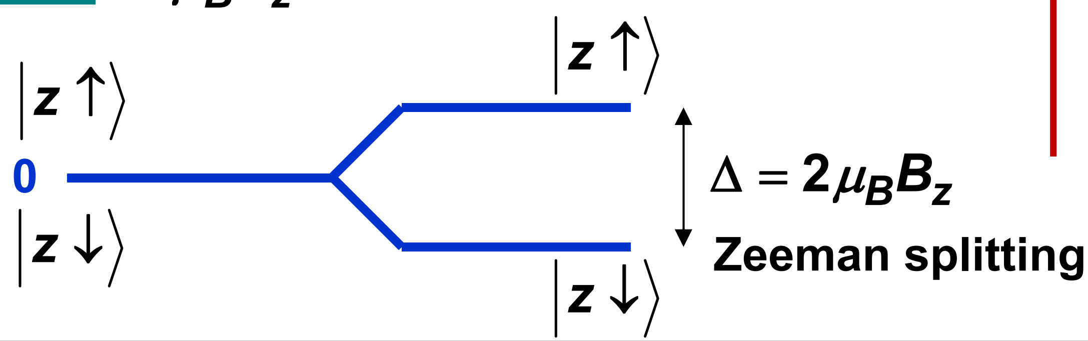
**The Hamiltonian as written in the $|z\uparrow\rangle, |z\downarrow\rangle$ basis is already diagonal so:**

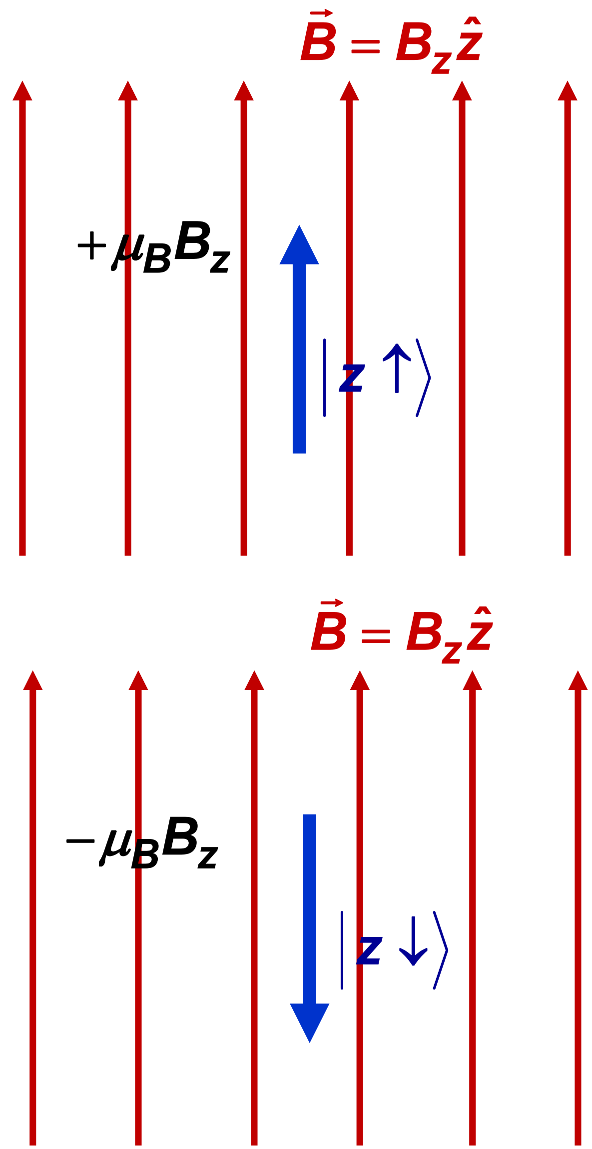
$$\hat{H}|z\uparrow\rangle = +\mu_B B_z |z\uparrow\rangle$$

$$\hat{H}|z\downarrow\rangle = -\mu_B B_z |z\downarrow\rangle$$

**The eigenvalues are:** $+\mu_B B_z, -\mu_B B_z$

**Energy splitting between the eigenvalues = Zeeman splitting = $2\mu_B B_z = \Delta$**



## Spin 1/2 Qubit in an External Magnetic Field
## An Example of a Two-Level System (TLS)

$$\hat{H} = \mu_B B_z \hat{\sigma}_z = \mu_B B_z \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

**This is an example of a two-level system (TLS)**

**A TLS has a Hilbert space of dimension two**

**Any 2x2 matrix $\hat{W}$ can be written in terms of the Pauli matrices as follows:**

$$\hat{W} = A\hat{\sigma}_x + B\hat{\sigma}_y + C\hat{\sigma}_z + D\hat{1}$$

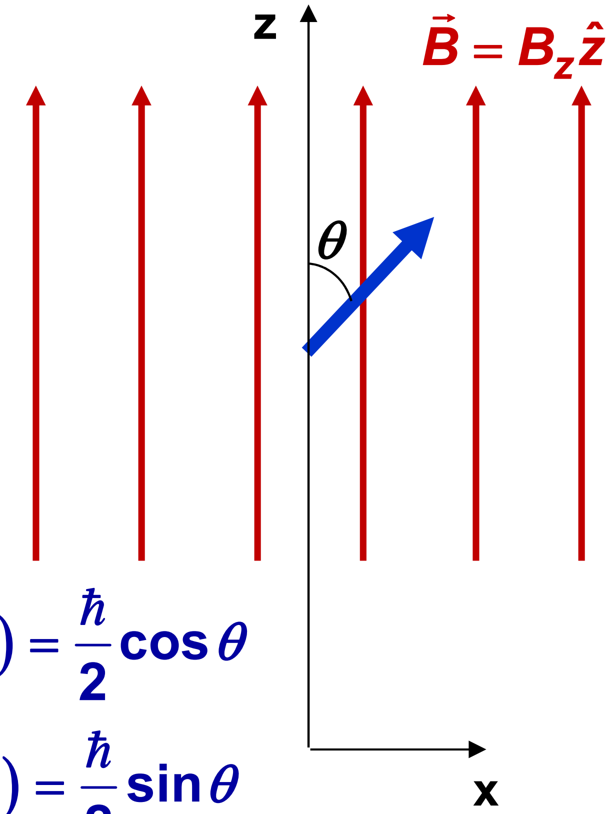
**With appropriate choice of the constants A, B, C, and D**

**Therefore, the Hamiltonian of a TLS can always be written as:**

$$\hat{H} = a\hat{\sigma}_x + b\hat{\sigma}_y + c\hat{\sigma}_z + d\hat{1}$$

## Spin 1/2 Qubit in an External Magnetic Field: TLS Dynamics

Consider an electron (spin 1/2) placed inside a uniform magnetic field, as shown




Suppose at time $t$=0, the spin state of the electron is:

$$|\psi(t=0)\rangle = \cos(\theta/2)|z\uparrow\rangle + \sin(\theta/2)|z\downarrow\rangle$$

**Question: Find the spin state for time $t > 0$**

$$\langle \hat{S}_z\rangle(t=0) = \frac{\hbar}{2}\cos\theta$$

$$\langle \hat{S}_x\rangle(t=0) = \frac{\hbar}{2}\sin\theta$$

We need to use:

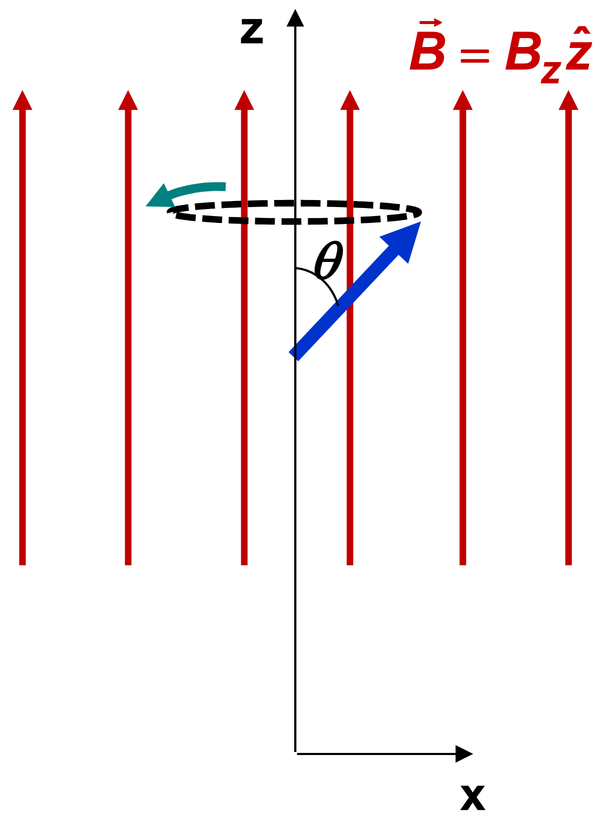
$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

$$\Rightarrow |\psi(t)\rangle = e^{-i\frac{\hat{H}}{\hbar}t}|\psi(t=0)\rangle$$

$$\Rightarrow |\psi(t)\rangle = e^{-i\frac{\hat{H}}{\hbar}t}\left[\cos(\theta/2)|z\uparrow\rangle + \sin(\theta/2)|z\downarrow\rangle\right] = \left[\cos(\theta/2)e^{-i\frac{\hat{H}}{\hbar}t}|z\uparrow\rangle + \sin(\theta/2)e^{-i\frac{\hat{H}}{\hbar}t}|z\downarrow\rangle\right]$$

$$\Rightarrow |\psi(t)\rangle = \cos(\theta/2)e^{-i\frac{\mu_B B_z}{\hbar}t}|z\uparrow\rangle + \sin(\theta/2)e^{+i\frac{\mu_B B_z}{\hbar}t}|z\downarrow\rangle$$

# Spin 1/2 Qubit in an External Magnetic Field: TLS Dynamics




$$\left|\psi(t)\right\rangle = \cos(\theta/2)\, e^{-i\frac{\mu_B B_z}{\hbar}t}\left|z\uparrow\right\rangle + \sin(\theta/2)\, e^{+i\frac{\mu_B B_z}{\hbar}t}\left|z\downarrow\right\rangle$$

$$\left\langle \hat{S}_x \right\rangle(t) = \left\langle \psi(t)\right|\hat{S}_x\left|\psi(t)\right\rangle = \frac{\hbar}{2}\sin\theta\,\cos\left(\frac{2\mu_B B_z}{\hbar}t\right)$$

$$\left\langle \hat{S}_y \right\rangle(t) = \left\langle \psi(t)\right|\hat{S}_y\left|\psi(t)\right\rangle = \frac{\hbar}{2}\sin\theta\,\sin\left(\frac{2\mu_B B_z}{\hbar}t\right)$$

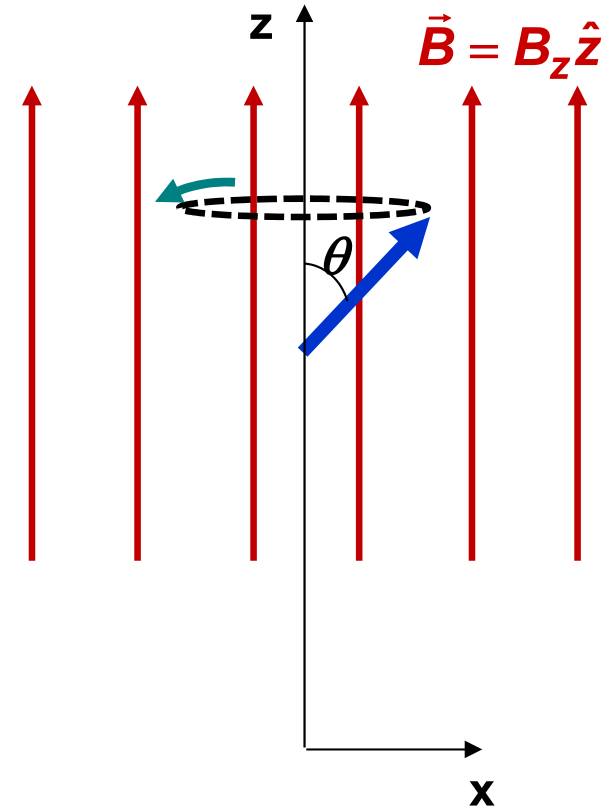
$$\left\langle \hat{S}_z \right\rangle(t) = \left\langle \psi(t)\right|\hat{S}_z\left|\psi(t)\right\rangle = \frac{\hbar}{2}\cos\theta$$

Precession frequency = Larmor frequency = $\omega = \dfrac{2\mu_B B_z}{\hbar} = \left|\gamma B_z\right|$

This is as close to the classical Larmor precession of magnetic moments in magnetic fields as you can get in quantum physics with spin 1/2

# Spin 1/2 Qubit in an External Magnetic Field: TLS Dynamics

$$\left|\psi(t)\right\rangle = \cos(\theta/2)\, e^{-i\frac{\mu_B B_z}{\hbar}t}\left|z\uparrow\right\rangle + \sin(\theta/2)\, e^{+i\frac{\mu_B B_z}{\hbar}t}\left|z\downarrow\right\rangle$$




**Suppose a measurement is made at time *t* to determine the z-component of the spin angular momentum**

**What are the possible values?**
**What are the a-priori probabilities?**

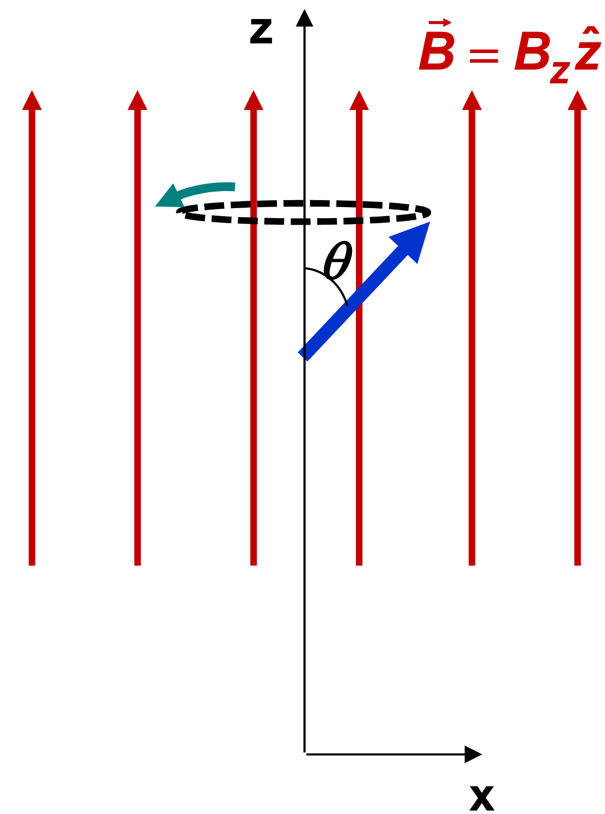
**The possible values are the eigenvalues of** $\hat{S}_z = \frac{\hbar}{2}\hat{\sigma}_z$ **which are** $+\hbar/2$ **and** $-\hbar/2$

**The a-priori probability of finding** $+\hbar/2$ **is:** $\left|\left\langle z\uparrow\middle|\psi(t)\right\rangle\right|^2 = \cos^2(\theta/2)$

**The a-priori probability of finding** $-\hbar/2$ **is:** $\left|\left\langle z\downarrow\middle|\psi(t)\right\rangle\right|^2 = \sin^2(\theta/2)$

## Spin 1/2 Qubit in an External Magnetic Field: TLS Dynamics

$$|\psi(t)\rangle = \cos(\theta/2)\, e^{-i\frac{\mu_B B_z}{\hbar}t}\,|z\uparrow\rangle + \sin(\theta/2)\, e^{+i\frac{\mu_B B_z}{\hbar}t}\,|z\downarrow\rangle$$




**Suppose a measurement is made at time *t* to determine the x-component of the spin angular momentum**

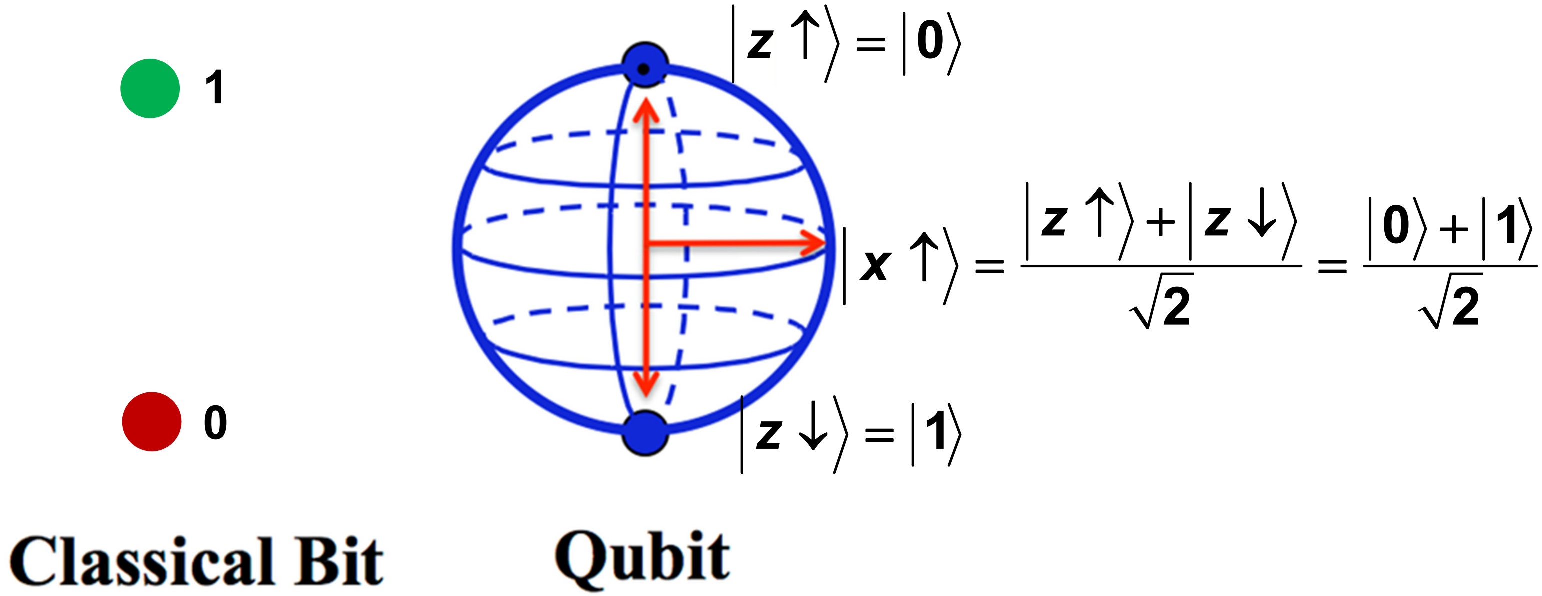
**What are the possible values?**
**What are the a-priori probabilities?**

**The possible values are the eigenvalues of** $\hat{S}_x = \frac{\hbar}{2}\hat{\sigma}_x$ **which are** $+\hbar/2$ **and** $-\hbar/2$

**The a-priori probability of finding** $+\hbar/2$ **is:** $\left|\langle x\uparrow|\psi(t)\rangle\right|^2 = \frac{1}{2}\left[1+\sin\theta\cos\left(\frac{2\mu_B B_z}{\hbar}t\right)\right]$

**The a-priori probability of finding** $-\hbar/2$ **is:** $\left|\langle x\downarrow|\psi(t)\rangle\right|^2 = \frac{1}{2}\left[1-\sin\theta\cos\left(\frac{2\mu_B B_z}{\hbar}t\right)\right]$

# The Classical Bit vs the Quantum Bit (Qubit)




**The classical bit (however realized physically) can deterministically take any one of two values, 0 or 1**

**The quantum bit (or qubit) can be realized by any physical thing whose quantum state resides in a Hilbert space of dimension two (like the Hilbert space of up spin state and down spin state of an electron).** ***The qubit, unlike the classical bit, can also be in a superposition of two states!***

# Visualizing the Quantum State of the Quantum Bit (Qubit)

Consider a qubit (not necessarily a spin qubit):

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$\langle\psi|\psi\rangle = 1 \quad \Rightarrow |\alpha|^2 + |\beta|^2 = 1$$

$\alpha$ and $\beta$ are complex numbers

We can write it using the spin notation:

$$|\psi\rangle = \alpha|z\uparrow\rangle + \beta|z\downarrow\rangle$$

**Writing a qubit using the spin notation gives a way to visualize the quantum state!**
**Lets see how that happens ………..**

Without losing generality we can assume:

$$\alpha = \cos(\theta/2)e^{-i\phi/2}$$
$$\beta = \sin(\theta/2)e^{i\phi/2}$$

$$\Longrightarrow \quad |\alpha|^2 + |\beta|^2 = 1$$

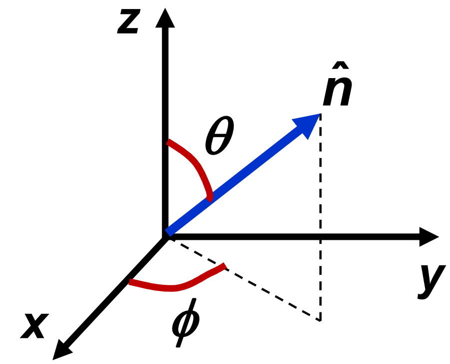
Then:

$$|\psi\rangle = \cos(\theta/2)e^{-i\phi/2}|z\uparrow\rangle + \sin(\theta/2)e^{i\phi/2}|z\downarrow\rangle$$

# Visualizing the Quantum State of the Quantum Bit (Qubit)

$$|\psi\rangle = \cos(\theta/2)e^{-i\phi/2}|z\uparrow\rangle + \sin(\theta/2)e^{i\phi/2}|z\downarrow\rangle = |\hat{n}\uparrow\rangle$$

The qubit state can now be visualized as a spin 1/2 state pointing in the direction of the unit vector $\hat{n}$ in 3D space:

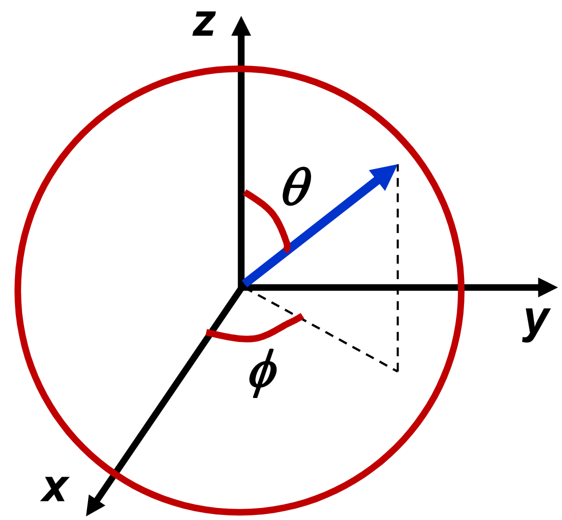



$$\hat{n} = \sin\theta\cos\phi e_x + \sin\theta\sin\phi e_y + \cos\theta e_z$$

Different possible qubit states are then seen as "spin states" pointing in different directions in 3D space

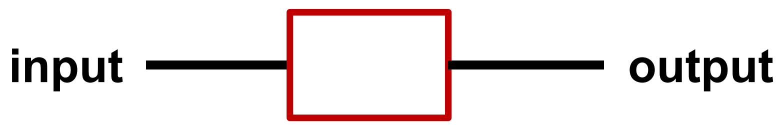
Different spin states are not orthogonal, and the dimension of the Hilbert space is still 2, but one can have an arbitrary superposition of the up and down spin states:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$



# Single-Bit Classical Logic Gates

**Classical single-bit logic gate takes one-bit input and produces a one-bit output**




**Only two single-bit logic gates are possible**

| Buffer | |
|---|---|
| Input | Output |
| 0 | 0 |
| 1 | 1 |

| NOT Gate | |
|---|---|
| Input | Output |
| 0 | 1 |
| 1 | 0 |

# Single-Qubit Quantum Gates

**Quantum single-qubit gate takes one-qubit input and produces a one-qubit output**

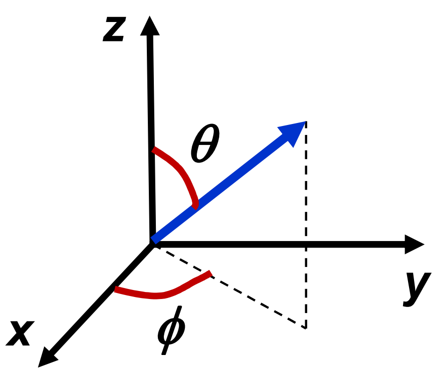
$$\left|\psi_{in}\right\rangle = \alpha\left|z\uparrow\right\rangle + \beta\left|z\downarrow\right\rangle = \alpha\left|0\right\rangle + \beta\left|1\right\rangle$$

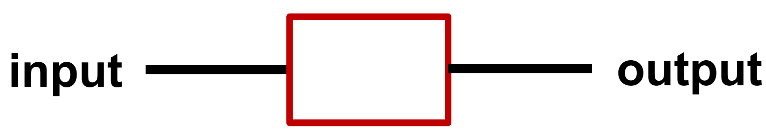

$$\left|\psi_{out}\right\rangle = \gamma\left|z\uparrow\right\rangle + \delta\left|z\downarrow\right\rangle = \gamma\left|0\right\rangle + \delta\left|1\right\rangle$$

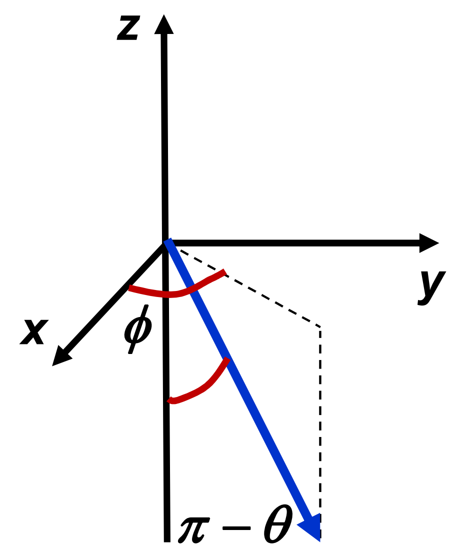
**Infinitely many single-qubit gates are possible**

**Example:**

**A gate that reflects a qubit along the horizontal plane**




**How does one physically realize these single-qubit quantum gates?**

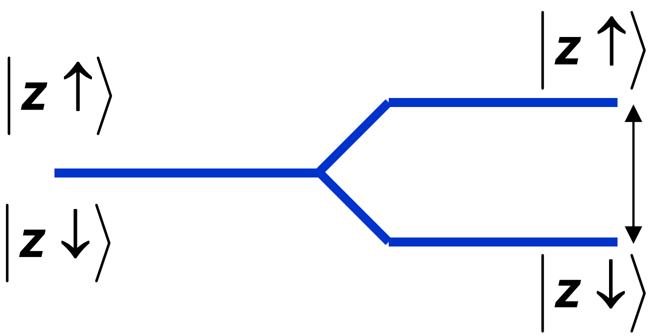
# Single Spin Qubit in a DC Magnetic Field: A Two Level System

Consider a spin 1/2 in a z-directed strong DC magnetic field:

$$\hat{H} = -\hat{\vec{m}} \,.\, \vec{B} = \mu_B B_z \hat{\sigma}_z = \mu_B B_z \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

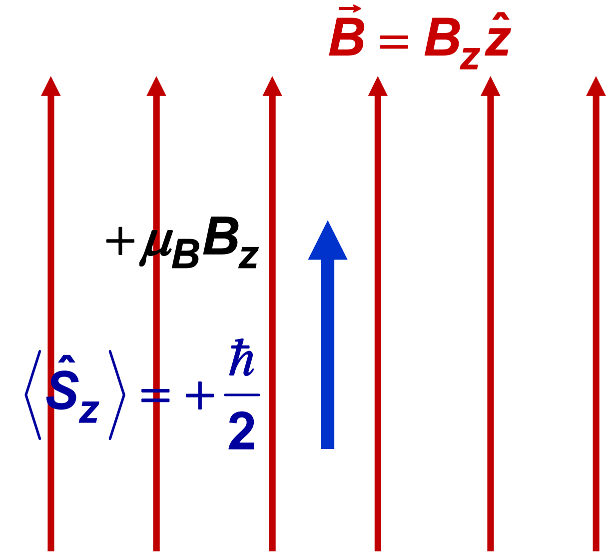
$$= \begin{bmatrix} \mu_B B_z & 0 \\ 0 & -\mu_B B_z \end{bmatrix}$$

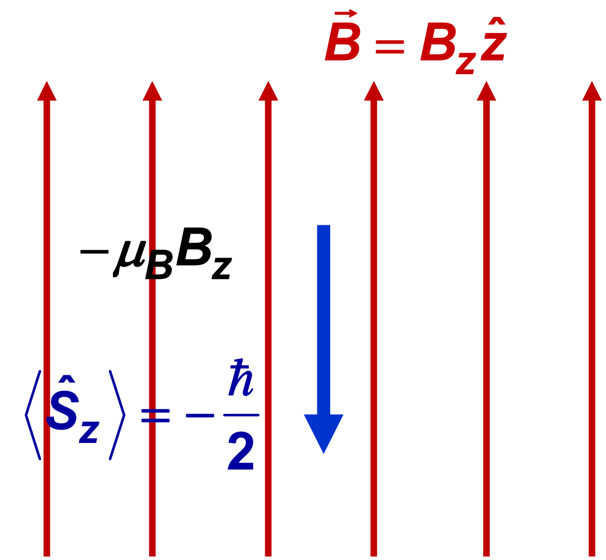
$$= \begin{bmatrix} \frac{\Delta}{2} & 0 \\ 0 & -\frac{\Delta}{2} \end{bmatrix} = \frac{\Delta}{2} \hat{\sigma}_z$$




$$\boxed{\Delta = 2\mu_B B_z}$$

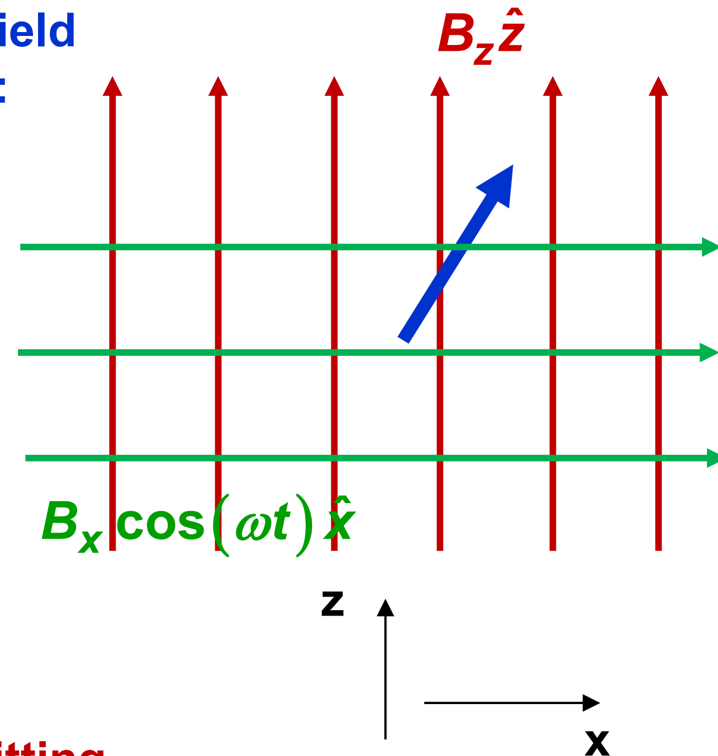
Zeeman splitting



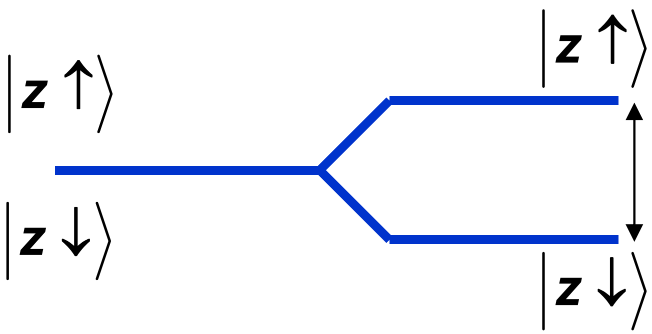
# Single Qubit Rotations: Time-Dependent Magnetic Field

Consider a spin 1/2 in a z-directed <u>strong</u> DC magnetic field and a x-directed <u>weak</u> AC magnetic field at frequency $\omega$ :




$$\vec{B}(t) = B_z e_z + B_x \cos(\omega t) e_x$$

Where: $\hbar\omega = \Delta = 2\mu_B B_z$




$$\boxed{\Delta = 2\mu_B B_z = \hbar\omega}$$

Zeeman splitting

The frequency of the AC field is tuned to the Zeeman splitting

The Hamiltonian (now time-dependent) becomes:

$$\left\{\begin{aligned} \hat{\vec{m}} &= -\mu_B\left[\hat{\sigma}_x e_x + \hat{\sigma}_y e_y + \hat{\sigma}_z e_z\right] \\ &= -\mu_B \hat{\vec{\sigma}} \end{aligned}\right.$$

$$\hat{H}(t) = -\hat{\vec{m}}\,.\,\vec{B} = \mu_B \hat{\vec{\sigma}}.\,\vec{B}(t) = \mu_B B_z \hat{\sigma}_z + \mu_B B_x \cos(\omega t)\hat{\sigma}_x$$

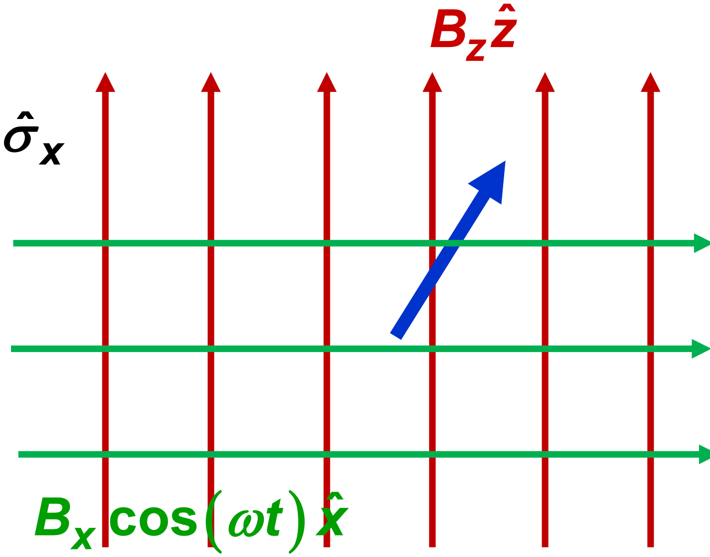
$$= \frac{\Delta}{2}\hat{\sigma}_z + \kappa\hat{\sigma}_x \cos(\omega t)$$

## Single Qubit Rotations: Time-Dependent Magnetic Field

$$\hat{H}(t) = -\hat{\vec{m}} \, . \, \vec{B} = \mu_B \hat{\vec{\sigma}} . \, \vec{B}(t) = \mu_B B_z \hat{\sigma}_z + \mu_B B_x \cos(\omega t)\, \hat{\sigma}_x$$

$$= \frac{\Delta}{2} \hat{\sigma}_z + \kappa \hat{\sigma}_x \cos(\omega t)$$

$$= \begin{bmatrix} \frac{\Delta}{2} & \kappa \cos(\omega t) \\ \kappa \cos(\omega t) & -\frac{\Delta}{2} \end{bmatrix}$$




**Suppose:**

$$|\psi(t=0)\rangle = |z \uparrow\rangle = |0\rangle = \text{qubit in state "0"}$$

**Question:**

$$|\psi(t)\rangle = ?$$

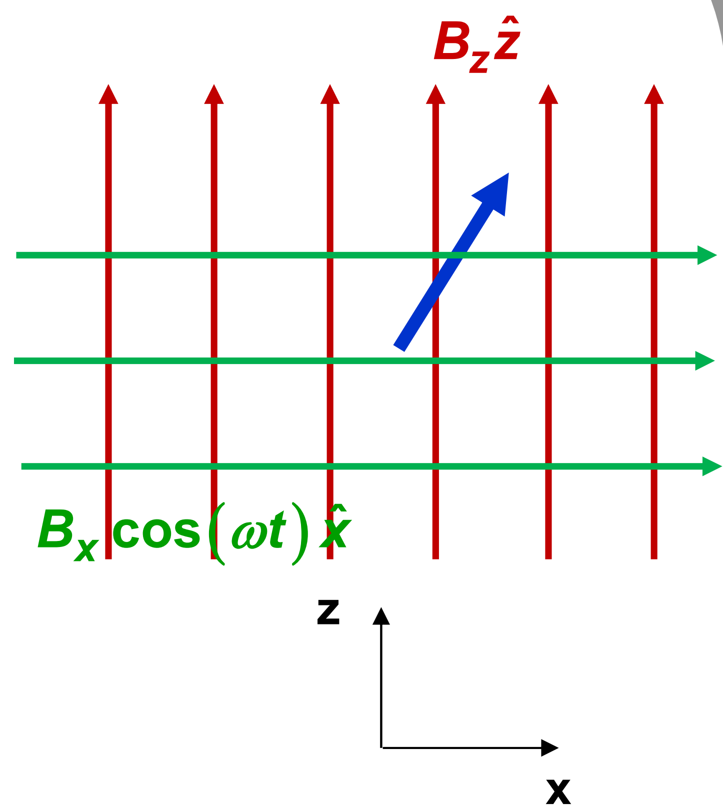
## Single Qubit Rotations: Time-Dependent Magnetic Field

**Suppose:**

$$\left|\psi(t=0)\right\rangle = \left|z\uparrow\right\rangle$$

**Then find:**

$$i\hbar\frac{\partial}{\partial t}\left|\psi(t)\right\rangle = \hat{H}(t)\left|\psi(t)\right\rangle$$




**Can we convert this time-dependent Hamiltonian to something that is time-independent?**

**Let:**

$$\left|\chi(t)\right\rangle = \begin{bmatrix} e^{+i\frac{\omega}{2}t} & 0 \\ 0 & e^{-i\frac{\omega}{2}t} \end{bmatrix}\left|\psi(t)\right\rangle = \hat{W}(t)\left|\psi(t)\right\rangle$$

$$\Rightarrow i\hbar\frac{\partial}{\partial t}\left|\chi(t)\right\rangle = \hat{W}(t)\,i\hbar\frac{\partial}{\partial t}\left|\psi(t)\right\rangle + i\hbar\frac{\partial\hat{W}(t)}{\partial t}\left|\psi(t)\right\rangle$$

$$\Rightarrow i\hbar\frac{\partial}{\partial t}\left|\chi(t)\right\rangle = \hat{W}(t)\hat{H}(t)\left|\psi(t)\right\rangle + i\hbar\frac{\partial\hat{W}(t)}{\partial t}\left|\psi(t)\right\rangle$$

$$\Rightarrow i\hbar\frac{\partial}{\partial t}\left|\chi(t)\right\rangle = \hat{W}(t)\hat{H}(t)\underbrace{\hat{W}^{\dagger}(t)\hat{W}(t)}\left|\psi(t)\right\rangle + i\hbar\frac{\partial\hat{W}(t)}{\partial t}\underbrace{\hat{W}^{\dagger}(t)\hat{W}(t)}\left|\psi(t)\right\rangle$$

# Single Qubit Rotations: Time-Dependent Magnetic Field

$$i\hbar\frac{\partial}{\partial t}\left|\chi(t)\right\rangle = \hat{W}(t)\hat{H}(t)\hat{W}^{\dagger}(t)\hat{W}(t)\left|\psi(t)\right\rangle + i\hbar\frac{\partial\hat{W}(t)}{\partial t}\hat{W}^{\dagger}(t)\hat{W}(t)\left|\psi(t)\right\rangle$$

$$\Rightarrow i\hbar\frac{\partial}{\partial t}\left|\chi(t)\right\rangle = \underbrace{\hat{W}(t)\hat{H}(t)\hat{W}^{\dagger}(t)}\left|\chi(t)\right\rangle + \underbrace{i\hbar\frac{\partial\hat{W}(t)}{\partial t}\hat{W}^{\dagger}(t)}\left|\chi(t)\right\rangle$$

$$\begin{bmatrix} e^{+i\frac{\omega}{2}t} & 0 \\ 0 & e^{-i\frac{\omega}{2}t} \end{bmatrix}\begin{bmatrix} \frac{\Delta}{2} & \kappa\cos(\omega t) \\ \kappa\cos(\omega t) & -\frac{\Delta}{2} \end{bmatrix}\begin{bmatrix} e^{-i\frac{\omega}{2}t} & 0 \\ 0 & e^{+i\frac{\omega}{2}t} \end{bmatrix} = \begin{bmatrix} \frac{\Delta}{2} & \kappa\cos(\omega t)e^{+i\omega t} \\ \kappa\cos(\omega t)e^{-i\omega t} & -\frac{\Delta}{2} \end{bmatrix}$$

$$i\hbar\frac{\partial\hat{W}(t)}{\partial t}\hat{W}^{\dagger}(t) = \begin{bmatrix} -\frac{\hbar\omega}{2}e^{+i\frac{\omega}{2}t} & 0 \\ 0 & +\frac{\hbar\omega}{2}e^{-i\frac{\omega}{2}t} \end{bmatrix}\begin{bmatrix} e^{-i\frac{\omega}{2}t} & 0 \\ 0 & e^{+i\frac{\omega}{2}t} \end{bmatrix} = \begin{bmatrix} -\frac{\hbar\omega}{2} & 0 \\ 0 & +\frac{\hbar\omega}{2} \end{bmatrix}$$

# Single Qubit Rotations: Time-Dependent Magnetic Field

$$\Rightarrow i\hbar\frac{\partial}{\partial t}\left|\chi(t)\right\rangle = \hat{W}(t)\hat{H}(t)\hat{W}^{\dagger}(t)\left|\chi(t)\right\rangle + i\hbar\frac{\partial\hat{W}(t)}{\partial t}\hat{W}^{\dagger}(t)\left|\chi(t)\right\rangle$$

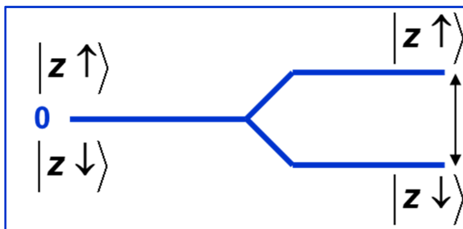
$$\Rightarrow i\hbar\frac{\partial}{\partial t}\left|\chi(t)\right\rangle \approx \begin{bmatrix} \frac{(\Delta-\hbar\omega)}{2} & \kappa\cos(\omega t)e^{+i\omega t} \\ \kappa\cos(\omega t)e^{-i\omega t} & -\frac{(\Delta-\hbar\omega)}{2} \end{bmatrix}\left|\chi(t)\right\rangle$$

Rotating wave approximation

$$\Rightarrow i\hbar\frac{\partial}{\partial t}\left|\chi(t)\right\rangle = \begin{bmatrix} \cancelto{0}{\frac{(\Delta-\hbar\omega)}{2}} & \frac{\kappa}{2} \\ \frac{\kappa}{2} & -\cancelto{0}{\frac{(\Delta-\hbar\omega)}{2}} \end{bmatrix}\left|\chi(t)\right\rangle$$

Zero-detuning

$$i\hbar\frac{\partial}{\partial t}\left|\chi(t)\right\rangle = \begin{bmatrix} 0 & \frac{\kappa}{2} \\ \frac{\kappa}{2} & 0 \end{bmatrix}\left|\chi(t)\right\rangle = \frac{\kappa}{2}\hat{\sigma}_x\left|\chi(t)\right\rangle$$




$\Delta = 2\mu_B B_z = \hbar\omega$

Zeeman splitting

The frequency of the AC field if tuned to the Zeeman splitting

# Single Qubit Rotations: Time-Dependent Magnetic Field

$$i\hbar\frac{\partial}{\partial t}\left|\chi(t)\right\rangle = \begin{bmatrix} 0 & \frac{\kappa}{2} \\ \frac{\kappa}{2} & 0 \end{bmatrix}\left|\chi(t)\right\rangle = \frac{\kappa}{2}\hat{\sigma}_x\left|\chi(t)\right\rangle \longrightarrow$$ Time-independent equation!!

**Boundary condition:**

$$\left|\chi(t=0)\right\rangle = \left|z\uparrow\right\rangle = \frac{1}{\sqrt{2}}\left[\left|x\uparrow\right\rangle + \left|x\downarrow\right\rangle\right]$$

$$\left\{\begin{aligned} \hat{\sigma}_x\left|x\uparrow\right\rangle &= +1\left|x\uparrow\right\rangle \\ \hat{\sigma}_x\left|x\downarrow\right\rangle &= -1\left|x\downarrow\right\rangle \end{aligned}\right.$$

**Solution of the above time-independent equation is:**

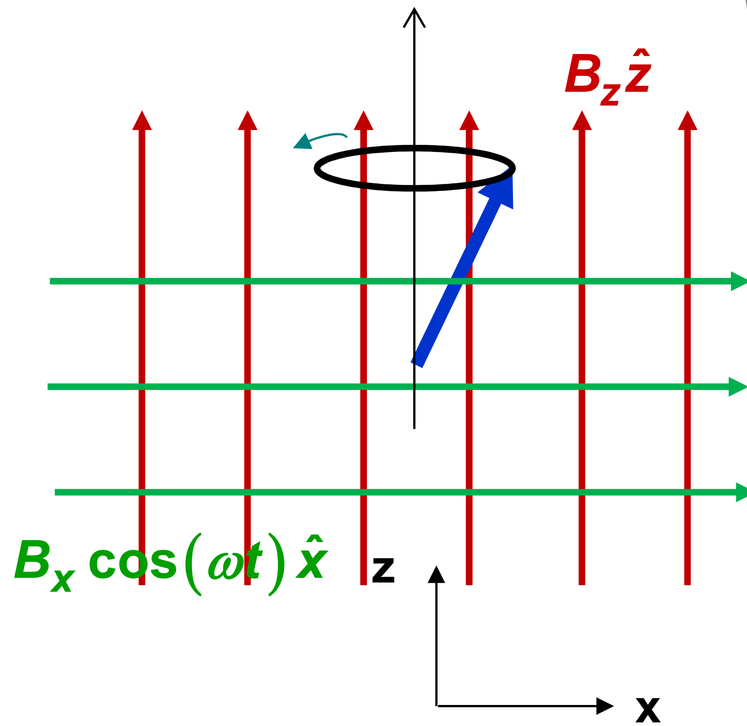
$$\left|\chi(t)\right\rangle = e^{-i\frac{\kappa}{2\hbar}\hat{\sigma}_x t}\left|\chi(t=0)\right\rangle = e^{-i\frac{\kappa}{2\hbar}\hat{\sigma}_x t}\left|z\uparrow\right\rangle = \frac{e^{-i\frac{\kappa}{2\hbar}\hat{\sigma}_x t}}{\sqrt{2}}\left[\left|x\uparrow\right\rangle + \left|x\downarrow\right\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[e^{-i\frac{\kappa}{2\hbar}t}\left|x\uparrow\right\rangle + e^{+i\frac{\kappa}{2\hbar}t}\left|x\downarrow\right\rangle\right] = \cos\left(\frac{\kappa}{2\hbar}t\right)\left|z\uparrow\right\rangle - i\sin\left(\frac{\kappa}{2\hbar}t\right)\left|z\downarrow\right\rangle$$

$$\left|\psi(t)\right\rangle = \hat{W}^{\dagger}(t)\left|\chi(t)\right\rangle = \begin{bmatrix} e^{-i\frac{\omega}{2}t} & 0 \\ 0 & e^{+i\frac{\omega}{2}t} \end{bmatrix}\left|\chi(t)\right\rangle = e^{-i\frac{\omega}{2}t}\cos\left(\frac{\kappa}{2\hbar}t\right)\left|z\uparrow\right\rangle - ie^{+i\frac{\omega}{2}t}\sin\left(\frac{\kappa}{2\hbar}t\right)\left|z\downarrow\right\rangle$$

# Single Qubit Rotations: Spin Rabi Oscillations

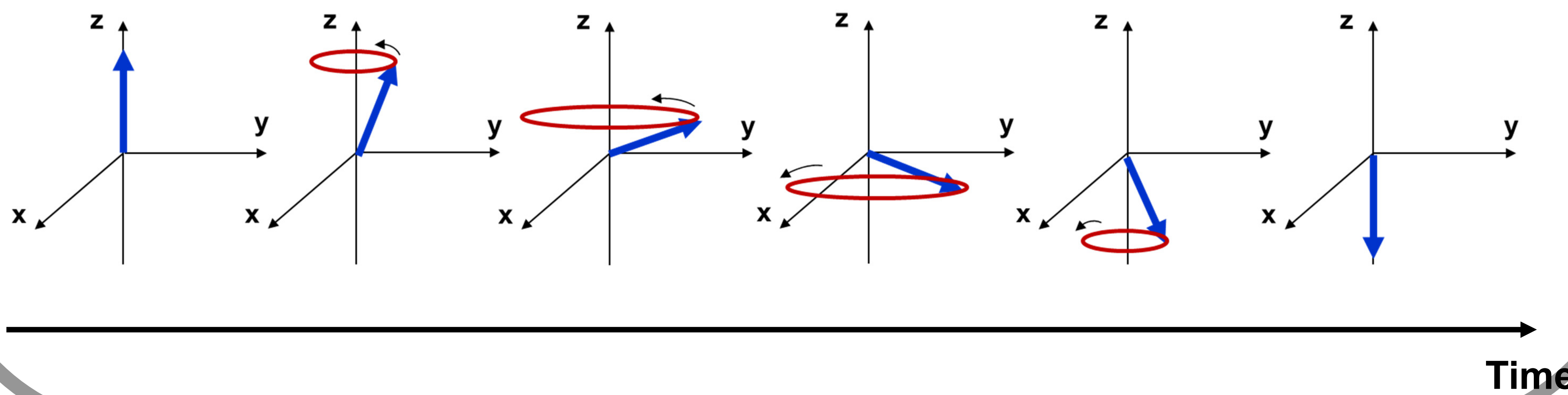
$$\left|\psi(t)\right\rangle = e^{-i\frac{\omega}{2}t}\cos\left(\frac{\kappa}{2\hbar}t\right)\left|z\uparrow\right\rangle - ie^{+i\frac{\omega}{2}t}\sin\left(\frac{\kappa}{2\hbar}t\right)\left|z\downarrow\right\rangle$$




$$\left\langle \hat{S}_x \right\rangle(t) = \left\langle\psi(t)\right|\hat{S}_x\left|\psi(t)\right\rangle = \frac{\hbar}{2}\sin\left(\frac{\kappa}{\hbar}t\right)\sin(\omega t)$$

$$\left\langle \hat{S}_y \right\rangle(t) = \left\langle\psi(t)\right|\hat{S}_y\left|\psi(t)\right\rangle = -\frac{\hbar}{2}\sin\left(\frac{\kappa}{\hbar}t\right)\cos(\omega t)$$

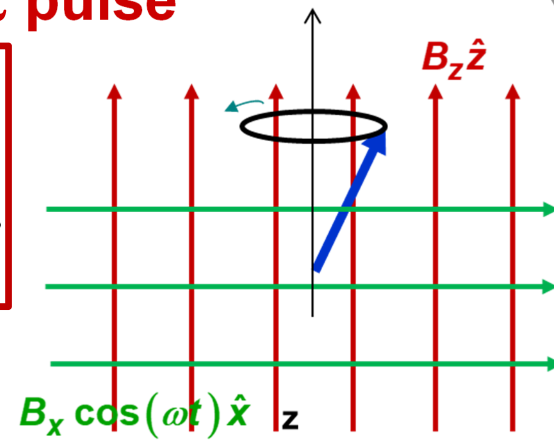
$$\left\langle \hat{S}_z \right\rangle(t) = \left\langle\psi(t)\right|\hat{S}_z\left|\psi(t)\right\rangle = \frac{\hbar}{2}\cos\left(\frac{\kappa}{\hbar}t\right)$$



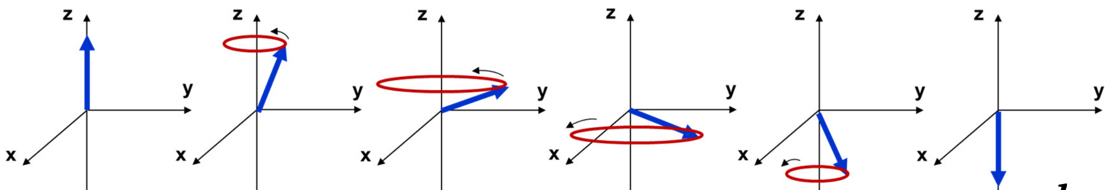
# Single Qubit Rotations: π/2 pulse and π pulse

$$\left|\psi(t=0)\right\rangle = \left|z\uparrow\right\rangle = \left|0\right\rangle$$

$$\left|\psi(t)\right\rangle = e^{-i\frac{\omega}{2}t}\cos\left(\frac{\kappa}{2\hbar}t\right)\left|z\uparrow\right\rangle - ie^{+i\frac{\omega}{2}t}\sin\left(\frac{\kappa}{2\hbar}t\right)\left|z\downarrow\right\rangle$$




If $\frac{\kappa t}{2\hbar} = \frac{\pi}{2}$ : $\left|\psi(t)\right\rangle \approx -ie^{+i\frac{\omega}{2}t}\left|z\downarrow\right\rangle = -ie^{+i\frac{\omega}{2}t}\left|1\right\rangle$

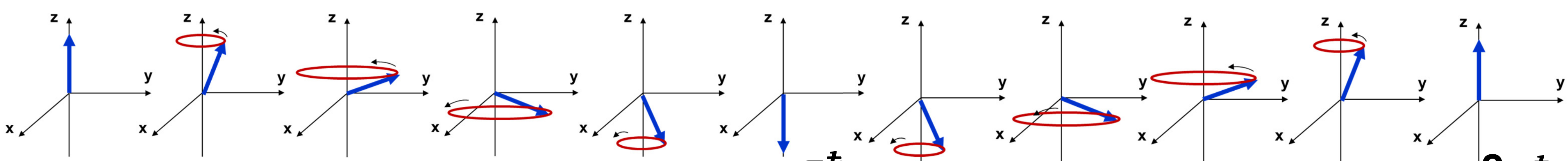


t=0

$t = \frac{\pi\hbar}{\kappa}$

Flip the qubit

If $\frac{\kappa t}{2\hbar} = \pi$ : $\left|\psi(t)\right\rangle \approx -e^{-i\frac{\omega}{2}t}\left|z\uparrow\right\rangle = -e^{-i\frac{\omega}{2}t}\left|0\right\rangle$

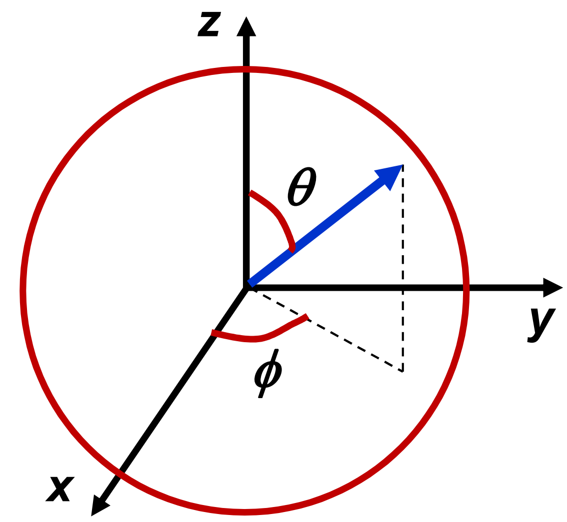


t=0

$t = \frac{\pi\hbar}{\kappa}$

$t = \frac{2\pi\hbar}{\kappa}$

# The Quantum Bit (Qubit): Notation




The quantum spin qubit can be written as:

$$|\psi\rangle = \alpha|\mathbf{z}\uparrow\rangle + \beta|\mathbf{z}\downarrow\rangle$$

Or one can slightly change the notation to match that of the classical bits:

$$|\psi\rangle = \alpha|\mathbf{0}\rangle + \beta|\mathbf{1}\rangle$$

But one should keep in mind that the dimension of the Hilbert space is 2 so any 2-dimensional representation would work.

In particular, we can use column vectors:

$$|\psi\rangle = \alpha\begin{bmatrix}\mathbf{1}\\ \mathbf{0}\end{bmatrix} + \beta\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix} = \begin{bmatrix}\alpha\\ \beta\end{bmatrix}$$

# Single-Qubit Quantum Gates

**Quantum single-qubit gate takes one-qubit input and produces a one-qubit output**

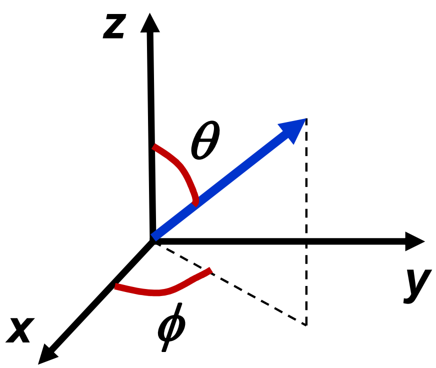
$$\left|\psi_{in}\right\rangle = \alpha\left|z\uparrow\right\rangle + \beta\left|z\downarrow\right\rangle$$
$$= \alpha\left|0\right\rangle + \beta\left|1\right\rangle$$

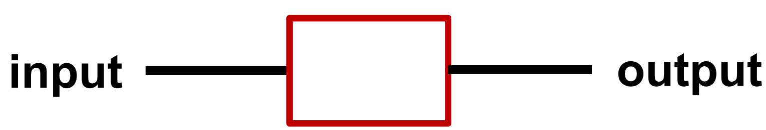

$$\left|\psi_{out}\right\rangle = \gamma\left|z\uparrow\right\rangle + \delta\left|z\downarrow\right\rangle$$
$$= \gamma\left|0\right\rangle + \delta\left|1\right\rangle$$

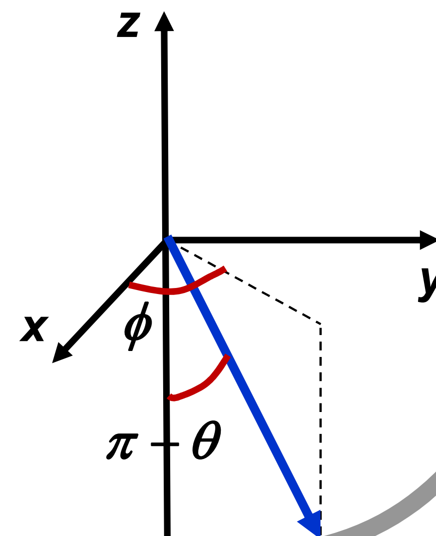
**Infinitely many single-qubit gates are possible corresponding to any rotation of the qubit**

**Example:**

**A gate that reflects a qubit along the horizontal plane**



# Single-Qubit Quantum Gates are Defined by Unitary Operators

**Quantum single-qubit gate takes one-qubit input and produces a one-qubit output by a unitary rotation**

$$\left|\psi_{in}\right\rangle = \left|\psi(t=0)\right\rangle = \alpha\left|z\uparrow\right\rangle + \beta\left|z\downarrow\right\rangle = \alpha\left|0\right\rangle + \beta\left|1\right\rangle$$

input —— $\boxed{\hat{U}}$ —— output

$$\left|\psi_{out}\right\rangle = \left|\psi(t=T)\right\rangle = \gamma\left|z\uparrow\right\rangle + \delta\left|z\downarrow\right\rangle = \gamma\left|0\right\rangle + \delta\left|1\right\rangle$$

$$\left|\psi_{out}\right\rangle = \hat{U}\left|\psi_{in}\right\rangle$$

**Unitary** $\Rightarrow \hat{U}^{\dagger}\hat{U} = \hat{1}$

**The action of the quantum gate on an input qubit is unitary:**

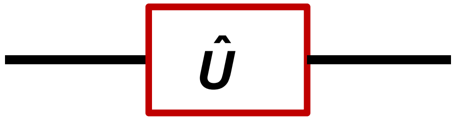
$$\left|\psi_{out}\right\rangle = \hat{U}\left|\psi_{in}\right\rangle = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}\left|\psi_{in}\right\rangle$$

**Any physical quantum gate can be represented by a unitary operator**
**Conversely, any unitary operator can be realized by a physical quantum gate**

# Single-Qubit Quantum Gates are Linear

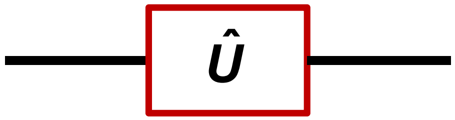
**Suppose we have a quantum gate that is defined by its actions on 0 and 1 qubits (or up and down qubits) separately as follows:**

$$|\psi_{in}\rangle = |0\rangle$$




$$|\psi_{out}\rangle = \hat{U}|\psi_{in}\rangle = \alpha|0\rangle + \beta|1\rangle$$

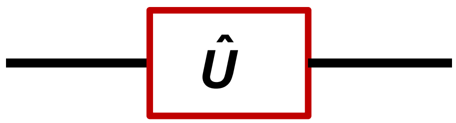
$$|\psi_{in}\rangle = |1\rangle$$




$$|\psi_{out}\rangle = \hat{U}|\psi_{in}\rangle = \gamma|0\rangle + \delta|1\rangle$$

**Then if the input becomes a superposition, the result is:**

$$|\psi_{in}\rangle = A|0\rangle + B|1\rangle$$



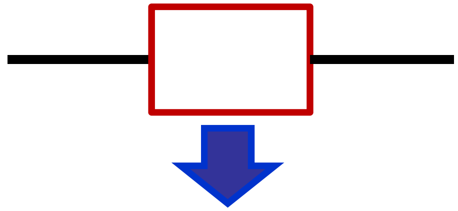

$$\begin{aligned} |\psi_{out}\rangle &= \hat{U}|\psi_{in}\rangle \\ &= \hat{U}\left[A|0\rangle + B|1\rangle\right] \\ &= A\hat{U}|0\rangle + B\hat{U}|1\rangle \\ &= A\left(\alpha|0\rangle + \beta|1\rangle\right) + B\left(\gamma|0\rangle + \delta|1\rangle\right) \\ &= \left(A\alpha + B\gamma\right)|0\rangle + \left(A\beta + B\delta\right)|1\rangle \end{aligned}$$

**Linearity**

**The linearity of the single qubit gates follows from the linearity of the quantum operators**

# Single-Qubit Quantum Gates: Pauli Matrices

**Quantum single-qubit gate takes one-qubit input and produces a one-qubit output**

$|\psi_{in}\rangle = \alpha|z\uparrow\rangle + \beta|z\downarrow\rangle$ **input**  **output** $|\psi_{out}\rangle = \gamma|z\uparrow\rangle + \delta|z\downarrow\rangle$

$$|\psi_{out}\rangle = \hat{U}|\psi_{in}\rangle$$

$$|\psi_{out}\rangle = \hat{U}|\psi_{in}\rangle = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}|\psi_{in}\rangle$$

**Note:**

- **Since the gate is an operator acting in a 2-dimensional Hilbert space, it can be represented as a 2x2 matrix**

- **Since any 2x2 matrix can be represented as linear combination of the 3 Pauli matrices and the identity matrix, one can write:**

$$\hat{U} = A\hat{\sigma}_x + B\hat{\sigma}_y + C\hat{\sigma}_z + D\hat{1}$$

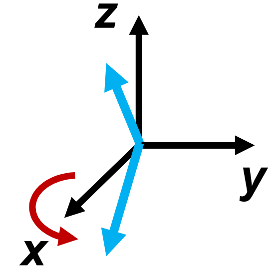
# Simple Single-Qubit Quantum Gates: X, Y, Z Gates

**The X-gate:**

$|\psi_{in}\rangle$ —[ X ]— $|\psi_{out}\rangle$

$$|\psi_{out}\rangle = \hat{U}|\psi_{in}\rangle = \hat{\sigma}_x|\psi_{in}\rangle$$

**Rotates the qubit by 180 degrees w.r.t the x-axis**

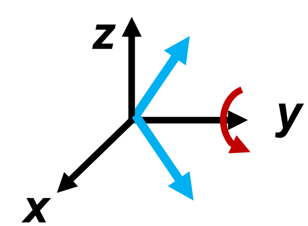



**The Y-gate:**

$|\psi_{in}\rangle$ —[ Y ]— $|\psi_{out}\rangle$

$$|\psi_{out}\rangle = \hat{U}|\psi_{in}\rangle = \hat{\sigma}_y|\psi_{in}\rangle$$

**Rotates the qubit by 180 degrees w.r.t the y-axis**

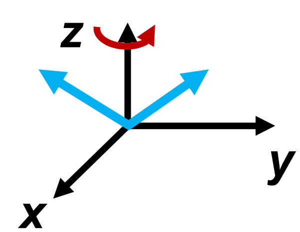



**The Z-gate:**

$|\psi_{in}\rangle$ —[ Z ]— $|\psi_{out}\rangle$

$$|\psi_{out}\rangle = \hat{U}|\psi_{in}\rangle = \hat{\sigma}_z|\psi_{in}\rangle$$

**Rotates the qubit by 180 degrees w.r.t the z-axis**



# Simple Single-Qubit Quantum Gates: The Hadamard Gate

**The H-gate:**

$$\left|\psi_{out}\right\rangle \; \text{——} \boxed{\;H\;} \text{——} \; \left|\psi_{in}\right\rangle$$

$$\left|\psi_{out}\right\rangle = \hat{U}\left|\psi_{in}\right\rangle = \frac{1}{\sqrt{2}}\left[\hat{\sigma}_z + \hat{\sigma}_x\right]\left|\psi_{in}\right\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\left|\psi_{in}\right\rangle$$

$$\left|\psi_{in}\right\rangle = \left|0\right\rangle = \left|z\uparrow\right\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\left|\psi_{out}\right\rangle = \frac{1}{\sqrt{2}}\left[\left|0\right\rangle + \left|1\right\rangle\right] = \frac{1}{\sqrt{2}}\left[\left|z\uparrow\right\rangle + \left|z\downarrow\right\rangle\right] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\left|\psi_{in}\right\rangle = \left|1\right\rangle = \left|z\downarrow\right\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\left|\psi_{out}\right\rangle = \frac{1}{\sqrt{2}}\left[\left|0\right\rangle - \left|1\right\rangle\right] = \frac{1}{\sqrt{2}}\left[\left|z\uparrow\right\rangle - \left|z\downarrow\right\rangle\right] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

# Simple Single-Qubit Quantum Gates: The S-Gate

**The S-gate:**

$$|\psi_{out}\rangle \;\text{———}\boxed{\;S\;}\text{———}\; |\psi_{in}\rangle$$

$$|\psi_{out}\rangle = \hat{U}|\psi_{in}\rangle = \left[\left(\frac{1-i}{2}\right)\hat{\sigma}_z + \left(\frac{1+i}{2}\right)\hat{1}\right]|\psi_{in}\rangle = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}|\psi_{in}\rangle$$

$$|\psi_{in}\rangle = |0\rangle = |z\uparrow\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|\psi_{out}\rangle = |0\rangle = |z\uparrow\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|\psi_{in}\rangle = |1\rangle = |z\downarrow\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$|\psi_{out}\rangle = i|1\rangle = i|z\downarrow\rangle = i\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

# Simple Single-Qubit Quantum Gates: The T-Gate

**The T-gate:**

$$|\psi_{out}\rangle \text{———} \boxed{T} \text{———} |\psi_{in}\rangle$$

$$|\psi_{out}\rangle = \hat{U}|\psi_{in}\rangle = \left[\left(\frac{1-e^{i\pi/4}}{2}\right)\hat{\sigma}_z + \left(\frac{1+e^{i\pi/4}}{2}\right)\hat{1}\right]|\psi_{in}\rangle = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}|\psi_{in}\rangle$$

$$|\psi_{in}\rangle = |0\rangle = |z\uparrow\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|\psi_{out}\rangle = |0\rangle = |z\uparrow\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|\psi_{in}\rangle = |1\rangle = |z\downarrow\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$|\psi_{out}\rangle = e^{i\pi/4}|1\rangle = e^{i\pi/4}|z\downarrow\rangle = e^{i\pi/4}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$