Lecture 13

The Ubiquitous Quantum Simple Harmonic Oscillator
(Superconducting Qubit, Light Quantization, and all that)

In this lecture you will learn:

• Particle in a quadratic potential
• Quantum simple harmonic oscillator
• Quantum superconducting qubit
• Quantum description of light and photons

Superconducting qubit
(Google, UCSB)
Classical SHO in 1D: A Spring-Mass System

\[ V(x) = \frac{1}{2} kx^2 \]

\[ H = \frac{p^2}{2m} + \frac{1}{2} kx^2 \]

Spring constant = \( k \)

Newton’s laws and classical equations:

\[ \frac{dp}{dt} = -\frac{\partial V}{\partial x} = -kx \]

\[ m\frac{dx}{dt} = p \]
Classical SHO in 1D: A Particle in a Quadratic Potential

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Newton’s Laws and Classical Equations:

\[ \frac{dp}{dt} = -\frac{\partial V}{\partial x} = -kx \]
\[ m \frac{dx}{dt} = p \]

Momentum and position are coupled and dynamics are described two coupled differential equations

Solutions:

\[ x(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t) \]
\[ p(t) = mB\omega_0 \cos(\omega_0 t) - m\omega_0 A \sin(\omega_0 t) \]

\[ \omega_0 = \sqrt{\frac{k}{m}} \]
A Particle in a Quadratic Potential: Change of Notation

\[ V(x) = \frac{1}{2} m \omega_0^2 x^2 \]

\[ H = \frac{p^2}{2m} + \frac{1}{2} m \omega_0^2 x^2 \]

Newton’s Laws and Classical Equations:

\[ \frac{dp}{dt} = - \frac{\partial V}{\partial x} = -m \omega_0^2 x \]

\[ m \frac{dx}{dt} = p \]

Momentum and position are coupled and dynamics are described two coupled differential equations.

Solutions:

\[ x(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t) \]

\[ p(t) = mB \omega_0 \cos(\omega_0 t) - m \omega_0 A \sin(\omega_0 t) \]

\[ \omega_0 = \sqrt{\frac{k}{m}} \]
Atoms in all Materials behave like Simple Harmonic Oscillators

All crystalline materials have atoms arranged in a periodic pattern

Atoms are held in place by electrostatic forces from neighbors

Every atom behaves like a simple harmonic oscillator that is coupled to its neighbors
Quantum MicroAcoustics

A new science of quantum sound

Acoustic microresonators are modeled as quantum simple harmonic oscillators

Konrad Lehnert
JILA
A Lossless Superconducting $LC$ Circuit

\[ H = \frac{1}{2} CV^2 + \frac{1}{2} LI^2 \]

Circuit equations:

\[ \frac{dV}{dt} = -\frac{I}{C} \]

\[ L \frac{dI}{dt} = V \]

\[ \omega_o = \sqrt{\frac{1}{LC}} \]
A Lossless Superconducting LC Circuit

\[ H = \frac{1}{2} CV^2 + \frac{1}{2} LI^2 \]

Switch circuit variables:

Define charge \( Q_s \) stored in the capacitor and the flux \( \lambda_s \) stored in the inductor as:

\[ Q = CV \]
\[ \lambda = LI \]

\[ \Rightarrow H = \frac{Q^2}{2C} + \frac{\lambda^2}{2L} \]
\[ = \frac{Q^2}{2C} + \frac{1}{2} C\omega_o^2 \lambda^2 \]

\[ \omega_o = \sqrt{\frac{1}{LC}} \]

Superconducting qubit (Google, UCSB)
A Lossless Superconducting LC Circuit

\[ H = \frac{Q^2}{2C} + \frac{\lambda^2}{2L} = \frac{Q^2}{2C} + \frac{1}{2} C \omega_0^2 \lambda^2 \]

Circuit equations:

\[ \frac{dQ}{dt} = -C \omega_0^2 \lambda \]

\[ C \frac{d\lambda}{dt} = Q \]

\[ \omega_0 = \sqrt{\frac{1}{LC}} \]

Compare with SHO:

\[ H = \frac{p^2}{2m} + \frac{1}{2} m \omega_0^2 x^2 \]

\[ \frac{dp}{dt} = -\frac{\partial V}{\partial x} = -m \omega_0^2 x \]

\[ m \frac{dx}{dt} = p \]

\[ Q \leftrightarrow p \]

\[ \lambda \leftrightarrow x \]

\[ C \leftrightarrow m \]

\[ \omega_0 \leftrightarrow \omega_0 \]
Modes in a Photonic Cavity

Different modes of a photonic cavity

Wave equation:

\[ \nabla \times \nabla \times \tilde{E}(\vec{r}, t) = \frac{1}{c^2} \frac{\partial^2 \tilde{E}(\vec{r}, t)}{\partial t^2} \]
\[ \nabla \cdot \tilde{E}(\vec{r}, t) = 0 \]

Let:

\[ \tilde{E}(\vec{r}, t) = A \tilde{U}_n(\vec{r}) e^{-i\omega_n t} \]

We get the following eigenvalue equation for the mode spatial profile:

\[ \nabla \times \nabla \times \tilde{U}_n(\vec{r}) = \frac{\omega_n^2}{c^2} \tilde{U}_n(\vec{r}) \]
\[ \nabla \cdot \tilde{U}_n(\vec{r}) = 0 \]

Mode orthogonality and normalization:

\[ \int d^3\vec{r} \ \tilde{U}_m(\vec{r}) \cdot \tilde{U}_n(\vec{r}) = \delta_{n,m} \]
An Electromagnetic Mode in a Photonic Cavity

Consider one single mode of frequency $\omega_0$ a photonic cavity:

$$\nabla \times \nabla \times \tilde{U}(\vec{r}) = \frac{\omega_0^2}{c^2} \tilde{U}(\vec{r})$$

Normalization:

$$\int d^3\vec{r} \tilde{U}(\vec{r}).\tilde{U}(\vec{r}) = 1$$

$$H = \int d^3\vec{r} \left[ \frac{1}{2} \varepsilon_o \tilde{E}(\vec{r},t).\tilde{E}(\vec{r},t) + \frac{1}{2} \mu_o \tilde{H}(\vec{r},t).\tilde{H}(\vec{r},t) \right]$$

$$= \frac{q_E^2(t)}{2\mu_o} + \frac{1}{2} \mu_o \omega_o^2 q_H^2(t)$$

Faraday’s law: $$\nabla \times \tilde{E}(\vec{r},t) = -\frac{\partial}{\partial t} \mu_o \tilde{H}(\vec{r},t)$$

$$\mu_o \frac{dq_H(t)}{dt} = q_E(t)$$

Ampere’s law: $$\nabla \times \tilde{H}(\vec{r},t) = \frac{\partial}{\partial t} \varepsilon_o \tilde{E}(\vec{r},t)$$

$$\frac{dq_E(t)}{dt} = -\mu_o \omega_o^2 q_H(t)$$
An Electromagnetic Mode in a Photonic Cavity

\[ \vec{E}(\vec{r}, t) = \frac{q_E(t)}{\sqrt{\mu_0 \varepsilon_0}} \vec{U}(\vec{r}) \]

\[ \vec{H}(\vec{r}, t) = -\frac{q_H(t)}{\sqrt{\mu_0 \varepsilon_0}} \nabla \times \vec{U}(\vec{r}) \]

Compare with SHO:

\[ H = \frac{p^2}{2m} + \frac{1}{2} m \omega_0^2 x^2 \]

\[ \frac{dp}{dt} = -\frac{\partial V}{\partial x} = -m \omega_0^2 x \]

\[ m \frac{dx}{dt} = p \]

\[ q_E \leftrightarrow p \]

\[ q_H \leftrightarrow x \]

\[ \mu_0 \leftrightarrow m \]

\[ \omega_0 \leftrightarrow \omega_0 \]
Quantum SHO: A Particle in a Quadratic Potential

The Hamiltonian is:

\[ \hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega_0^2 \hat{x}^2 \]

We assume:

\[ \psi(x,t) = \phi(x) e^{-\frac{iEt}{\hbar}} \]

And get:

\[ -\frac{\hbar^2}{2m} \frac{\partial^2 \phi(x)}{\partial x^2} + \frac{1}{2} m \omega_0^2 x^2 \phi(x) = E \phi(x) \]

How do we solve it??
Quantum SHO: A Particle in a Quadratic Potential

\[ V(x) = \frac{1}{2} m\omega^2 x^2 \]

Factor the operator on the LHS

Try a solution that satisfies:

\[ \left[ \sqrt{\frac{\hbar}{2m\omega}} \frac{\partial}{\partial x} + \sqrt{\frac{m\omega}{2\hbar}} x \right] \phi(x) = 0 \]

Solution is:

\[ \phi(x) = Ae^{-\frac{m\omega x^2}{2\hbar}} \]

Need to normalize

If we find one, its energy will be:

\[ E = \frac{1}{2} \hbar \omega \]
Quantum SHO: A Particle in a Quadratic Potential

\[ V(x) = \frac{1}{2} m\omega_0^2 x^2 \]

One solution is:

\[ \phi_0(x) = \left( \frac{m\omega_0}{\pi\hbar} \right)^{\frac{1}{4}} e^{-\frac{m\omega_0}{2\hbar} x^2} \]

Is this the lowest energy solution?
What are the other solutions?
New Operators

\[ V(x) = \frac{1}{2}m\omega_0^2x^2 \]

\[ -\frac{\hbar^2}{2m}\frac{\partial^2\phi(x)}{\partial x^2} + \frac{1}{2}m\omega_0^2x^2\phi(x) = E\phi(x) \]

\[ \Rightarrow \hbar\omega_0 \left[ -\sqrt{\frac{\hbar}{2m\omega_0}} \frac{\partial}{\partial x} + \sqrt{\frac{m\omega_0}{2\hbar}} x \right] \left[ \sqrt{\frac{\hbar}{2m\omega_0}} \frac{\partial}{\partial x} + \sqrt{\frac{m\omega_0}{2\hbar}} x \right] \phi(x) + \frac{1}{2}\hbar\omega_0\phi(x) = E\phi(x) \]

Define:

\[ \hat{a} = i \sqrt{\frac{1}{2m\hbar\omega_0}} \hat{p} + \sqrt{\frac{m\omega_0}{2\hbar}} \hat{x} \]

\[ \hat{a}^\dagger = -i \sqrt{\frac{1}{2m\hbar\omega_0}} \hat{p} + \sqrt{\frac{m\omega_0}{2\hbar}} \hat{x} \]

Commutation relation

\[ [\hat{a}, \hat{a}^\dagger] = 1 \]

With a little abuse of notation
Hamiltonian in Terms of the New Operators

\[ V(x) = \frac{1}{2} m \omega_o^2 x^2 \]

\[ \hat{a} = i \sqrt{\frac{1}{2m\hbar \omega_o}} \hat{p} + \sqrt{\frac{m \omega_o}{2\hbar}} \hat{x} \]

\[ \hat{a}^\dagger = -i \sqrt{\frac{1}{2m\hbar \omega_o}} \hat{p} + \sqrt{\frac{m \omega_o}{2\hbar}} \hat{x} \]

\[ \hat{\hat{x}} = \sqrt{\frac{\hbar}{2m \omega_o}} (\hat{a} + \hat{a}^\dagger) \]

\[ \hat{\hat{p}} = \sqrt{\frac{m \hbar \omega_o}{2}} \left( \frac{\hat{a} - \hat{a}^\dagger}{i} \right) \]

Substitute the above in the Hamiltonian operator to get:

\[ \hat{H} = \frac{\hat{\hat{p}}^2}{2m} + \frac{1}{2} m \omega_o^2 \hat{x}^2 = \frac{\hbar \omega_o}{2} (\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger) = \hbar \omega_o \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \]

\[ [\hat{\hat{a}}, \hat{\hat{a}}^\dagger] = 1 \]
The energy eigenstates are given by:

\[ \hat{H} |\phi\rangle = E |\phi\rangle \]

\Rightarrow \hbar \omega_0 \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right) |\phi\rangle = E |\phi\rangle

Consider the operator (called the “number operator”):

\[ \hat{n} = \hat{a}^\dagger \hat{a} \]

If we can find eigenstates and eigenvalues of the this operator, then we have the eigenstates and eigenvalues of the Hamiltonian.
Creation and Destruction Operators

\[ \hat{n} = \hat{a} \dagger \hat{a} \quad \left[ \hat{a}, \hat{a} \dagger \right] = 1 \]

1) The operator \( \hat{n} = \hat{a} \dagger \hat{a} \) can only have non-negative eigenvalues

Suppose \( |v\rangle \) is an eigenstate of \( \hat{n} = \hat{a} \dagger \hat{a} \) and \( \lambda \) is the corresponding eigenvalue. Now consider the norm of the state: \( |u\rangle = \hat{a} |v\rangle \)

\[ \langle u | u \rangle \geq 0 \]
\[ \Rightarrow \langle v | \hat{a} \dagger \hat{a} | v \rangle \geq 0 \]
\[ \Rightarrow \langle v | \hat{n} | v \rangle \geq 0 \]
\[ \Rightarrow \lambda \langle v | v \rangle \geq 0 \]
\[ \Rightarrow \lambda \geq 0 \]

2) If \( |v\rangle \) is an eigenstate of \( \hat{n} \) with eigenvalue \( \lambda \), then \( \hat{a} |v\rangle \) is also an eigenstate with eigenvalue \( \lambda - 1 \)

\[ \hat{n} (\hat{a} |v\rangle) = \hat{a} \dagger \hat{a} \hat{a} |v\rangle = (\hat{a} \dagger \hat{a}) \hat{a} |v\rangle = (\hat{a} \hat{a} \dagger - 1) \hat{a} |v\rangle = \hat{a}(\hat{n} - 1) |v\rangle = (\lambda - 1) \hat{a} |v\rangle \]

\( \hat{a} \) is called the “destruction” operator
Creation and Destruction Operators

\[ \hat{n} = \hat{a}^\dagger \hat{a} \quad [\hat{a},\hat{a}^\dagger] = 1 \]

3) If \( |\psi\rangle \) is an eigenstate of \( \hat{n} \) with eigenvalue \( \lambda \) then \( \hat{a}^\dagger |\psi\rangle \) is also an eigenstate with eigenvalue \( \lambda + 1 \)

\[ \hat{n} (\hat{a}^\dagger |\psi\rangle) = \hat{a}^\dagger \hat{a} \hat{a}^\dagger |\psi\rangle = \hat{a}^\dagger (\hat{a} \hat{a}^\dagger) |\psi\rangle = \hat{a}^\dagger (\hat{a}^\dagger \hat{a} + 1) |\psi\rangle = \hat{a}^\dagger (\hat{n} + 1) |\psi\rangle = (\lambda + 1) \hat{a}^\dagger |\psi\rangle \]

\( \hat{a}^\dagger \) is called the “creation” operator
Properties of the Number Operator

4) The smallest eigenvalue of \( \hat{n} \) is 0 and all eigenvalues of \( \hat{n} \) are integers

\[
\hat{n} |\psi\rangle = \lambda |\psi\rangle
\]

\[
\Rightarrow \hat{n}(\hat{a} |\psi\rangle) = (\lambda - 1)(\hat{a} |\psi\rangle)
\]

\[
\Rightarrow \hat{n}(\hat{a}^2 |\psi\rangle) = (\lambda - 2)(\hat{a}^2 |\psi\rangle)
\]

\[
\Rightarrow \hat{n}(\hat{a}^3 |\psi\rangle) = (\lambda - 3)(\hat{a}^3 |\psi\rangle)
\]

\[
\Rightarrow \hat{n}(\hat{a}^{p-1} |\psi\rangle) = (\lambda - p + 1)(\hat{a}^{p-1} |\psi\rangle)
\]

\[
\Rightarrow \hat{n}(\hat{a}^p |\psi\rangle) = (\lambda - p)(\hat{a}^p |\psi\rangle)
\]

For some integer \( p \) we will have:

\[
(\lambda - p) < 0 \quad \text{Not allowed!!}
\]

So it must be that acting with \( \hat{a} \) on the state \( \hat{a}^{p-1} |\psi\rangle \) must not give another state in the Hilbert space but give a zero instead:

\[
\hat{a}(\hat{a}^{p-1} |\psi\rangle) = 0 \quad \Rightarrow \hat{a}^{\dagger}\hat{a}(\hat{a}^{p-1} |\psi\rangle) = 0
\]

\[
\Rightarrow \hat{n}(\hat{a}^{p-1} |\psi\rangle) = 0
\]

That can only happen if \( (\lambda - p + 1) = 0 \) for some \( p \)

That means \( \lambda \) was an integer and all eigenvalues of \( \hat{n} \) are integers!!!
Properties of the Number Operator

\[ \hat{n} = \hat{a}^{\dagger} \hat{a} \quad \left[ \hat{a}, \hat{a}^{\dagger} \right] = 1 \]

5) The eigenstates of \( \hat{n} \) are written as \( |n\rangle \) and are labeled by their eigenvalue \( n \)

\[ \hat{n} |n\rangle = n |n\rangle \quad \{ n = 0, 1, 2, 3, ....... \} \]

6) Since \( \hat{n} \) is Hermitian, its eigenstates are orthonormal and form a complete set:

\[ \langle n | m \rangle = \delta_{nm} \quad \sum_{n=0}^{\infty} |n \rangle \langle n | = \hat{1} \quad \{ n = 0, 1, 2, 3, ....... \} \]

7) The smallest eigenvalue of \( \hat{n} \) is 0 and the corresponding eigenstate is \( |0\rangle \)

\[ \hat{a} |0\rangle = 0 \]

\[ \Rightarrow \hat{n} |0\rangle = 0 \]

8) From properties (2) and (3):

\[ \hat{n} \left( \hat{a} |n\rangle \right) = (n - 1) \left( \hat{a} |n\rangle \right) \]

\[ \hat{n} \left( \hat{a}^{\dagger} |n\rangle \right) = (n + 1) \left( \hat{a}^{\dagger} |n\rangle \right) \]
Properties of the Number Operator

\[ \hat{n} = \hat{a}^\dagger \hat{a} \quad \left[ \hat{a}, \hat{a}^\dagger \right] = 1 \]

8) Given: \( \hat{n} \ket{n} = n \ket{n} \) we know that: \( \hat{a}^\dagger \ket{n} \propto \ket{n + 1} \)

Let: \( \hat{a}^\dagger \ket{n} = A \ket{n + 1} \)

\[ \Rightarrow \bra{n} \hat{a} \hat{a}^\dagger \ket{n} = A^2 \bra{n + 1} \ket{n + 1} = A^2 \]

\[ \Rightarrow A^2 = \bra{n} \hat{a} \hat{a}^\dagger \ket{n} = \bra{n} \hat{a}^\dagger \hat{a} + 1 \ket{n} = n + 1 \]

\[ \Rightarrow A = \sqrt{n + 1} \]

\[ \hat{a}^\dagger \ket{n} = \sqrt{n + 1} \ket{n + 1} \]

9) Given: \( \hat{n} \ket{n} = n \ket{n} \) we know that: \( \hat{a} \ket{n} \propto \ket{n - 1} \)

Let: \( \hat{a} \ket{n} = A \ket{n - 1} \)

\[ \Rightarrow \bra{n} \hat{a}^\dagger \hat{a} \ket{n} = A^2 \bra{n - 1} \ket{n - 1} = A^2 \]

\[ \Rightarrow A^2 = \bra{n} \hat{a}^\dagger \hat{a} \ket{n} = n \]

\[ \Rightarrow A = \sqrt{n} \]

\[ \hat{a} \ket{n} = \sqrt{n} \ket{n - 1} \quad \hat{a} \ket{0} = 0 \]
Properties of the Number Operator

\[ \hat{n} = \hat{a}^\dagger \hat{a} \quad \left[\hat{a}, \hat{a}^\dagger\right] = 1 \]

10) All eigenstates of \( \hat{n} \) can be written as:

\[ |0\rangle \]

\[ \Rightarrow |1\rangle = \hat{a}^\dagger |0\rangle \]

\[ \Rightarrow |2\rangle = \frac{\left(\hat{a}^\dagger\right)^2}{\sqrt{2!}} |0\rangle \]

\[ \Rightarrow |3\rangle = \frac{\left(\hat{a}^\dagger\right)^3}{\sqrt{3!}} |0\rangle \]

\[ \vdots \]

\[ \Rightarrow |n\rangle = \frac{\left(\hat{a}^\dagger\right)^n}{\sqrt{n!}} |0\rangle \]
Quantum SHO: Summary of Results

\[ \hat{n} = \hat{a}^\dagger \hat{a} \quad \left[ \hat{a}, \hat{a}^\dagger \right] = 1 \]

\[
\hat{a} = i \sqrt{\frac{1}{2m\hbar \omega_0}} \hat{p} + \sqrt{\frac{m\omega_0}{2\hbar}} \; \hat{x}
\]

\[
\hat{a}^\dagger = -i \sqrt{\frac{1}{2m\hbar \omega_0}} \hat{p} + \sqrt{\frac{m\omega_0}{2\hbar}} \; \hat{x}
\]

\[
\hat{x} = \sqrt{\frac{\hbar}{2m\omega_0}} \left( \hat{a} + \hat{a}^\dagger \right)
\]

\[
\hat{p} = \sqrt{\frac{m\hbar \omega_0}{2}} \left( \frac{\hat{a} - \hat{a}^\dagger}{i} \right)
\]

Eigenstates and Eigenvalues:

\[ \hat{n} |n\rangle = n |n\rangle \quad \{ n = 0, 1, 2, 3, \ldots \} \]

\[ \langle n | m \rangle = \delta_{nm} \quad \sum_{n=0}^{\infty} |n\rangle \langle n| = \hat{1} \quad \{ n = 0, 1, 2, 3, \ldots \} \]

Actions of Creation and Destruction Operators:

\[ \hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle \]

\[ \hat{a} |n\rangle = \sqrt{n} |n-1\rangle \quad \rightarrow \quad \hat{a} |0\rangle = 0 \]
Quantum SHO: Hamiltonian and Energy Eigenstates

\[ E_3 = \left(3 + \frac{1}{2}\right)\hbar\omega_o \]
\[ E_2 = \left(2 + \frac{1}{2}\right)\hbar\omega_o \]
\[ E_1 = \left(1 + \frac{1}{2}\right)\hbar\omega_o \]
\[ E_0 = \frac{1}{2}\hbar\omega_o \]

The eigenstates \( |n\rangle \) of the number operator \( \hat{n} \) are also the eigenstates of the Hamiltonian:

\[ \hat{H} = \hbar\omega_o \left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) = \hbar\omega_o \left(\hat{n} + \frac{1}{2}\right) \]

\[ \hat{H} |n\rangle = \hbar\omega_o \left(\hat{n} + \frac{1}{2}\right) |n\rangle = \hbar\omega_o \left(n + \frac{1}{2}\right) |n\rangle \quad \{n = 0, 1, 2, 3 \ldots\} \]

Therefore, the eigenvalues of the Hamiltonian are:

\[ \frac{1}{2}\hbar\omega_o, \left(1 + \frac{1}{2}\right)\hbar\omega_o, \left(2 + \frac{1}{2}\right)\hbar\omega_o, \left(3 + \frac{1}{2}\right)\hbar\omega_o \ldots \]
The lowest energy eigenstate satisfies:

\[ \hat{a} |0\rangle = 0 \]

This means:

\[ \langle x | \hat{a} |0\rangle = 0 \]

\[ \langle x | i \sqrt{\frac{1}{2m\hbar\omega_0}} \hat{p} + \sqrt{\frac{m\omega_0}{2\hbar}} \hat{x} |0\rangle = 0 \]

\[ \Rightarrow \left[ \sqrt{\frac{\hbar}{2m\omega_0}} \frac{\partial}{\partial x} + \sqrt{\frac{m\omega_0}{2\hbar}} x \right] \phi_0(x) = 0 \]

\[ \Rightarrow \phi_0(x) = \left( \frac{m\omega_o}{\pi\hbar} \right)^{1/4} e^{-\frac{m\omega_o}{2\hbar} x^2} \]

\[ \int dx |\phi_0(x)|^2 = 1 \]
Quantum SHO: Wavefunctions

The $n$-th eigenstate is:

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle$$

$$\Rightarrow \phi_n(x) = \langle x | n \rangle = \frac{1}{\sqrt{n!}} \langle x | (\hat{a}^\dagger)^n |0\rangle = \frac{1}{\sqrt{n!}} \langle x | -i \sqrt{\frac{1}{2m\hbar\omega_o}} \hat{p} + \sqrt{\frac{m\omega_o}{2\hbar}} \hat{x} \rangle^n |0\rangle$$

$$\Rightarrow \phi_n(x) = \frac{1}{\sqrt{n!}} \left[-\sqrt{\frac{\hbar}{2m\omega_o}} \frac{\partial}{\partial x} + \sqrt{\frac{m\omega_o}{2\hbar}} x\right]^n \phi_0(x)$$

$$\Rightarrow \phi_n(x) = \left(\frac{m\omega_o}{\pi\hbar}\right)^4 e^{-\frac{m\omega_o}{2\hbar}x^2} \left[\frac{1}{\sqrt{2^n n!}} H_n \left(\sqrt{\frac{m\omega_o}{\hbar}} x\right)\right]$$

Hermite-Gaussians

Hermite polynomials
Quantum SHO: Wavefunctions

\[ \phi_n(x) = \phi_0(x) \left[ \frac{1}{\sqrt{2^n n!}} H_n \left( \sqrt{\frac{m\omega_o}{\hbar}} x \right) \right] \]

Hermite-Gaussians

Note: Wavefunctions have even or odd parity
Quantum SHO: Wavefunctions

Wavefunction in position basis:

\[ \langle x | 0 \rangle = \phi_0 (x) = \left( \frac{m\omega_0}{\pi\hbar} \right)^{\frac{1}{4}} e^{-\frac{m\omega_0 x^2}{2\hbar}} \]

\[ \Rightarrow \langle \Delta x^2 \rangle = \langle 0 | \Delta x^2 | 0 \rangle = \frac{\hbar}{2m\omega_0} \]

Wavefunction in momentum basis:

\[ \langle p | 0 \rangle = \phi_0 (p) = \int_{-\infty}^{\infty} dx \phi_0 (x) e^{-i\frac{p}{\hbar}x} = \sqrt{2\pi} \left( \frac{1}{\pi m\hbar\omega_0} \right)^{\frac{1}{4}} e^{-\frac{1}{2m\hbar\omega_0}p^2} \]

\[ \Rightarrow \langle \Delta p^2 \rangle = \langle 0 | \Delta p^2 | 0 \rangle = \frac{m\hbar\omega_0}{2} \]

Position-momentum uncertainty product:

\[ \Rightarrow \langle \Delta x^2 \rangle \langle \Delta p^2 \rangle = \frac{\hbar^2}{4} \]

Min value allowed by the Heisenberg relation

\[ \int \frac{dp}{2\pi} |\phi_0 (p)|^2 = 1 \]
Hamiltonian:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega_0^2 \hat{x}^2$$

Use the lowest two circuit states as your qubit!!

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

Problems:
- All states have equal energy spacings
- This makes qubit operations impossible (one can accidentally put the circuit in one of the higher energy states during computation)
- Need to have just two energy levels (unless multilevel qubit logic is desired)
- How do we perform single qubit logic operations?
A Lossless Superconducting $LC$ Circuit

$$H = \frac{Q^2}{2C} + \frac{\lambda^2}{2L}$$
$$= \frac{Q^2}{2C} + \frac{1}{2} C \omega_o^2 \lambda^2$$

Circuit equations:

$$\frac{dQ}{dt} = -C \omega_o^2 \lambda$$
$$C \frac{d\lambda}{dt} = Q$$

$$\omega_o = \sqrt{\frac{1}{LC}}$$

Compare with SHO:

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega_o^2 x^2$$

$$\frac{dp}{dt} = -\frac{\partial V}{\partial x} = -m \omega_o^2 x$$

$$m \frac{dx}{dt} = p$$

$Q \leftrightarrow p$

$\lambda \leftrightarrow x$

$C \leftrightarrow m$

$\omega_o \leftrightarrow \omega_o$
A Quantum Lossless Superconducting \( LC \) Circuit

The macroscopic quantum state of the circuit is described by a vector \( |\psi(t)\rangle \) in a Hilbert space.

Charge and flux, and voltage and current, are observables and the corresponding operators are:

\[
\hat{Q} = CV \quad \hat{\lambda} = LI
\]

A circuit has many measurable physical degrees of freedom:

- There are billions and billions of electrons and atoms in the wires, capacitor plates, etc. Each electron/atom has position, momentum, spin, etc, as observables. These are the \textit{microscopic} degrees of freedom of the circuit.

- We are interested here in only the \textit{macroscopic} electrical degrees of freedom of the circuit and will make a quantum description of only these degrees of freedom. That such a macroscopic quantum description is possible, without taking into account all the microscopic degrees of freedom, is quite remarkable.

\[ \omega_o = \sqrt{\frac{1}{LC}} \]
A Quantum Lossless Superconducting LC Circuit

The quantum state of the circuit is described by a vector \( |\psi(t)\rangle \) in a Hilbert space.

Charge and flux, and voltage and current, are observables and the corresponding operators are:

\[
\hat{Q} = CV \quad \hat{\lambda} = LI
\]

The energy becomes the Hamiltonian operator:

\[
\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{1}{2} C\omega_o^2 \hat{\lambda}^2
\]

Postulate the following commutation relation:

\[
[\hat{\lambda}, \hat{Q}] = i\hbar
\]

Compare with:

\[
\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega_o^2 \hat{x}^2
\]

\[
\langle x | p \rangle = \frac{e^{ipx}}{\sqrt{\hbar}}
\]

\[
[\hat{x}, \hat{p}] = i\hbar
\]

\[
\hat{Q} \leftrightarrow \hat{p}
\]

\[
\hat{\lambda} \leftrightarrow \hat{x}
\]

\[
C \leftrightarrow m
\]

\[
\omega_o \leftrightarrow \omega_0
\]
A Quantum Lossless Superconducting LC Circuit

Charge and flux, and voltage and current, being all observables, become operators

\[ \hat{Q} = C\hat{V} \quad \hat{\lambda} = L\hat{I} \]

Postulate complete basis states formed by charge and flux eigenstates:

\[ \hat{\lambda} |\lambda\rangle = \lambda |\lambda\rangle \quad \hat{Q} |Q\rangle = Q |Q\rangle \]

\[ \int_{-\infty}^{+\infty} d\lambda |\lambda\rangle \langle \lambda | = \hat{1} \quad \int_{-\infty}^{+\infty} \frac{dQ}{2\pi} |Q\rangle \langle Q | = \hat{1} \]

\[ \left[ \hat{\lambda}, \hat{Q} \right] = i\hbar \quad \langle \lambda | Q \rangle = \frac{e^{i\frac{Q}{\hbar}\lambda}}{\sqrt{\hbar}} \]

Example:

If \( |\psi\rangle = |\lambda\rangle \) then the inductor flux is certain but the capacitor charge is very uncertain (because the quantum state \( |\psi\rangle \) is a superposition of different capacitor charge states)

\[ \begin{align*}
\hat{Q} & \leftrightarrow \hat{\rho} \\
\hat{\lambda} & \leftrightarrow \hat{x} \\
C & \leftrightarrow m \\
\omega_o & \leftrightarrow \omega_o
\end{align*} \]
A Quantum Lossless Superconducting LC Circuit

The energy becomes the Hamiltonian operator:

\[ \hat{H} = \frac{\hat{Q}^2}{2C} + \frac{1}{2} C \omega_o^2 \hat{\lambda}^2 \]

\[ [\hat{\lambda}, \hat{Q}] = i\hbar \]

Define:

\[ \hat{a} = i \sqrt{\frac{1}{2C\hbar \omega_o}} \hat{Q} + \sqrt{\frac{C \omega_o}{2\hbar}} \hat{\lambda} \]

\[ \hat{a}^\dagger = -i \sqrt{\frac{1}{2C\hbar \omega_o}} \hat{Q} + \sqrt{\frac{C \omega_o}{2\hbar}} \hat{\lambda} \]

\[ \Rightarrow [\hat{a}, \hat{a}^\dagger] = 1 \]

Compare with SHO:

\[ \hat{H} = \frac{\hat{\rho}^2}{2m} + \frac{1}{2} m \omega_o^2 \hat{x}^2 \]

\[ [\hat{x}, \hat{\rho}] = i\hbar \]

\[ \hat{a} = i \sqrt{\frac{1}{2m\hbar \omega_o}} \hat{\rho} + \sqrt{\frac{m \omega_o}{2\hbar}} \hat{x} \]

\[ \hat{a}^\dagger = -i \sqrt{\frac{1}{2m\hbar \omega_o}} \hat{\rho} + \sqrt{\frac{m \omega_o}{2\hbar}} \hat{x} \]

\[ \hat{Q} \leftrightarrow \hat{\rho} \]

\[ \hat{\lambda} \leftrightarrow \hat{x} \]

\[ C \leftrightarrow m \]

\[ \omega_o \leftrightarrow \omega_o \]
The Hamiltonian operator is:

\[ \hat{H} = \frac{\hat{Q}^2}{2C} + \frac{1}{2} C \omega_0^2 \hat{\lambda}^2 \]

The Hamiltonian operator becomes:

\[ \hat{H} = \hbar \omega_0 \left( \hat{n} + \frac{1}{2} \right) = \hbar \omega_0 \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \]

\[ \Rightarrow [\hat{a}, \hat{a}^\dagger] = 1 \]

The eigenstates and eigenvalues are:

\[ \hat{H} |n\rangle = \hbar \omega_0 \left( \hat{n} + \frac{1}{2} \right) |n\rangle = \hbar \omega_0 \left( n + \frac{1}{2} \right) |n\rangle = E_n |n\rangle \]

\[ \{ n = 0, 1, 2, 3, \ldots \} \]

\[ \omega_0 = \sqrt{\frac{1}{LC}} \]

The circuit has been quantized !!!
A Lossless Superconducting LC Circuit

Suppose the circuit is in its lowest energy state:

\[
|\psi\rangle = |0\rangle
\]

\[
\hat{H}|0\rangle = \frac{1}{2}\hbar \omega_o |0\rangle
\]

The inductor flux is measured in the circuit. What is the a-priori probability of finding the result "\(\lambda\)"

\[
|\langle \lambda | \psi \rangle|^2 = |\langle \lambda | 0 \rangle|^2 = |\phi_0(\lambda)|^2 = \left(\frac{C\omega_o}{\pi\hbar}\right)^2 e^{-\frac{C\omega_o}{\hbar} \lambda^2}
\]

\[
\int d\lambda |\langle \lambda | \psi \rangle|^2 = 1
\]

Probability distribution of flux

Zero-point quantum fluctuations in flux

Compare with SHO:

\[
\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega_o^2 \hat{x}^2
\]

\[
\hat{Q} \leftrightarrow \hat{p}
\]

\[
\hat{\lambda} \leftrightarrow \hat{x}
\]

\[
C \leftrightarrow m
\]

\[
\omega_o \leftrightarrow \omega_o
\]

\[
|\langle x | 0 \rangle|^2 = \left(\frac{m\omega_o}{2\pi\hbar}\right)^4 e^{-\frac{m\omega_o}{\hbar} x^2}
\]
A Lossless Superconducting LC Circuit

Suppose the circuit is in its lowest energy state:

\[ |\psi\rangle = |0\rangle \]

\[ \hat{H}|0\rangle = \frac{1}{2}\hbar \omega_0 |0\rangle \]

The inductor current is measured in the circuit. What is the a-priori probability of finding the result “\(I\)”?

Since the inductor current and inductor flux operators are related by a constant: \( \hat{\lambda} = L\hat{I} \)

\[ |I\rangle = \sqrt{L} |\lambda = LI\rangle \]

The probability distribution function for the current will be:

\[ \left| \langle I | \psi \rangle \right|^2 = L \left| \langle \lambda | \psi \rangle \right|^2 \bigg|_{\lambda = LI} = \left( \frac{L}{\pi \hbar \omega_0} \right)^2 e^{-\frac{L}{\hbar \omega_0} I^2} \]

\[ \int dI \left| \langle I | \psi \rangle \right|^2 = 1 \]

Zero-point quantum fluctuations in current
A Lossless Superconducting LC Circuit

Suppose the circuit is in its lowest energy state:

$$|\psi\rangle = |0\rangle$$

$$\hat{H}|0\rangle = \frac{1}{2}\hbar\omega_0 |0\rangle$$

The capacitor voltage is measured in the circuit. What is the a-priori probability of finding the result “V”:

Since the voltage and the charge operators are related by a constant: $$\hat{Q} = C\hat{V}$$

$$|V\rangle = \sqrt{C}|Q\rangle = CV$$

The probability distribution function for the voltage will be:

$$\left|\langle V | \psi\rangle\right|^2 = C\left|\langle Q | \psi\rangle\right|^2_{Q=CV} = 2\pi\left(\frac{C}{\pi\hbar\omega_0}\right)^2 e^{-\frac{c}{\hbar\omega_0}V^2}$$

$$\int \frac{dV}{2\pi} \left|\langle V | \psi\rangle\right|^2 = 1$$
A Lossless Superconducting \( LC \) Circuit

Suppose the circuit is in its lowest energy state:

- \( \omega_0 = \sqrt{\frac{1}{LC}} = 10 \text{ GHz} \)
- \( \omega_c = 2\pi \times 10^{10} \text{ rad/s} \)
- \( L = 0.25 \text{ nH} \)
- \( C = 1 \text{ pF} \)

RMS zero-point current quantum fluctuations:

\[
\sigma_I = \sqrt{\langle \Delta \hat{I}^2 \rangle} = \sqrt{\frac{\hbar \omega_0}{2L}} \sim 0.12 \mu A
\]

RMS zero-point voltage quantum fluctuations:

\[
\sigma_V = \sqrt{\langle \Delta \hat{V}^2 \rangle} = \sqrt{\frac{\hbar \omega_0}{2C}} \sim 1.8 \mu V
\]
A Lossless Superconducting LC Circuit: Commutation Relations

Voltage and current are non-commuting operators:

\[ \hat{Q} = CV \quad \hat{\lambda} = LI \]

\[ \left[ \hat{\lambda}, \hat{Q} \right] = i\hbar \]

\[ \Rightarrow \left[ \hat{i}, \hat{V} \right] = \frac{i\hbar}{LC} \]

Accurate simultaneous measurement of both the charge and flux is not possible.

Accurate simultaneous measurement of both the current and voltage is not possible.

For the ground state:

\[ \left\langle \Delta i^2 \right\rangle \geq \frac{\hbar^2}{4L^2C^2} = \frac{\hbar^2 \omega_o^2}{4LC} = \frac{\hbar^2 \omega_o^4}{4} \]

\[ \left\langle \Delta V^2 \right\rangle = \frac{\hbar \omega_o}{2C} \]
A Lossless Superconducting LC Circuit as a Qubit

Circuit Hamiltonian:

\[ \hat{H} = \frac{\hat{Q}^2}{2C} + \frac{1}{2} C\omega_o^2 \lambda^2 \]

\[ \hat{H} = \hbar\omega_o \left( \hat{n} + \frac{1}{2} \right) = \hbar\omega_o \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \]

Use the lowest two circuit states as your qubit!!

\[ |\psi\rangle = \alpha |0\rangle + \beta |1\rangle \]

Problem:
- All states have equal energy spacings
- This makes qubit operations impossible (one can accidentally put the circuit in one of the higher energy states during computation)
- Need to have just two energy levels (unless multilevel qubit logic is desired)
- How do we perform single qubit logic operations?
The Inductively Shunted Josephson Junction Qubit

Circuit Hamiltonian:

\[ \hat{H} = \frac{\hat{Q}^2}{2C} + \frac{1}{2} C\omega_0^2 \hat{\lambda}^2 \]

Flux quantum

\[ \lambda_o = \frac{\pi \hbar}{e} \]

Potential

Total potential

\[ \hat{H} = \frac{\hat{Q}^2}{2C} + \frac{1}{2} C\omega_0^2 \hat{\lambda}^2 - E_J \cos \left( \frac{2\pi \hat{\lambda}}{\lambda_o} \right) + E_J \]
The Inductively Shunted Josephson Junction Qubit

Circuit Hamiltonian:

\[
\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{1}{2} C \omega_o^2 \hat{\lambda}^2
\]

Only two confined energy levels near the potential minimum. Problem solved!

\[
E_0 = \frac{1}{2} \hbar \omega_o
\]

\[
E_1 = \left(1 + \frac{1}{2}\right) \hbar \omega_o
\]

\[
E_2 = \left(2 + \frac{1}{2}\right) \hbar \omega_o
\]

\[
E_3 = \left(3 + \frac{1}{2}\right) \hbar \omega_o
\]
An Electromagnetic Mode in a Photonic Cavity

\[ \vec{E}(\vec{r}, t) = \frac{q_E(t)}{\sqrt{\mu_0 \varepsilon_0}} \vec{U}(\vec{r}) \]
\[ \vec{H}(\vec{r}, t) = -\frac{q_H(t)}{\sqrt{\mu_0 \varepsilon_0}} \nabla \times \vec{U}(\vec{r}) \]

Wave equation:
\[ \nabla \times \nabla \times \vec{U}(\vec{r}) = \frac{\omega_o^2}{c^2} \vec{U}(\vec{r}) \]

Compare with:
\[ H = \frac{p^2}{2m} + \frac{1}{2} m \omega_o^2 x^2 \]
\[ \frac{dp}{dt} = -\frac{\partial V}{\partial x} = -m \omega_o^2 x \]
\[ m \frac{dx}{dt} = p \]
\[ q_E \leftrightarrow p \]
\[ q_H \leftrightarrow x \]
\[ \mu_0 \leftrightarrow m \]
\[ \omega_o \leftrightarrow \omega_o \]
A Quantized Electromagnetic Mode in a Photonic Cavity

The quantum state of the electromagnetic mode is described by a vector $|\psi(t)\rangle$ in a Hilbert space.

Electric and magnetic field of the mode are observables and the corresponding operators are:

$$\hat{E}(\vec{r}) = \frac{\hat{q}_E}{\sqrt{\mu_0 \varepsilon_0}} \vec{U}(\vec{r})$$

$$\hat{H}(\vec{r}) = -\frac{\hat{q}_H}{\sqrt{\mu_0 \varepsilon_0}} \nabla \times \vec{U}(\vec{r})$$

The energy becomes the Hamiltonian operator:

$$\hat{H} = \frac{\hat{q}_E^2}{2\mu_0} + \frac{1}{2} \mu_0 \omega_o^2 \hat{q}_H^2$$

Postulate the following commutation relation:

$$[\hat{q}_H, \hat{q}_E] = i\hbar$$

Compare with:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega_o^2 \hat{x}^2$$

$$[\hat{x}, \hat{p}] = i\hbar$$

$$\langle x | p \rangle = \frac{e^{-\frac{i p x}{\hbar}}}{\sqrt{\hbar}}$$

$$\hat{q}_E \leftrightarrow \hat{p}$$

$$\hat{q}_H \leftrightarrow \hat{x}$$

$$\mu_0 \leftrightarrow m$$

$$\omega_o \leftrightarrow \omega_o$$
A Quantized Electromagnetic Mode in a Photonic Cavity

Fields become operators and so do $q_E$ and $q_H$:

$$\hat{E}(\vec{r}) = \frac{\hat{q}_E}{\sqrt{\mu_0 \varepsilon_0}} \tilde{U}(\vec{r})$$
$$\hat{H}(\vec{r}) = -\frac{\hat{q}_H}{\sqrt{\mu_0 \varepsilon_0}} \nabla \times \tilde{U}(\vec{r})$$

Postulate complete basis states formed by $q_E$ and $q_H$ eigenstates:

$$\hat{q}_H | q_H \rangle = q_H | q_H \rangle$$
$$\hat{q}_E | q_E \rangle = q_E | q_E \rangle$$

$$\int_{-\infty}^{+\infty} dq_H | q_H \rangle \langle q_H | = \hat{1}$$
$$\int_{-\infty}^{+\infty} \frac{dq_E}{2\pi} | q_E \rangle \langle q_E | = \hat{1}$$

$$[\hat{q}_H, \hat{q}_E] = i\hbar$$

$$\langle q_H | q_E \rangle = \frac{e^{i\frac{q_E}{\hbar} q_H}}{\sqrt{\hbar}}$$

Compare with:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_0^2 \hat{x}^2$$

$$[\hat{x}, \hat{p}] = i\hbar$$

$$\langle x | p \rangle = \frac{e^{i\frac{p}{\hbar} x}}{\sqrt{\hbar}}$$

$$\hat{q}_E \leftrightarrow \hat{p}$$
$$\hat{q}_H \leftrightarrow \hat{x}$$
$$\mu_0 \leftrightarrow m$$
$$\omega_0 \leftrightarrow \omega_0$$
A Quantized Electromagnetic Mode in a Photonic Cavity

Fields become operators:

\[
\hat{E}(\vec{r}) = \frac{\hat{q}_E}{\sqrt{\mu_0 \varepsilon_0}} \tilde{U}(\vec{r})
\]

\[
\hat{H}(\vec{r}) = -\frac{\hat{q}_H}{\sqrt{\mu_0 \varepsilon_0}} \nabla \times \tilde{U}(\vec{r})
\]

The energy becomes the Hamiltonian operator:

\[
\hat{H} = \frac{\hat{q}_E^2}{2 \mu_0} + \frac{1}{2} \frac{\mu_0 \omega_0^2 \hat{q}_H^2}{\varepsilon_0}
\]

Define:

\[
\hat{a} = i \sqrt{\frac{1}{2 \mu_0 \hbar \omega_0}} \hat{q}_E + \sqrt{\frac{\mu_0 \omega_0}{2 \hbar}} \hat{q}_H
\]

\[
\hat{a}^\dagger = -i \sqrt{\frac{1}{2 \mu_0 \hbar \omega_0}} \hat{q}_E + \sqrt{\frac{\mu_0 \omega_0}{2 \hbar}} \hat{q}_H
\]

\[\Rightarrow \left[ \hat{a}, \hat{a}^\dagger \right] = 1\]

Compare with SHO:

\[
\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega_o^2 \hat{x}^2
\]

\[
\left[ \hat{x}, \hat{p} \right] = i \hbar
\]

\[
\hat{a} = i \sqrt{\frac{1}{2m \hbar \omega_0}} \hat{p} + \sqrt{\frac{m \omega_0}{2 \hbar}} \hat{x}
\]

\[
\hat{a}^\dagger = -i \sqrt{\frac{1}{2m \hbar \omega_0}} \hat{p} + \sqrt{\frac{m \omega_0}{2 \hbar}} \hat{x}
\]

\[\hat{q}_E \leftrightarrow \hat{p}\]

\[\hat{q}_H \leftrightarrow \hat{x}\]

\[\mu_0 \leftrightarrow m\]

\[\omega_0 \leftrightarrow \omega_o\]
The electromagnetic mode has quantized energies!!!
The eigenstates and eigenvalues are:

\[ \hat{H} |n\rangle = \hbar \omega_0 \left( \hat{n} + \frac{1}{2} \right) |n\rangle = \hbar \omega_0 \left( n + \frac{1}{2} \right) |n\rangle \]

\[ |0\rangle \quad \Rightarrow \quad |1\rangle = \hat{a}^\dagger |0\rangle \quad \Rightarrow \quad |2\rangle = \frac{(\hat{a}^\dagger)^2}{\sqrt{2!}} |0\rangle \]

\[ |n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} |0\rangle \quad \Rightarrow \quad \left( \frac{1}{2} + n \right) \hbar \omega_0 \]

State with no photons (vacuum!!)
State with one photon
State with two photons
State with \( n \) photon

Why does a state with no photons – the vacuum - has energy \( \frac{1}{2} \hbar \omega_0 \)?
The Field Operators; Quantum Field Theory

Fields become operators:

\[ \hat{E}(\vec{r}) = \frac{\hat{q}_E}{\sqrt{\mu_0 \varepsilon_0}} \tilde{U}(\vec{r}) \quad \hat{H}(\vec{r}) = -\frac{\hat{q}_H}{\sqrt{\mu_0 \varepsilon_0}} \nabla \times \tilde{U}(\vec{r}) \]

Where:

\[ \hat{E}(\vec{r}) = \sqrt{\frac{\hbar \omega_0}{2 \varepsilon_0}} \left( \frac{\hat{a} - \hat{a}^\dagger}{i} \right) \tilde{U}(\vec{r}) \]

\[ \hat{H}(\vec{r}) = -\frac{1}{\mu_0 \sqrt{2 \varepsilon_0 \omega_0}} \left( \hat{a} + \hat{a}^\dagger \right) \nabla \times \tilde{U}(\vec{r}) \]

Mean value of the fields in the zero photon number state:

\[ \langle 0 | \hat{E}(\vec{r}) | 0 \rangle = \langle 0 | \hat{H}(\vec{r}) | 0 \rangle = 0 \]

Mean value of the fields in any photon number state:

\[ \langle n | \hat{E}(\vec{r}) | n \rangle = \langle n | \hat{H}(\vec{r}) | n \rangle = 0 \]
**A Quantized Electromagnetic Mode: Vacuum Fluctuations**

$|0\rangle$: Why does a state with no photons have energy $\frac{1}{2} \hbar \omega_0$?

The Hamiltonian operator is:

$$
\hat{H} = \int d^3\vec{r} \left[ \frac{1}{2} \varepsilon_0 \hat{E}(\vec{r}) \cdot \hat{E}(\vec{r}) + \frac{1}{2} \mu_0 \hat{H}(\vec{r}) \cdot \hat{H}(\vec{r}) \right]
$$

$$
= \frac{\hat{q}_E^2}{2\mu_0} + \frac{1}{2} \mu_0 \omega_0^2 \hat{q}_H^2 = \hbar \omega_0 \left( \hat{n} + \frac{1}{2} \right)
$$

$$
\langle 0 | \hat{H} | 0 \rangle = \langle 0 | \int d^3\vec{r} \left[ \frac{1}{2} \varepsilon_0 \hat{E}(\vec{r}) \cdot \hat{E}(\vec{r}) + \frac{1}{2} \mu_0 \hat{H}(\vec{r}) \cdot \hat{H}(\vec{r}) \right] | 0 \rangle
$$

$$
= \langle 0 | \frac{\hat{q}_E^2}{2\mu_0} + \frac{1}{2} \mu_0 \omega_0^2 \hat{q}_H^2 | 0 \rangle = \langle 0 | \hbar \omega_0 \left( \hat{n} + \frac{1}{2} \right) | 0 \rangle = \frac{1}{2} \hbar \omega_0
$$

The vacuum is not exactly a vacuum!!
It has fluctuating electric and magnetic fields!!
A Quantized Electromagnetic Mode: Vacuum Fluctuations

\[
\hat{H} = \int d^3 \vec{r} \left[ \frac{1}{2} \varepsilon_0 \hat{E}(\vec{r}) \cdot \hat{E}(\vec{r}) + \frac{1}{2} \mu_0 \hat{H}(\vec{r}) \cdot \hat{H}(\vec{r}) \right]
\]

\[= \frac{\hat{q}_E^2}{2\mu_o} + \frac{1}{2} \mu_0 \omega_o^2 \hat{q}_H^2 = \hbar \omega_o \left( \hat{n} + \frac{1}{2} \right) \]

Suppose the quantum state of the mode was \( |0\rangle \)

If a measurement is made of the magnetic field amplitude what is the a-priori probability of measuring \( q_H \)?

Answer:

\[|\langle q_H | \psi \rangle|^2 = |\langle q_H | 0 \rangle|^2 = \phi_0 (q_H)^2 = \left( \frac{\mu_0 \omega_o}{\pi \hbar} \right)^2 e^{-\frac{\mu_0 \omega_o q_H^2}{\hbar}}\]

\[\langle q_H | 0 \rangle^2\]
A Quantized Electromagnetic Mode: Commutation Relations

The commutation relation between the field amplitudes is:

\[ \left[ \hat{q}_H, \hat{q}_E \right] = i\hbar \]

Accurate simultaneous measurement of both the electric and the magnetic field amplitudes is not impossible.

\[ \Rightarrow \left[ \hat{q}_H, \hat{q}_E \right] = i\hbar \]

\[ \left\langle \Delta \hat{q}_H^2 \right\rangle \left\langle \Delta \hat{q}_E^2 \right\rangle \geq \frac{\hbar^2}{4} \]

\[ \hat{E}(\vec{r}) = \frac{\hat{q}_E}{\sqrt{\mu_0 \varepsilon_0}} \vec{U}(\vec{r}) \]

\[ \hat{H}(\vec{r}) = -\frac{\hat{q}_H}{\sqrt{\mu_0 \varepsilon_0}} \nabla \times \vec{U}(\vec{r}) \]
Appendix: The Josephson Junction - I

The Josephson junction relations:

1) \[ I(t) = I_1(t) + I_2(t) = C \frac{dV(t)}{dt} + \frac{2\pi}{\lambda_0} E_J \sin(\Delta \varphi(t)) \]

2) \[ \frac{d\Delta \varphi(t)}{dt} = \frac{2\pi}{\lambda_0} V(t) \]
### Appendix: The Josephson Junction - II

The superconductor phase difference can be related to the trapped flux

$$\Delta \varphi(t) = 2\pi \frac{\lambda(t)}{\lambda_0} + 2\pi n$$

Energy stored in the junction:

$$\Rightarrow I(t) = C \frac{dV(t)}{dt} + \frac{2\pi}{\lambda_0} E_J \sin\left(2\pi \frac{\lambda(t)}{\lambda_0}\right)$$
Appendix: The Capacitively Shunted Josephson Junction Qubit: Transmon

\[ \hat{Q} = \hat{Q}_J - \hat{Q}_g = C_J \hat{V} - C_g \hat{V} \]

\[ \hat{H} = \frac{\hat{Q}^2}{2 \left[ C_J + C_g \right]} - E_J \cos(\Delta \phi) + E_J \]

\[ \left[ \Delta \phi, \hat{Q} \right] = 2 |e| i \]

Transmon limit:

\[ E_C = \frac{e^2}{2 \left( C_J + C_g \right)} \ll E_J \]

\[ \Rightarrow \hat{H} = \frac{\hat{Q}^2}{2 \left[ C_J + C_g \right]} - E_J \cos(\Delta \phi) + E_J \approx \frac{\hat{Q}^2}{2 \left[ C_J + C_g \right]} + \frac{E_J}{2} \Delta \phi^2 \]