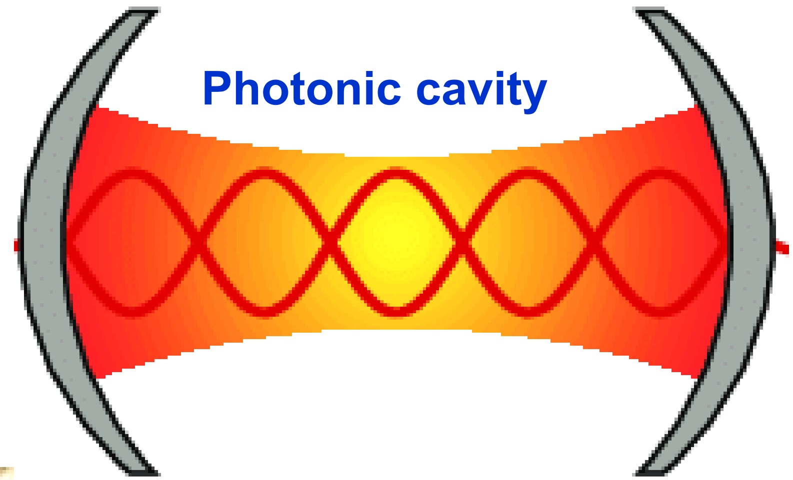
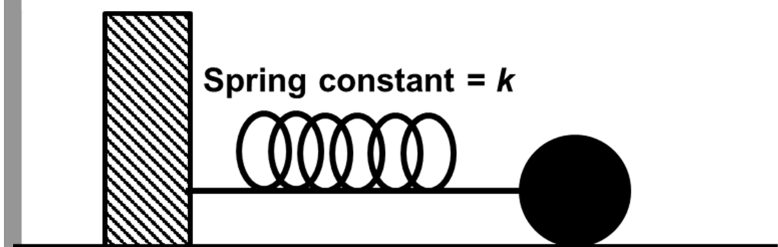
# Lecture 13

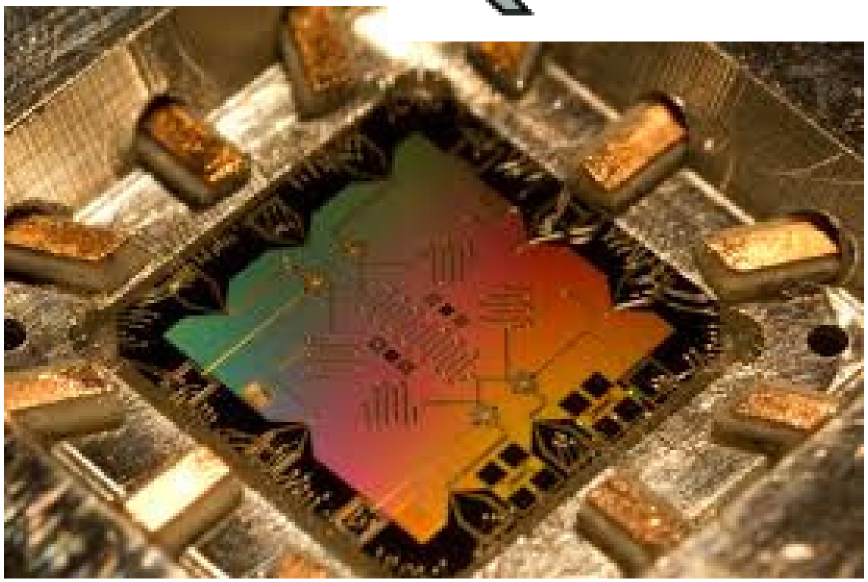
## The Ubiquitous Quantum Simple Harmonic Oscillator
## (Superconducting Qubit, Light Quantization, and all that)

**In this lecture you will learn:**




- Particle in a quadratic potential
- Quantum simple harmonic oscillator
- Quantum superconducting qubit
- Quantum description of light and photons

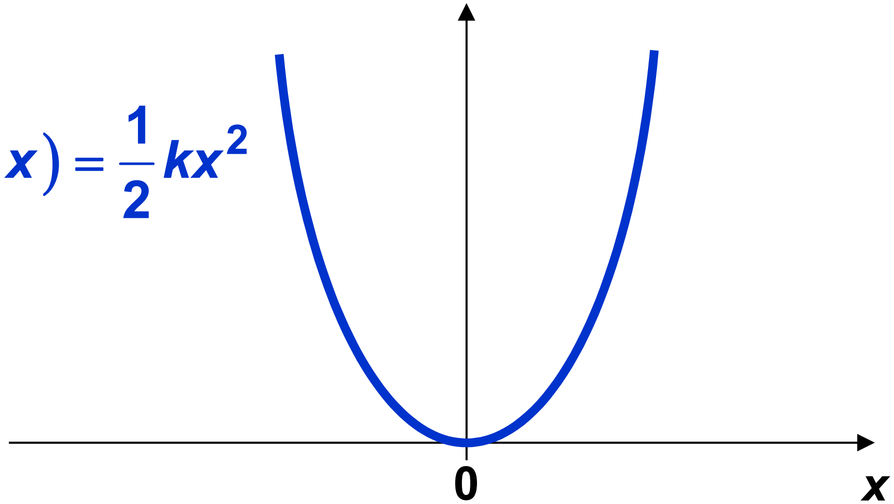


Superconducting qubit
(Google, UCSB)

# Classical SHO in 1D: A Spring-Mass System

$$V(x) = \frac{1}{2}kx^2$$

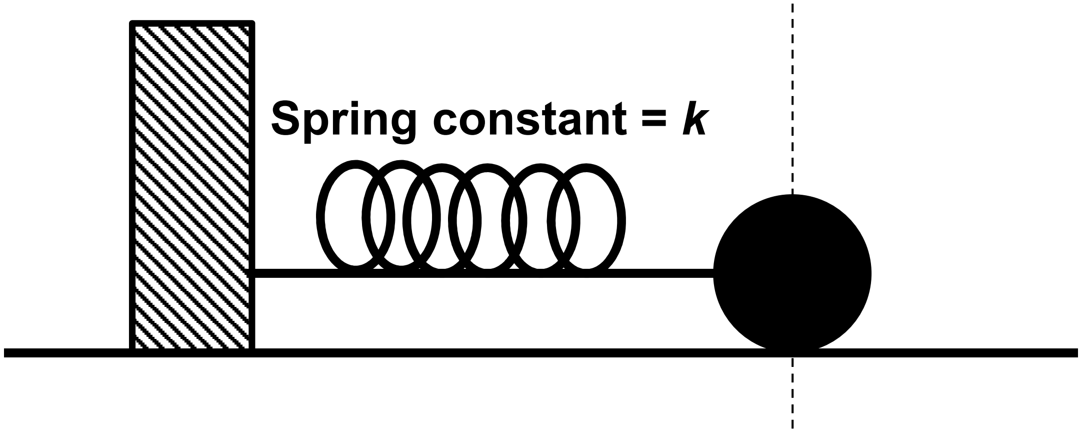



$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

**Newton's laws and classical equations:**

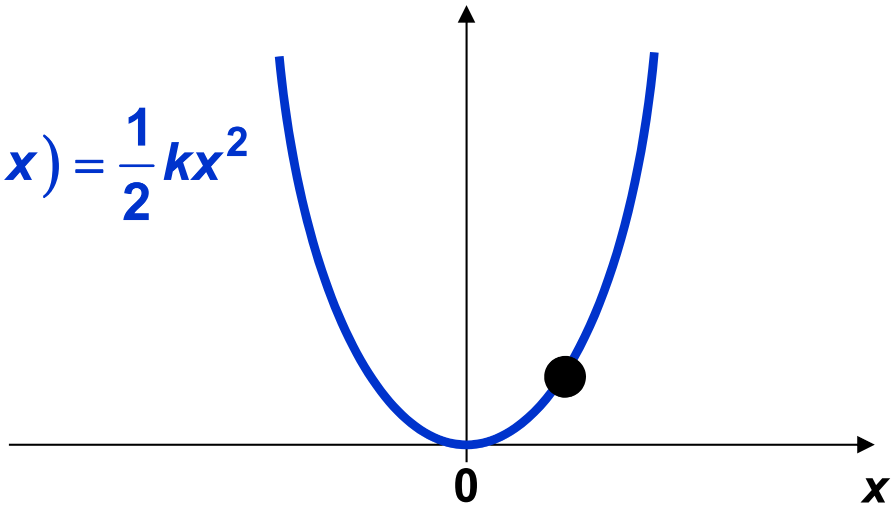
$$\frac{dp}{dt} = -\frac{\partial V}{\partial x} = -kx$$

$$m\frac{dx}{dt} = p$$



# Classical SHO in 1D: A Particle in a Quadratic Potential

$$V(x) = \frac{1}{2}kx^2$$




$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

## Newton's Laws and Classical Equations:

$$\frac{dp}{dt} = -\frac{\partial V}{\partial x} = -kx$$

$$m\frac{dx}{dt} = p$$

Momentum and position are coupled and dynamics are described two coupled differential equations

## Solutions:

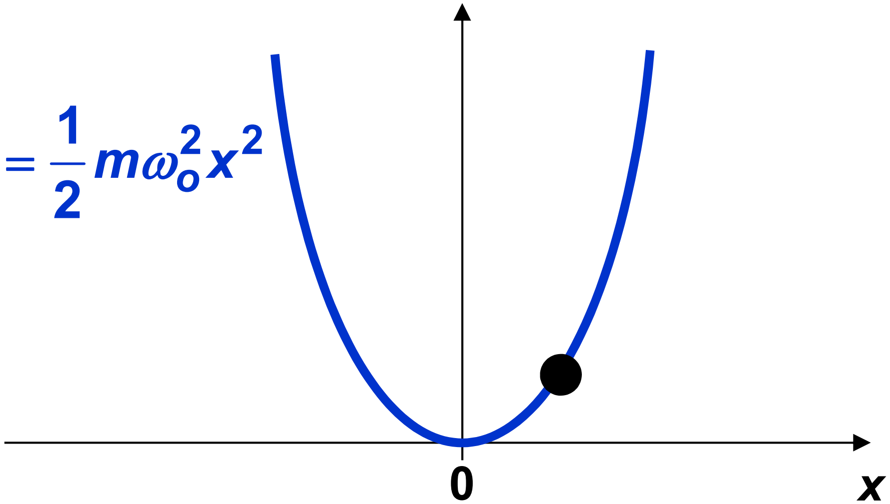
$$x(t) = A\cos(\omega_o t) + B\sin(\omega_o t)$$

$$p(t) = mB\omega_o \cos(\omega_o t) - m\omega_o A\sin(\omega_o t)$$

$$\omega_o = \sqrt{\frac{k}{m}}$$

# A Particle in a Quadratic Potential: Change of Notation

$$V(x) = \frac{1}{2} m\omega_o^2 x^2$$




$$\left\{ \omega_o = \sqrt{\frac{k}{m}} \right.$$

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega_o^2 x^2$$

**Newton's Laws and Classical Equations:**

$$\frac{dp}{dt} = -\frac{\partial V}{\partial x} = -m\omega_o^2 x$$

$$m\frac{dx}{dt} = p$$

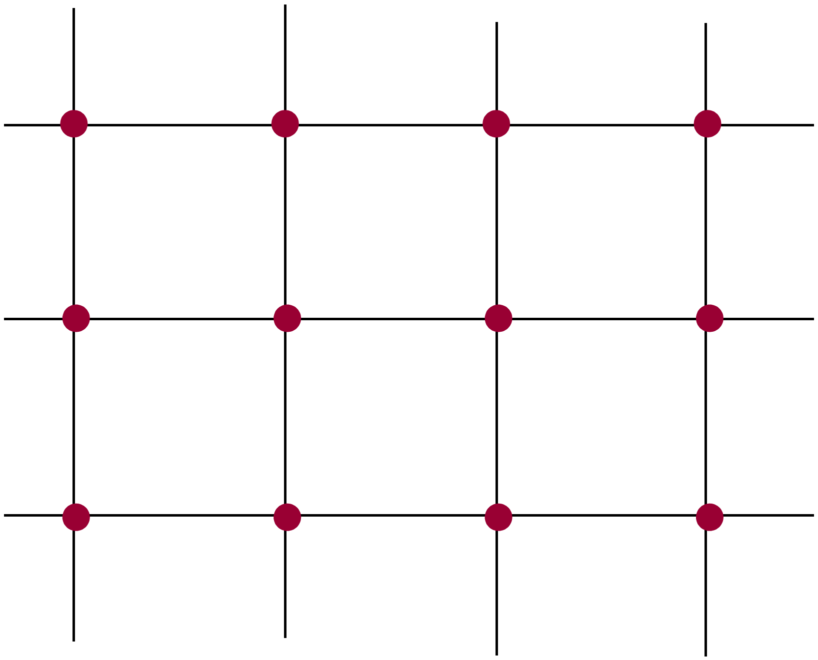
Momentum and position are coupled and dynamics are described two coupled differential equations

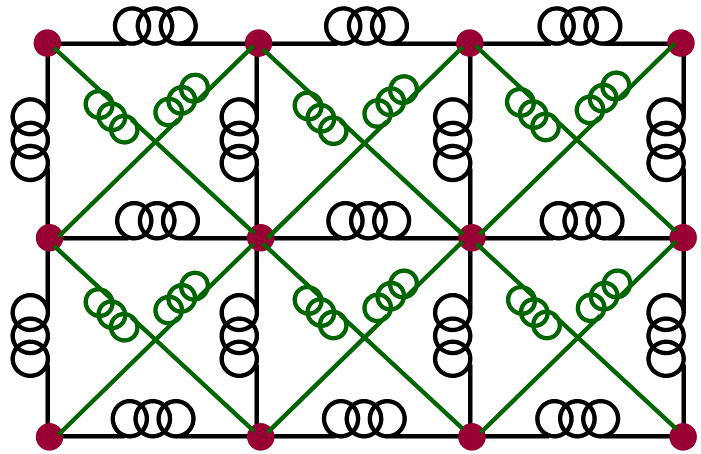
**Solutions:**

$$x(t) = A\cos(\omega_o t) + B\sin(\omega_o t)$$

$$p(t) = mB\omega_o \cos(\omega_o t) - m\omega_o A\sin(\omega_o t)$$

# Atoms in all Materials behave like Simple Harmonic Oscillators





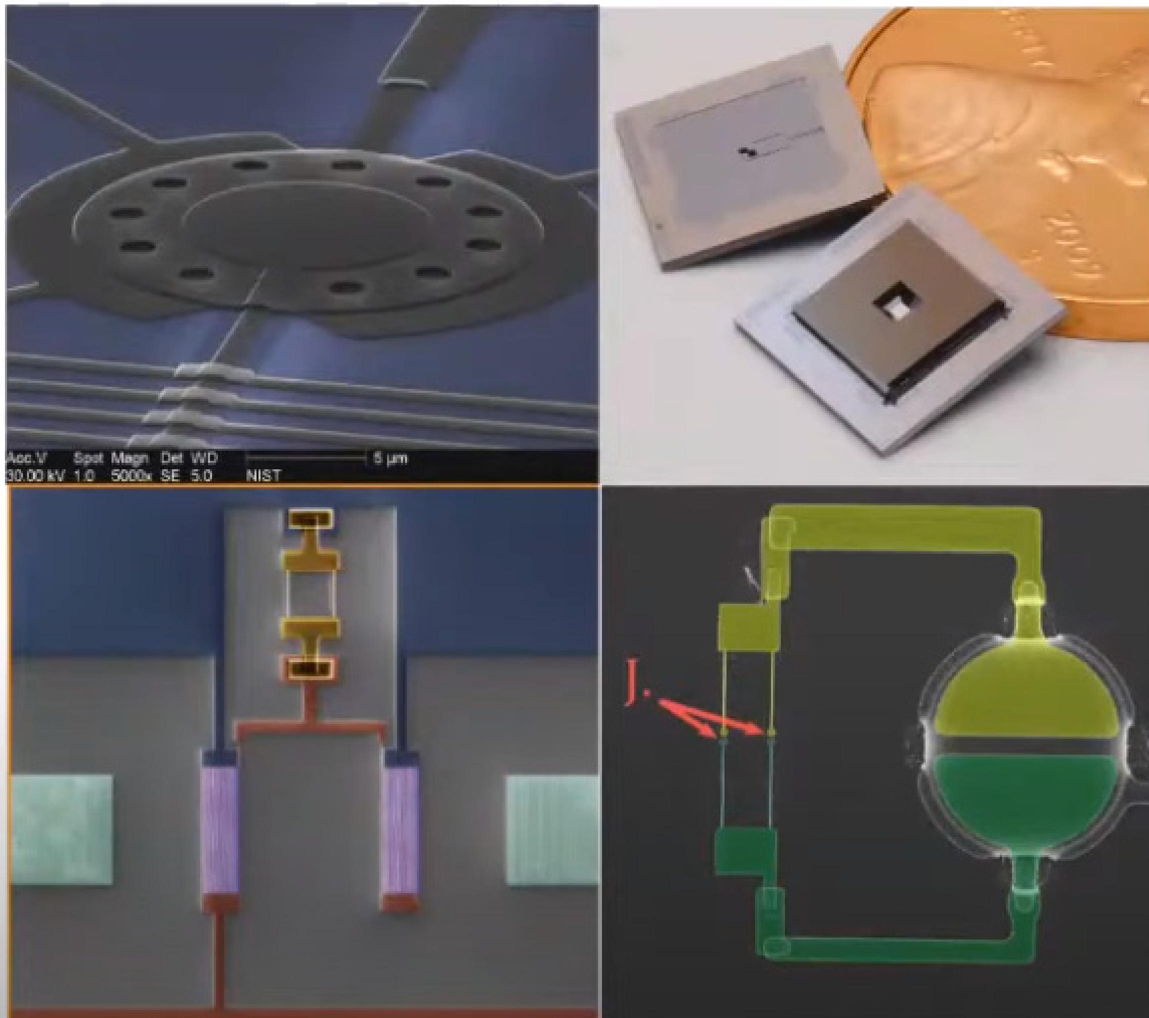
**All crystalline materials have atoms arranged in a periodic pattern**

**Atoms are held in place by electrostatic forces from neighbors**

**Every atom behaves like a simple harmonic oscillator that is coupled to its neighbors**

**Quantum MicroAcoustics**

# A new science of quantum sound

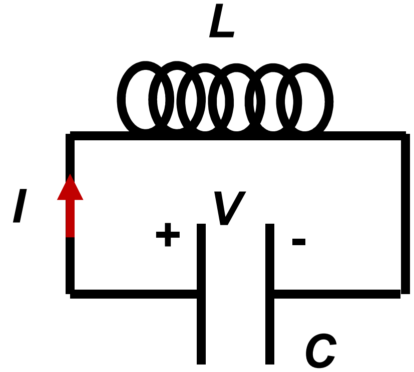



Acoustic microresonators are modeled as quantum simple harmonic oscillators

Konrad Lehnert
JILA

# A Lossless Superconducting *LC* Circuit

$$H = \frac{1}{2}CV^2 + \frac{1}{2}LI^2$$

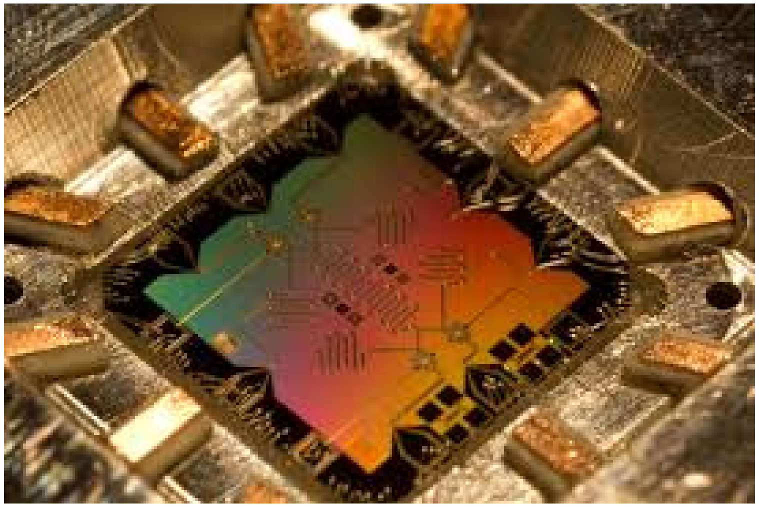



$$\left\{ \omega_o = \sqrt{\frac{1}{LC}} \right.$$

**Circuit equations:**

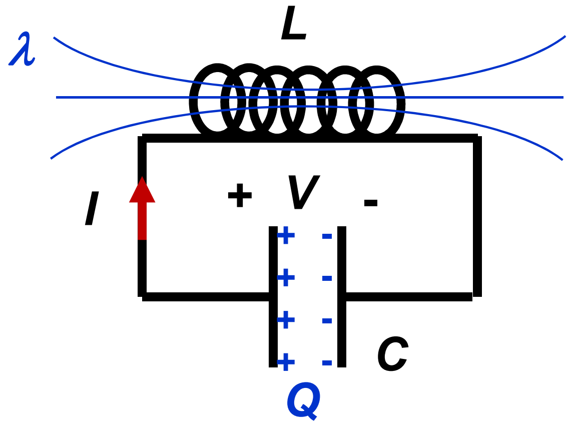
$$\frac{dV}{dt} = -\frac{I}{C}$$

$$L\frac{dI}{dt} = V$$



Superconducting qubit
(Google, UCSB)

# A Lossless Superconducting *LC* Circuit




$$H = \frac{1}{2}CV^2 + \frac{1}{2}LI^2$$

$$\left\{ \omega_o = \sqrt{\frac{1}{LC}} \right.$$

**Switch circuit variables:**

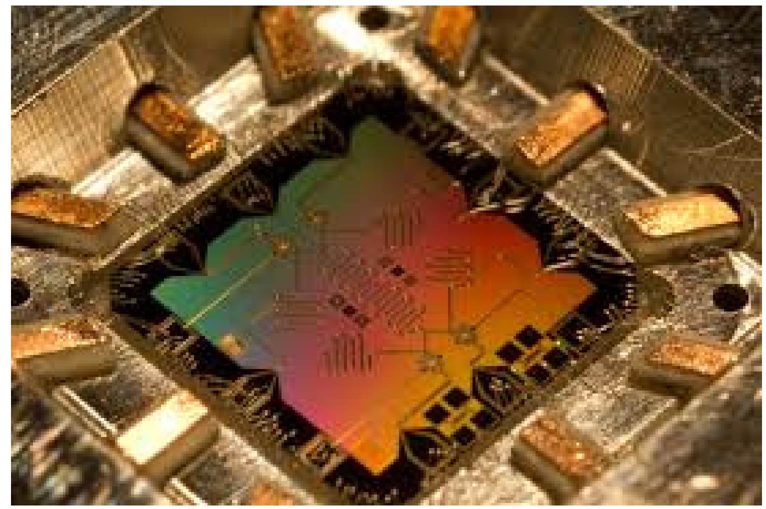
**Define charge $Q_s$ stored in the capacitor and the flux $\lambda_s$ stored in the inductor as:**

$$Q = CV$$

$$\lambda = LI$$

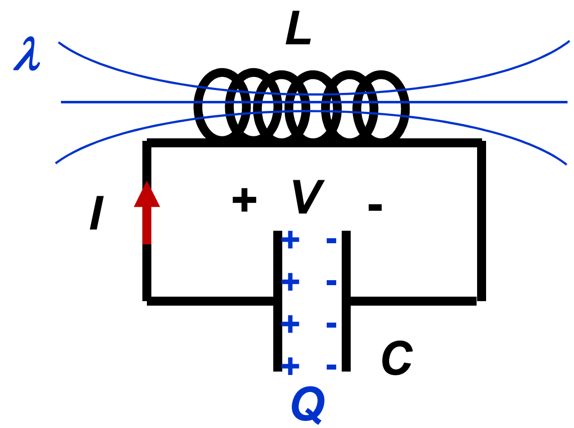
$$\Rightarrow H = \frac{Q^2}{2C} + \frac{\lambda^2}{2L}$$

$$= \frac{Q^2}{2C} + \frac{1}{2}C\omega_o^2\lambda^2$$



Superconducting qubit (Google, UCSB)

# A Lossless Superconducting *LC* Circuit




$$\omega_o = \sqrt{\frac{1}{LC}}$$

$$H = \frac{Q^2}{2C} + \frac{\lambda^2}{2L}$$
$$= \frac{Q^2}{2C} + \frac{1}{2} C \omega_o^2 \lambda^2$$

**Circuit equations:**

$$\frac{dQ}{dt} = -C\omega_o^2 \lambda$$

$$C \frac{d\lambda}{dt} = Q$$

**Compare with SHO:**

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega_o^2 x^2$$

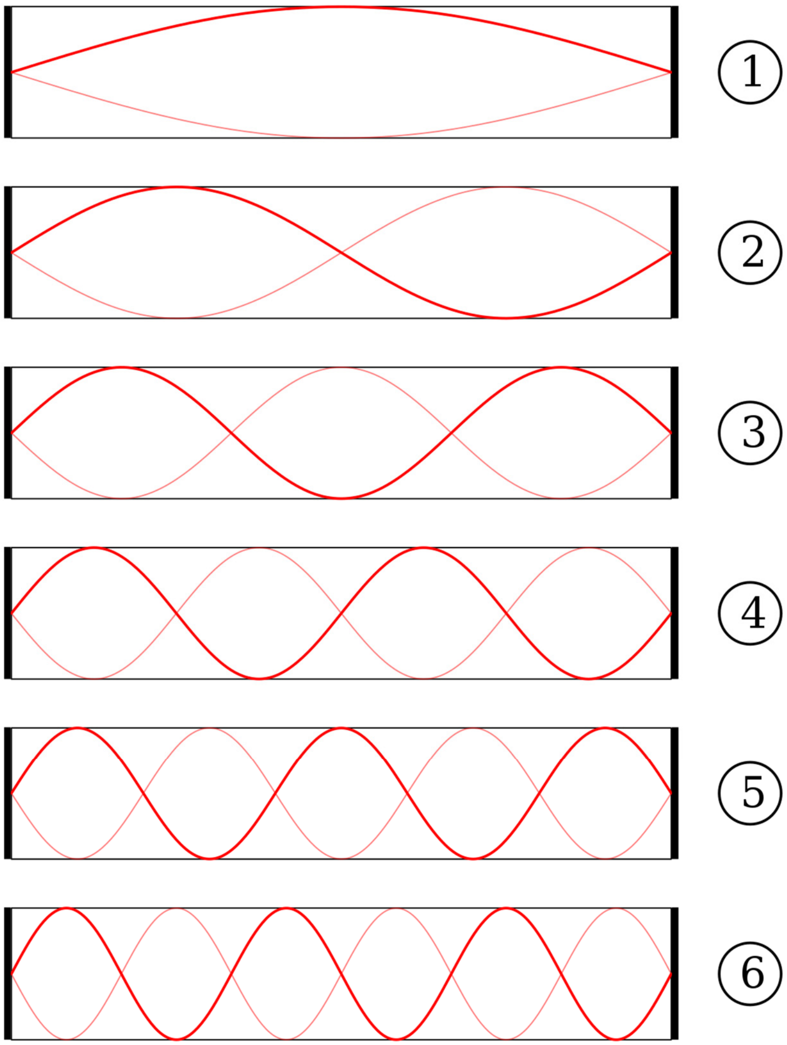
$$\frac{dp}{dt} = -\frac{\partial V}{\partial x} = -m\omega_o^2 x$$

$$m \frac{dx}{dt} = p$$

$$Q \leftrightarrow p$$
$$\lambda \leftrightarrow x$$
$$C \leftrightarrow m$$
$$\omega_o \leftrightarrow \omega_o$$

# Modes in a Photonic Cavity

**Different modes of a photonic cavity**




**Wave equation:**

$$\nabla \times \nabla \times \vec{E}(\vec{r},t) = \frac{1}{c^2}\frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2}$$

$$\nabla . \vec{E}(\vec{r},t) = 0$$

**Let:**

$$\vec{E}(\vec{r},t) = A\,\vec{U}_n\left(\vec{r}\right)e^{-i\omega_n t}$$

**We get the following eigenvalue equation for the mode spatial profile:**

$$\nabla \times \nabla \times \vec{U}_n(\vec{r}) = \frac{\omega_n^2}{c^2}\vec{U}_n(\vec{r})$$

$$\nabla . \vec{U}_n(\vec{r}) = 0$$

**Mode orthogonality and normalization:**

$$\int d^3\vec{r}\;\vec{U}_m(\vec{r}).\vec{U}_n(\vec{r}) = \delta_{n,m}$$

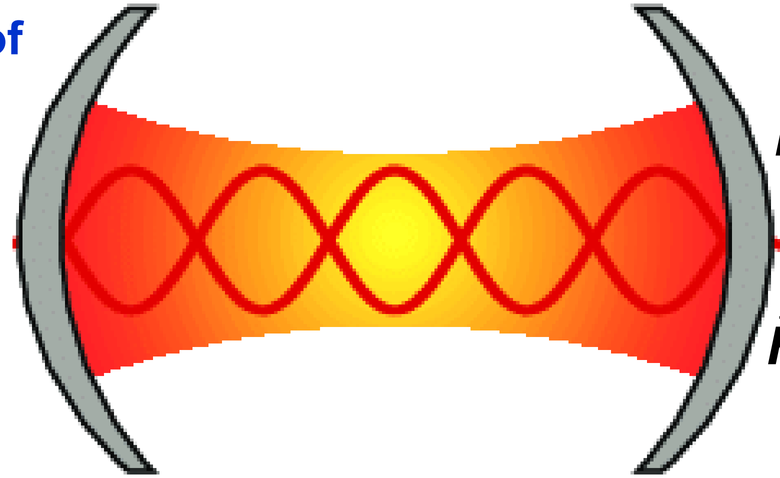
# An Electromagnetic Mode in a Photonic Cavity

Consider one single mode of frequency $\omega_o$ a photonic cavity:

$$\nabla \times \nabla \times \vec{U}(\vec{r}) = \frac{\omega_o^2}{c^2}\vec{U}(\vec{r})$$

Normalization:

$$\int d^3\vec{r}\ \vec{U}(\vec{r}).\vec{U}(\vec{r}) = 1$$



$$\vec{E}(\vec{r},t) = \frac{q_E(t)}{\sqrt{\mu_o \varepsilon_o}}\vec{U}(\vec{r})$$

$$\vec{H}(\vec{r},t) = -\frac{q_H(t)}{\sqrt{\mu_o \varepsilon_o}}\nabla \times \vec{U}(\vec{r})$$

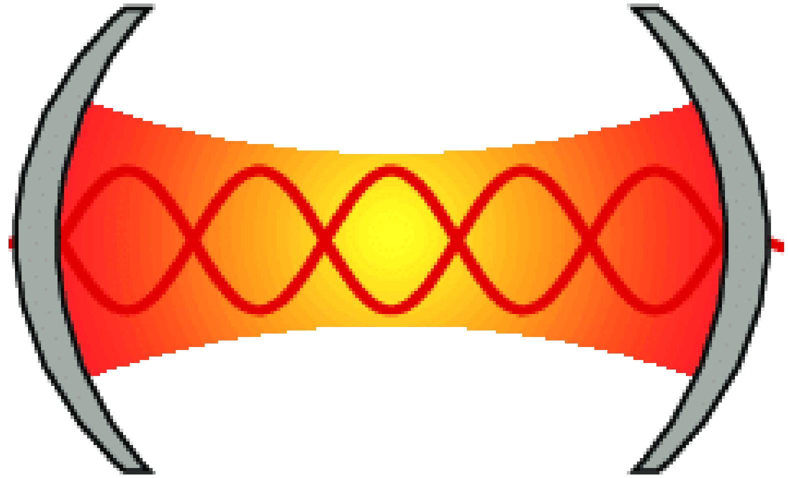
$$H = \int d^3\vec{r}\left[\frac{1}{2}\varepsilon_o\vec{E}(\vec{r},t).\vec{E}(\vec{r},t) + \frac{1}{2}\mu_o\vec{H}(\vec{r},t).\vec{H}(\vec{r},t)\right]$$

$$= \frac{q_E^2(t)}{2\mu_o} + \frac{1}{2}\mu_o\omega_o^2 q_H^2(t)$$

Faraday's law: $\nabla \times \vec{E}(\vec{r},t) = -\frac{\partial}{\partial t}\mu_o\vec{H}(\vec{r},t) \Longrightarrow \mu_o\frac{dq_H(t)}{dt} = q_E(t)$

Ampere's law: $\nabla \times \vec{H}(\vec{r},t) = \frac{\partial}{\partial t}\varepsilon_o\vec{E}(\vec{r},t) \Longrightarrow \frac{dq_E(t)}{dt} = -\mu_o\omega_o^2 q_H(t)$

# An Electromagnetic Mode in a Photonic Cavity



$$\vec{E}(\vec{r},t) = \frac{q_E(t)}{\sqrt{\mu_o \varepsilon_o}} \vec{U}(\vec{r})$$

$$\vec{H}(\vec{r},t) = -\frac{q_H(t)}{\sqrt{\mu_o \varepsilon_o}} \nabla \times \vec{U}(\vec{r})$$

$$H = \frac{q_E^2(t)}{2\mu_o} + \frac{1}{2}\mu_o \omega_o^2 q_H^2(t)$$

$$\frac{dq_E(t)}{dt} = -\mu_o \omega_o^2 q_H(t)$$

$$\mu_o \frac{dq_H(t)}{dt} = q_E(t)$$

**Compare with SHO:**

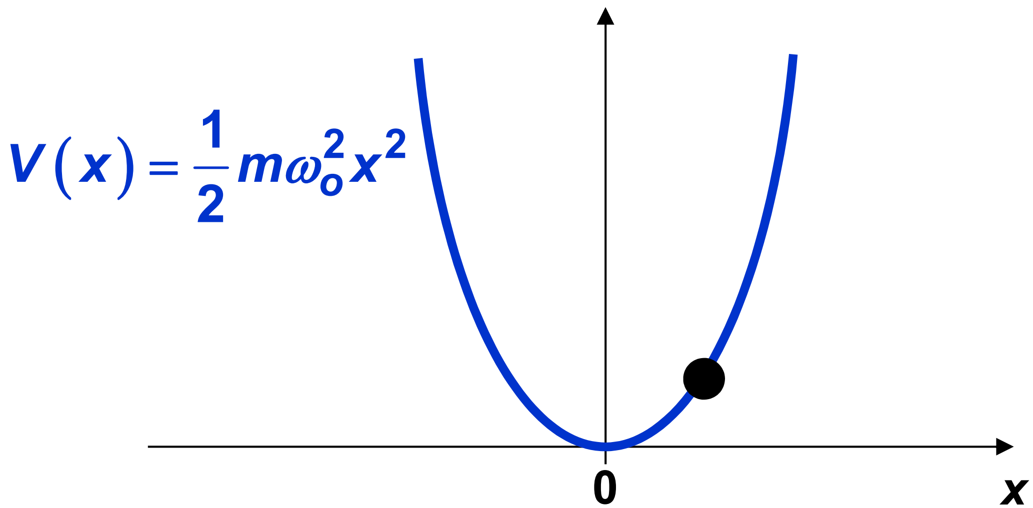
$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega_o^2 x^2$$

$$\frac{dp}{dt} = -\frac{\partial V}{\partial x} = -m\omega_o^2 x$$

$$m\frac{dx}{dt} = p$$

$$q_E \leftrightarrow p$$
$$q_H \leftrightarrow x$$
$$\mu_o \leftrightarrow m$$
$$\omega_o \leftrightarrow \omega_o$$

# Quantum SHO: A Particle in a Quadratic Potential




**The Hamiltonian is:**

$$\hat{H}=\frac{\hat{p}^2}{2m}+\frac{1}{2}m\omega_o^2\hat{x}^2 \qquad \left[\hat{x},\hat{p}\right]=i\hbar \quad \Longrightarrow \quad \langle x|p\rangle=\frac{e^{i\frac{p}{\hbar}x}}{\sqrt{\hbar}}$$

**We assume:**

$$\psi(x,t)=\phi(x)e^{-i\frac{E}{\hbar}t}$$

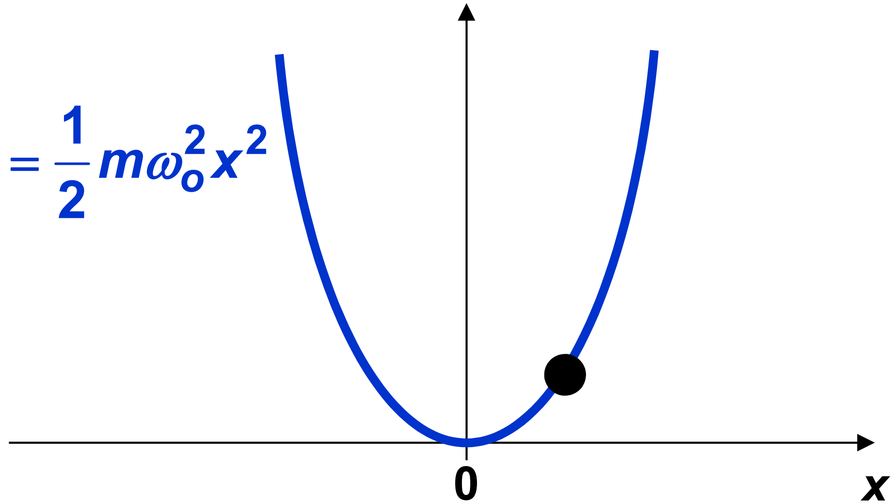
**And get:**

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi(x)}{\partial x^2}+\frac{1}{2}m\omega_o^2x^2\phi(x)=E\phi(x)$$

**How do we solve it??**

# Quantum SHO: A Particle in a Quadratic Potential

$$V(x) = \frac{1}{2} m\omega_o^2 x^2$$




$$-\frac{\hbar^2}{2m}\frac{\partial^2 \phi(x)}{\partial x^2} + \frac{1}{2} m\omega_o^2 x^2 \phi(x) = E\phi(x)$$

→ Factor the operator on the LHS

$$\Rightarrow \hbar\omega_o \left[-\sqrt{\frac{\hbar}{2m\omega_o}}\frac{\partial}{\partial x} + \sqrt{\frac{m\omega_o}{2\hbar}}\, x\right]\left[\sqrt{\frac{\hbar}{2m\omega_o}}\frac{\partial}{\partial x} + \sqrt{\frac{m\omega_o}{2\hbar}}\, x\right]\phi(x) + \frac{1}{2}\hbar\omega_o\phi(x) = E\phi(x)$$

Try a solution that satisfies:

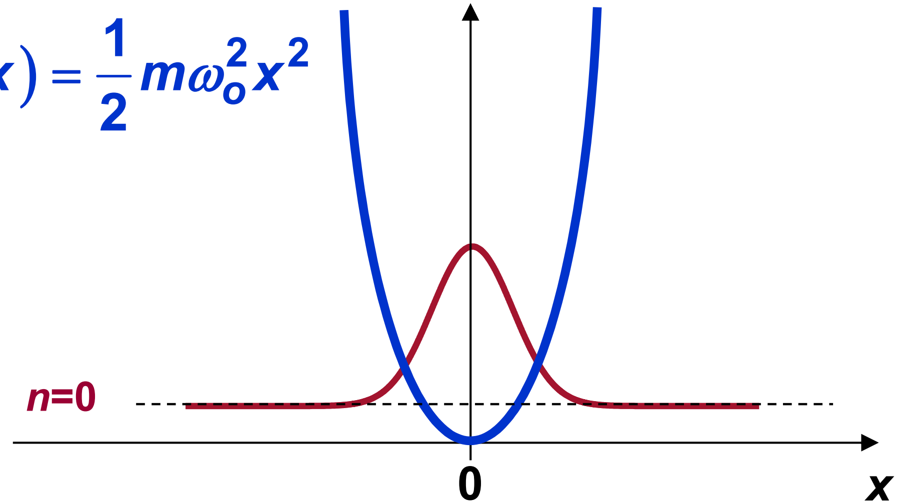
$$\left[\sqrt{\frac{\hbar}{2m\omega_o}}\frac{\partial}{\partial x} + \sqrt{\frac{m\omega_o}{2\hbar}}\, x\right]\phi(x) = 0$$

If we find one, its energy will be:

$$E = \frac{1}{2}\hbar\omega_o$$

Solution is: $\phi(x) = Ae^{-\frac{m\omega_o}{2\hbar}x^2}$ → Need to normalize

# Quantum SHO: A Particle in a Quadratic Potential



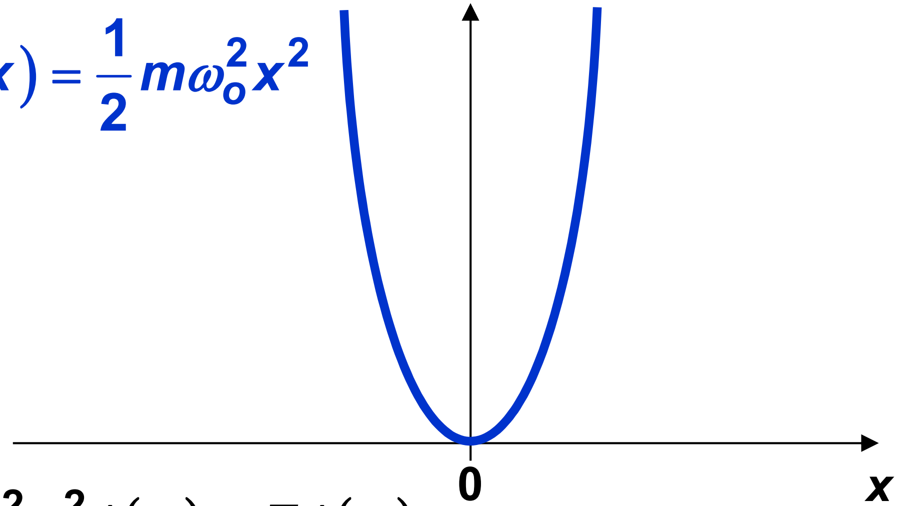

$$\hbar\omega_o\left[-\sqrt{\frac{\hbar}{2m\omega_o}}\frac{\partial}{\partial x}+\sqrt{\frac{m\omega_o}{2\hbar}}\,x\right]\left[\sqrt{\frac{\hbar}{2m\omega_o}}\frac{\partial}{\partial x}+\sqrt{\frac{m\omega_o}{2\hbar}}\,x\right]\phi(x)+\frac{1}{2}\hbar\omega_o\phi(x)=E\phi(x)$$

One solution is: $\phi_0(x) = \left(\frac{m\omega_o}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega_o}{2\hbar}x^2}$ $\longrightarrow$ $E_0 = \frac{1}{2}\hbar\omega_o$

Is this the lowest energy solution?
What are the other solutions?

# New Operators

$$V(x) = \frac{1}{2} m\omega_o^2 x^2$$




$$-\frac{\hbar^2}{2m}\frac{\partial^2 \phi(x)}{\partial x^2} + \frac{1}{2} m\omega_o^2 x^2 \phi(x) = E\phi(x)$$

$$\Rightarrow \hbar\omega_o \underbrace{\left[-\sqrt{\frac{\hbar}{2m\omega_o}}\frac{\partial}{\partial x} + \sqrt{\frac{m\omega_o}{2\hbar}}\, x\right]}_{\hat{a}^\dagger} \underbrace{\left[\sqrt{\frac{\hbar}{2m\omega_o}}\frac{\partial}{\partial x} + \sqrt{\frac{m\omega_o}{2\hbar}}\, x\right]}_{\hat{a}} \phi(x) + \frac{1}{2}\hbar\omega_o \phi(x) = E\phi(x)$$

**Define:**

$$\hat{a} = i\sqrt{\frac{1}{2m\hbar\omega_o}}\hat{p} + \sqrt{\frac{m\omega_o}{2\hbar}}\,\hat{x}$$

$$\hat{a}^\dagger = -i\sqrt{\frac{1}{2m\hbar\omega_o}}\hat{p} + \sqrt{\frac{m\omega_o}{2\hbar}}\,\hat{x}$$

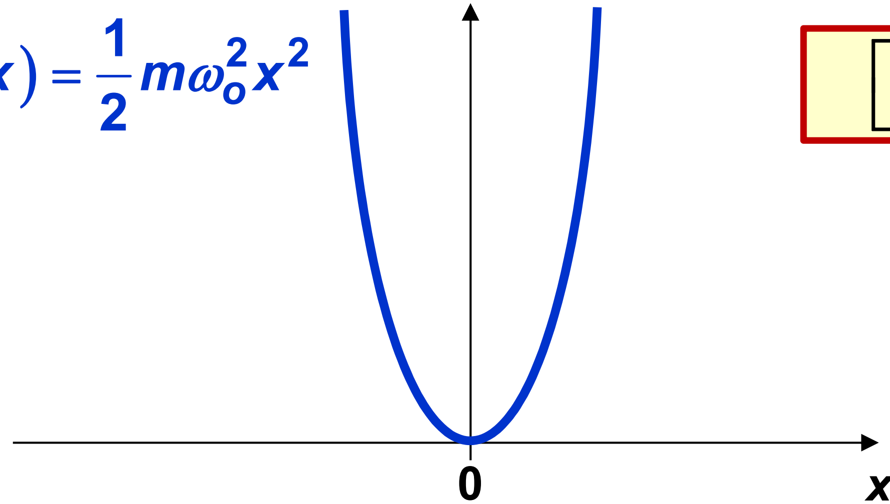
$\Rightarrow$ **Commutation relation**

$$\left[\hat{a}, \hat{a}^\dagger\right] = \hat{1} = 1$$

With a little abuse of notation

# Hamiltonian in Terms of the New Operators

$$V(x) = \frac{1}{2} m\omega_o^2 x^2$$




$$\left[\hat{a}, \hat{a}^\dagger\right] = 1$$

$$\hat{a} = i\sqrt{\frac{1}{2m\hbar\omega_o}}\,\hat{p} + \sqrt{\frac{m\omega_o}{2\hbar}}\,\hat{x}$$

$$\hat{a}^\dagger = -i\sqrt{\frac{1}{2m\hbar\omega_o}}\,\hat{p} + \sqrt{\frac{m\omega_o}{2\hbar}}\,\hat{x}$$

$\Rightarrow$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega_o}}\left(\hat{a} + \hat{a}^\dagger\right)$$

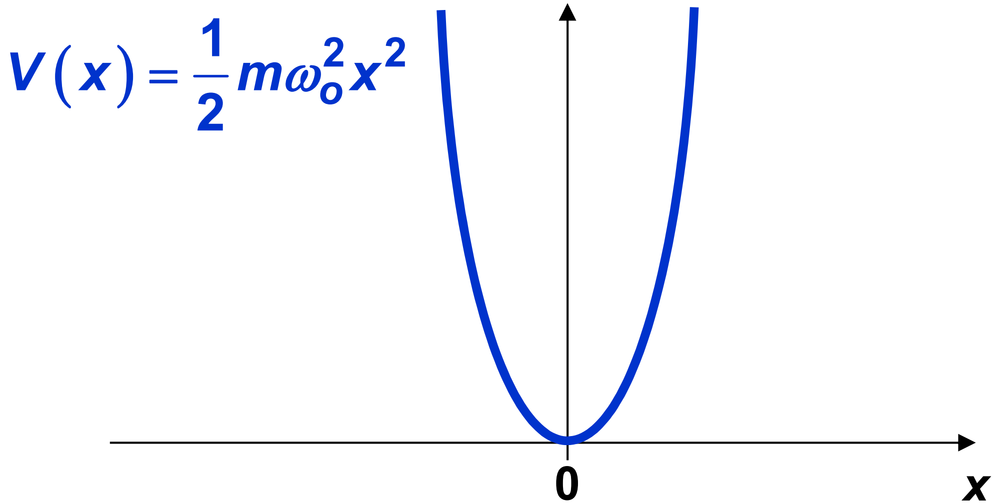
$$\hat{p} = \sqrt{\frac{m\hbar\omega_o}{2}}\left(\frac{\hat{a} - \hat{a}^\dagger}{i}\right)$$

Substitute the above in the Hamiltonian operator to get:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega_o^2 \hat{x}^2 = \frac{\hbar\omega_o}{2}\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right) = \hbar\omega_o\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1$$

# The Number Operator




$$\hat{H} = \hbar\omega_o\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

**The energy eigenstates are given by:**

$$\hat{H}|\phi\rangle = E|\phi\rangle$$

$$\Rightarrow \hbar\omega_o\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)|\phi\rangle = E|\phi\rangle$$

**Consider the operator (called the "number operator"):**

$$\hat{n} = \hat{a}^\dagger\hat{a}$$

**If we can find eigenstates and eigenvalues of the this operator, then we have the eigenstates and eigenvalues of the Hamiltonian**

# Creation and Destruction Operators

$$\hat{n} = \hat{a}^\dagger \hat{a} \qquad \left[\hat{a}, \hat{a}^\dagger\right] = 1$$

1) The operator $\hat{n} = \hat{a}^\dagger \hat{a}$ can only have non-negative eigenvalues

Suppose $|v\rangle$ is an eigenstate of $\hat{n} = \hat{a}^\dagger \hat{a}$ and $\lambda$ is the corresponding eigenvalue. Now consider the norm of the state: $|u\rangle = \hat{a}|v\rangle$

$$\langle u|u\rangle \geq 0$$

$$\Rightarrow \langle v|\hat{a}^\dagger \hat{a}|v\rangle \geq 0$$

$$\Rightarrow \langle v|\hat{n}|v\rangle \geq 0$$

$$\Rightarrow \lambda \langle v|v\rangle \geq 0$$

$$\Rightarrow \lambda \geq 0$$

2) If $|v\rangle$ is an eigenstate of $\hat{n}$ with eigenvalue $\lambda$ then $\hat{a}|v\rangle$ is also an eigenstate with eigenvalue $\lambda - 1$

$$\hat{n}\left(\hat{a}|v\rangle\right) = \hat{a}^\dagger \hat{a}\hat{a}|v\rangle = \left(\hat{a}^\dagger \hat{a}\right)\hat{a}|v\rangle = \left(\hat{a}\hat{a}^\dagger - 1\right)\hat{a}|v\rangle = \hat{a}\left(\hat{n} - 1\right)|v\rangle = (\lambda - 1)\hat{a}|v\rangle$$

$\hat{a}$ is called the "destruction" operator

# Creation and Destruction Operators

$$\hat{n} = \hat{a}^\dagger \hat{a} \qquad \left[\hat{a}, \hat{a}^\dagger\right] = 1$$

3) If $|v\rangle$ is an eigenstate of $\hat{n}$ with eigenvalue $\lambda$ then $\hat{a}^\dagger |v\rangle$ is also an eigenstate with eigenvalue $\lambda + 1$

$$\hat{n}\left(\hat{a}^\dagger |v\rangle\right) = \hat{a}^\dagger \hat{a} \hat{a}^\dagger |v\rangle = \hat{a}^\dagger \left(\hat{a}\hat{a}^\dagger\right)|v\rangle = \hat{a}^\dagger \left(\hat{a}^\dagger \hat{a} + 1\right)|v\rangle = \hat{a}^\dagger (\hat{n} + 1)|v\rangle = (\lambda + 1)\hat{a}^\dagger |v\rangle$$

$\hat{a}^\dagger$ is called the "creation" operator

# Properties of the Number Operator

4) The smallest eigenvalue of $\hat{n}$ is **0** and all eigenvalues of $\hat{n}$ are integers

$$\hat{n}|v\rangle = \lambda|v\rangle$$

$$\Rightarrow \hat{n}\left(\hat{a}|v\rangle\right) = (\lambda - 1)\left(\hat{a}|v\rangle\right)$$

$$\Rightarrow \hat{n}\left(\hat{a}^2|v\rangle\right) = (\lambda - 2)\left(\hat{a}^2|v\rangle\right)$$

$$\Rightarrow \hat{n}\left(\hat{a}^3|v\rangle\right) = (\lambda - 3)\left(\hat{a}^3|v\rangle\right)$$

$$\vdots$$

$$\Rightarrow \hat{n}\left(\hat{a}^{p-1}|v\rangle\right) = (\lambda - p + 1)\left(\hat{a}^{p-1}|v\rangle\right)$$

$$\Rightarrow \hat{n}\left(\hat{a}^{p}|v\rangle\right) = (\lambda - p)\left(\hat{a}^{p}|v\rangle\right)$$

$\Longrightarrow$ For some integer $p$ we will have: $(\lambda - p) < 0$ Not allowed!!

So it must be that acting with $\hat{a}$ on the state $\hat{a}^{p-1}|v\rangle$ must not give another state in the Hilbert space but give a zero instead:

$$\hat{a}\left(\hat{a}^{p-1}|v\rangle\right) = 0 \qquad \Rightarrow \hat{a}^\dagger\hat{a}\left(\hat{a}^{p-1}|v\rangle\right) = 0$$

$$\Rightarrow \hat{n}\left(\hat{a}^{p-1}|v\rangle\right) = 0$$

That can only happen if $(\lambda - p + 1) = 0$ for some $p$

That means $\lambda$ was an integer and all eigenvalues of $\hat{n}$ are integers!!!

# Properties of the Number Operator

$$\hat{n} = \hat{a}^{\dagger}\hat{a} \qquad \left[\hat{a},\hat{a}^{\dagger}\right] = 1$$

5) The eigenstates of $\hat{n}$ are written as $|n\rangle$ and are labeled by their eigenvalue $n$

$$\hat{n}|n\rangle = n|n\rangle \qquad \{n = 0,1,2,3.......$$

6) Since $\hat{n}$ is Hermitian, its eigenstates are orthonormal and form a complete set:

$$\langle n|m\rangle = \delta_{nm} \qquad \sum_{n=0}^{\infty} |n\rangle\langle n| = \hat{1} \qquad \{n = 0,1,2,3.......$$

7) The smallest eigenvalue of $\hat{n}$ is $0$ and the corresponding eigenstate is $|0\rangle$

$$\hat{a}|0\rangle = 0$$

$$\Rightarrow \hat{n}|0\rangle = 0$$

8) From properties (2) and (3):

$$\hat{n}\left(\hat{a}|n\rangle\right) = (n-1)\left(\hat{a}|n\rangle\right)$$

$$\hat{n}\left(\hat{a}^{\dagger}|n\rangle\right) = (n+1)\left(\hat{a}^{\dagger}|n\rangle\right)$$

# Properties of the Number Operator

$$\hat{n} = \hat{a}^\dagger \hat{a} \qquad \left[\hat{a}, \hat{a}^\dagger\right] = 1$$

8) **Given:** $\hat{n}|n\rangle = n|n\rangle$ **we know that:** $\hat{a}^\dagger|n\rangle \propto |n+1\rangle$

Let: $\hat{a}^\dagger|n\rangle = A|n+1\rangle$

$$\Rightarrow \langle n|\hat{a}\hat{a}^\dagger|n\rangle = |A|^2\langle n+1|n+1\rangle = |A|^2$$

$$\Rightarrow |A|^2 = \langle n|\hat{a}\hat{a}^\dagger|n\rangle = \langle n|\hat{a}^\dagger\hat{a}+1|n\rangle = n+1$$

$$\Rightarrow A = \sqrt{n+1}$$

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

9) **Given:** $\hat{n}|n\rangle = n|n\rangle$ **we know that:** $\hat{a}|n\rangle \propto |n-1\rangle$

Let: $\hat{a}|n\rangle = A|n-1\rangle$

$$\Rightarrow \langle n|\hat{a}^\dagger\hat{a}|n\rangle = |A|^2\langle n-1|n-1\rangle = |A|^2$$

$$\Rightarrow |A|^2 = \langle n|\hat{a}^\dagger\hat{a}|n\rangle = n$$

$$\Rightarrow A = \sqrt{n}$$

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \Longrightarrow \hat{a}|0\rangle = 0$$

## Properties of the Number Operator

$$\hat{n} = \hat{a}^{\dagger}\hat{a} \qquad \left[\hat{a}, \hat{a}^{\dagger}\right] = 1$$

**10)** All eigenstates of $\hat{n}$ can be written as:

$$|0\rangle$$

$$\Rightarrow |1\rangle = \hat{a}^{\dagger}|0\rangle$$

$$\Rightarrow |2\rangle = \frac{\left(\hat{a}^{\dagger}\right)^2}{\sqrt{2!}}|0\rangle$$

$$\Rightarrow |3\rangle = \frac{\left(\hat{a}^{\dagger}\right)^3}{\sqrt{3!}}|0\rangle$$

$$\vdots$$

$$\Rightarrow |n\rangle = \frac{\left(\hat{a}^{\dagger}\right)^n}{\sqrt{n!}}|0\rangle$$

# Quantum SHO: Summary of Results

$$\hat{n} = \hat{a}^{\dagger}\hat{a} \qquad \left[\hat{a}, \hat{a}^{\dagger}\right] = 1$$

$$\hat{a} = i\sqrt{\frac{1}{2m\hbar\omega_o}}\,\hat{p} + \sqrt{\frac{m\omega_o}{2\hbar}}\,\hat{x}$$

$$\hat{a}^{\dagger} = -i\sqrt{\frac{1}{2m\hbar\omega_o}}\,\hat{p} + \sqrt{\frac{m\omega_o}{2\hbar}}\,\hat{x}$$

$$\Rightarrow$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega_o}}\left(\hat{a} + \hat{a}^{\dagger}\right)$$

$$\hat{p} = \sqrt{\frac{m\hbar\omega_o}{2}}\left(\frac{\hat{a} - \hat{a}^{\dagger}}{i}\right)$$

## Eigenstates and Eigenvalues:

$$\hat{n}\left|n\right\rangle = n\left|n\right\rangle \qquad \{n = 0, 1, 2, 3\ldots\ldots$$

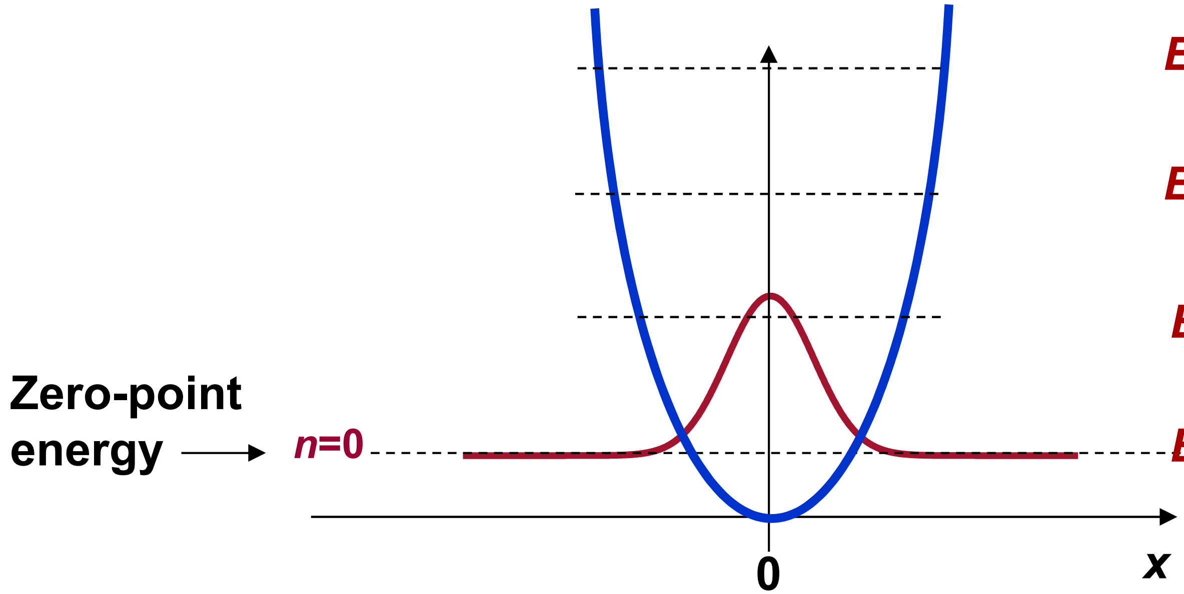
$$\left\langle n | m \right\rangle = \delta_{nm} \qquad \sum_{n=0}^{\infty} \left|n\right\rangle\left\langle n\right| = \hat{1} \qquad \{n = 0, 1, 2, 3\ldots\ldots$$

## Actions of Creation and Destruction Operators:

$$\hat{a}^{\dagger}\left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle$$

$$\hat{a}\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle \quad \Rightarrow \quad \hat{a}\left|0\right\rangle = 0$$

# Quantum SHO: Hamiltonian and Energy Eigenstates




$$E_3 = \left(3 + \frac{1}{2}\right)\hbar\omega_o$$

$$E_2 = \left(2 + \frac{1}{2}\right)\hbar\omega_o$$

$$E_1 = \left(1 + \frac{1}{2}\right)\hbar\omega_o$$

$$E_0 = \frac{1}{2}\hbar\omega_o$$

$$\hat{H} = \hbar\omega_o\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) = \hbar\omega_o\left(\hat{n} + \frac{1}{2}\right)$$

The eigenstates $|n\rangle$ of the number operator $\hat{n}$ are also the eigenstates of the Hamiltonian:

$$\hat{H}|n\rangle = \hbar\omega_o\left(\hat{n} + \frac{1}{2}\right)|n\rangle = \hbar\omega_o\left(n + \frac{1}{2}\right)|n\rangle \qquad \{n = 0,1,2,3.......$$

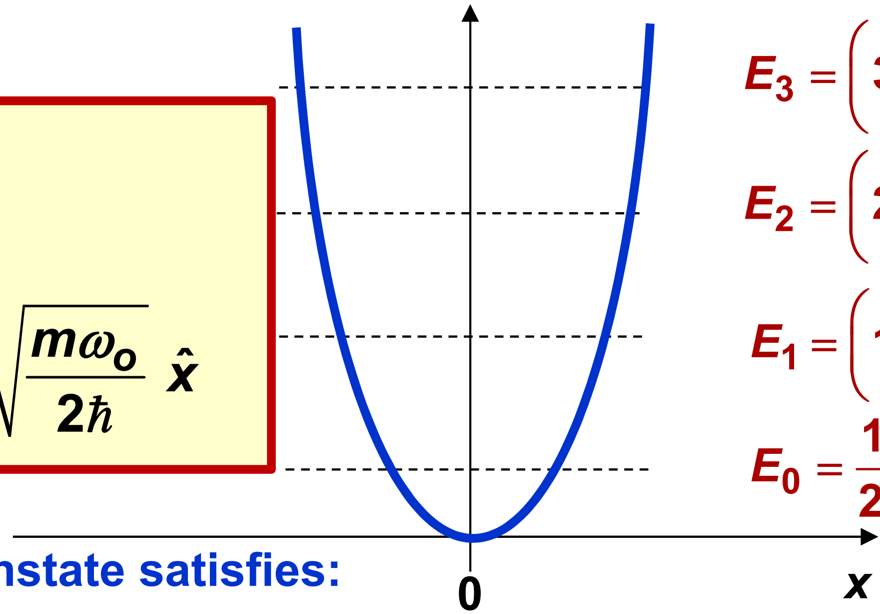
Therefore, the eigenvalues of the Hamiltonian are:

$$\frac{1}{2}\hbar\omega_o, \left(1 + \frac{1}{2}\right)\hbar\omega_o, \left(2 + \frac{1}{2}\right)\hbar\omega_o, \left(3 + \frac{1}{2}\right)\hbar\omega_o......$$

# Quantum SHO: Wavefunctions

$$\phi_n(x) = \langle x | n \rangle$$

$$\hat{a} = i\sqrt{\frac{1}{2m\hbar\omega_o}}\,\hat{p} + \sqrt{\frac{m\omega_o}{2\hbar}}\,\hat{x}$$




$$E_3 = \left(3 + \frac{1}{2}\right)\hbar\omega_o$$

$$E_2 = \left(2 + \frac{1}{2}\right)\hbar\omega_o$$

$$E_1 = \left(1 + \frac{1}{2}\right)\hbar\omega_o$$

$$E_0 = \frac{1}{2}\hbar\omega_o$$

The lowest energy eigenstate satisfies:

$$\hat{a}|0\rangle = 0$$

This means:

$$\langle x | \hat{a} | 0 \rangle = 0$$

$$\langle x | i\sqrt{\frac{1}{2m\hbar\omega_o}}\,\hat{p} + \sqrt{\frac{m\omega_o}{2\hbar}}\,\hat{x} | 0 \rangle = 0$$

$$\Rightarrow \left[\sqrt{\frac{\hbar}{2m\omega_o}}\frac{\partial}{\partial x} + \sqrt{\frac{m\omega_o}{2\hbar}}\,x\right]\phi_0(x) = 0$$

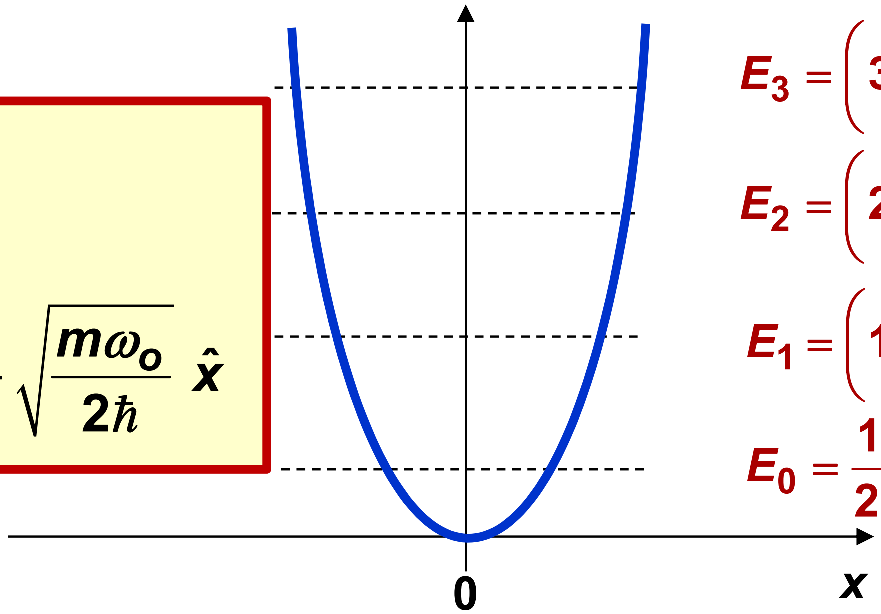
$$\Rightarrow \phi_0(x) = \left(\frac{m\omega_o}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega_o}{2\hbar}x^2}$$

$$\int dx\,|\phi_0(x)|^2 = 1$$

# Quantum SHO: Wavefunctions

$$\phi_n(x) = \langle x | n \rangle$$

$$\hat{a}^\dagger = -i\sqrt{\frac{1}{2m\hbar\omega_o}}\hat{p} + \sqrt{\frac{m\omega_o}{2\hbar}}\,\hat{x}$$




$$E_3 = \left(3 + \frac{1}{2}\right)\hbar\omega_o$$

$$E_2 = \left(2 + \frac{1}{2}\right)\hbar\omega_o$$

$$E_1 = \left(1 + \frac{1}{2}\right)\hbar\omega_o$$

$$E_0 = \frac{1}{2}\hbar\omega_o$$

The $n$-th eigenstate is:

$$|n\rangle = \frac{\left(\hat{a}^\dagger\right)^n}{\sqrt{n!}}|0\rangle$$

$$\Rightarrow \phi_n(x) = \langle x | n \rangle = \frac{1}{\sqrt{n!}}\langle x|\left(\hat{a}^\dagger\right)^n|0\rangle = \frac{1}{\sqrt{n!}}\langle x|\left[-i\sqrt{\frac{1}{2m\hbar\omega_o}}\hat{p} + \sqrt{\frac{m\omega_o}{2\hbar}}\,\hat{x}\right]^n|0\rangle$$

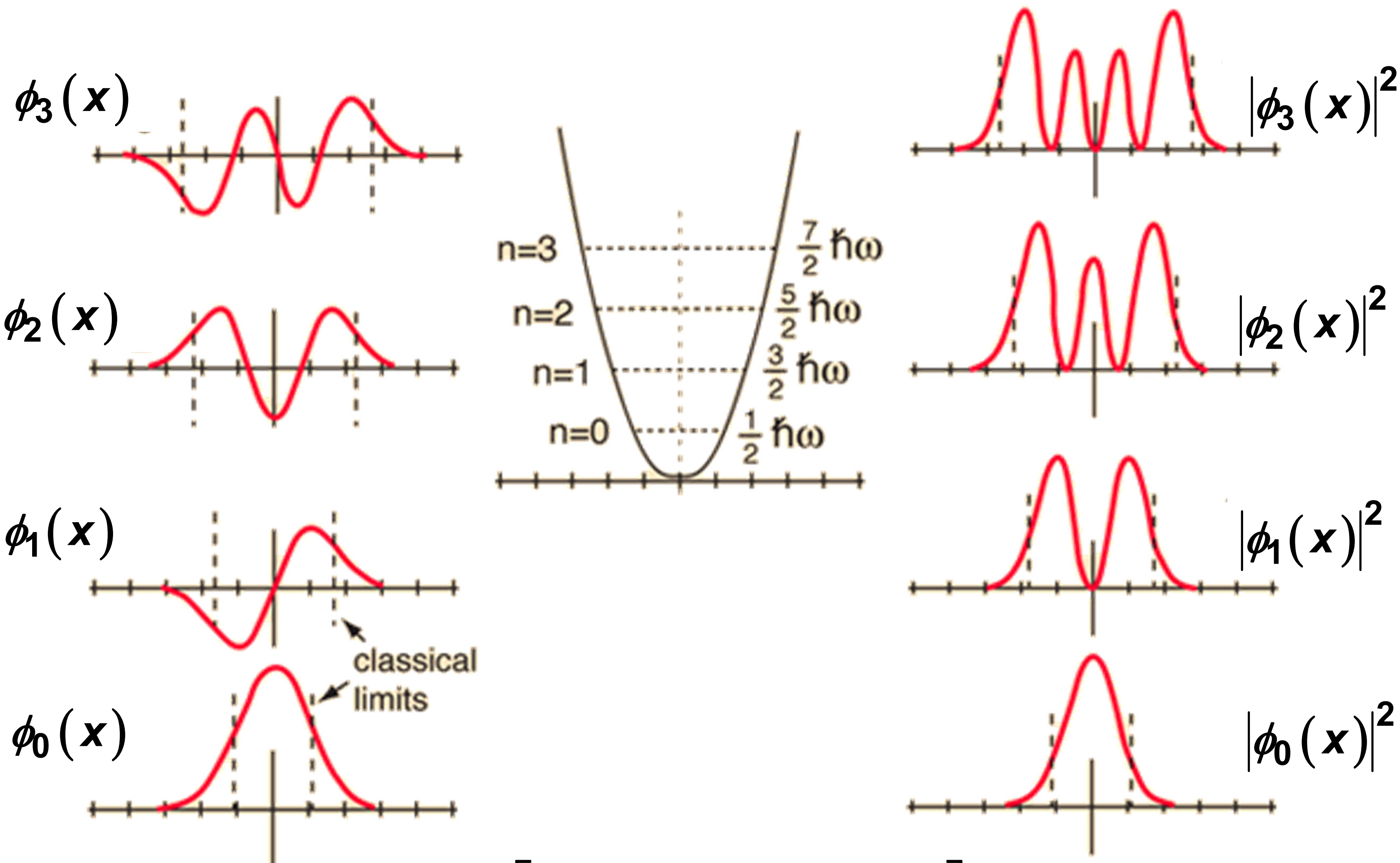
$$\Rightarrow \phi_n(x) = \frac{1}{\sqrt{n!}}\left[-\sqrt{\frac{\hbar}{2m\omega_o}}\frac{\partial}{\partial x} + \sqrt{\frac{m\omega_o}{2\hbar}}\,x\right]^n \phi_0(x)$$

$$\Rightarrow \phi_n(x) = \left(\frac{m\omega_o}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega_o}{2\hbar}x^2}\left[\frac{1}{\sqrt{2^n n!}}H_n\left(\sqrt{\frac{m\omega_o}{\hbar}}x\right)\right]$$

Hermite polynomials

Hermite-Gaussians

# Quantum SHO: Wavefunctions




$$\phi_n(x) = \phi_0(x)\left[\frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{\frac{m\omega_o}{\hbar}}x\right)\right]$$ Hermite-Gaussians

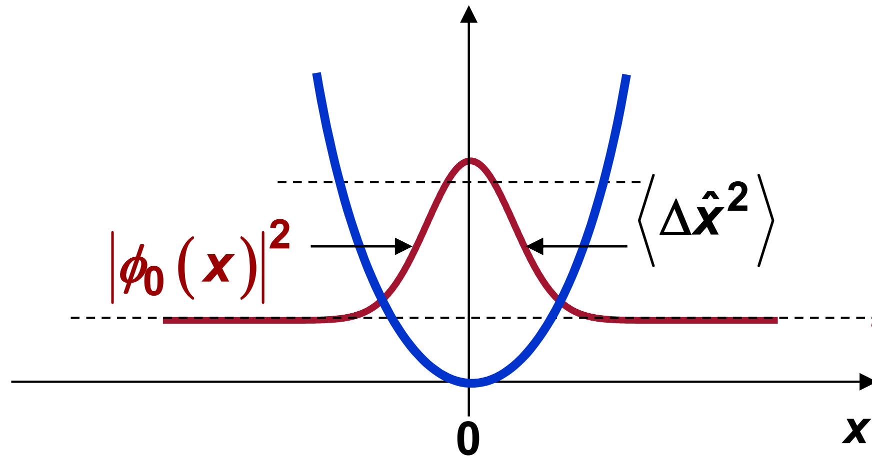
Note: Wavefunctions have even or odd parity

# Quantum SHO: Wavefunctions

**Wavefunction in position basis:**

$$\langle x|0\rangle = \phi_0(x) = \left(\frac{m\omega_o}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega_o}{2\hbar}x^2}$$

$$\Rightarrow \langle \Delta\hat{x}^2\rangle = \langle 0|\Delta\hat{x}^2|0\rangle = \frac{\hbar}{2m\omega_o}$$

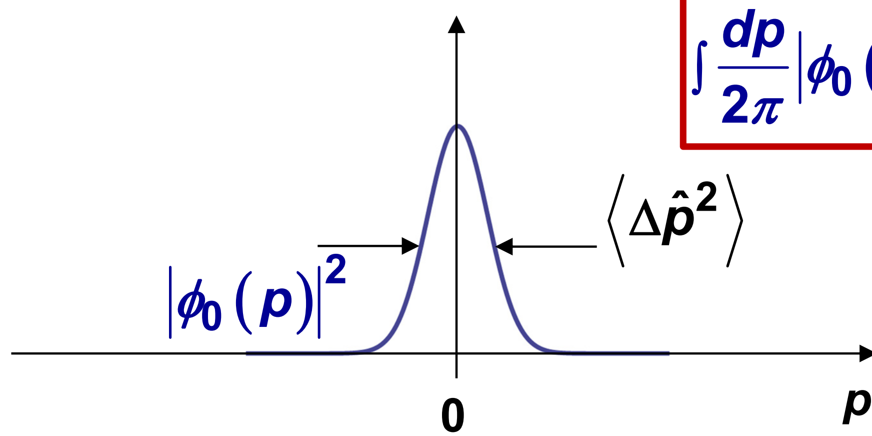



**Wavefunction in momentum basis:**

$$\langle p|0\rangle = \phi_0(p) = \int_{-\infty}^{\infty} dx\, \phi_0(x)\frac{e^{-i\frac{p}{\hbar}x}}{\sqrt{\hbar}} = \sqrt{2\pi}\left(\frac{1}{\pi m\hbar\omega_o}\right)^{\frac{1}{4}} e^{-\frac{1}{2m\hbar\omega_o}p^2}$$

$$\Rightarrow \langle \Delta\hat{p}^2\rangle = \langle 0|\Delta\hat{p}^2|0\rangle = \frac{m\hbar\omega_o}{2}$$

$$\int \frac{dp}{2\pi}|\phi_0(p)|^2 = 1$$




**Position-momentum uncertainty product:**

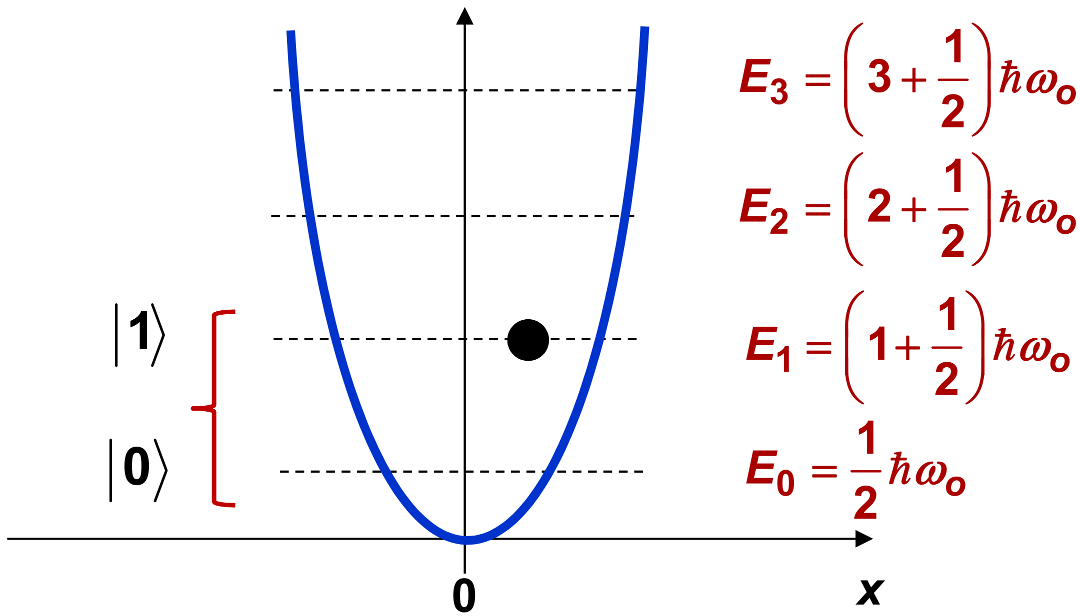
$$\Rightarrow \langle \Delta\hat{x}^2\rangle\langle \Delta\hat{p}^2\rangle = \frac{\hbar^2}{4}$$

Min value allowed by the Heisenberg relation

# A SHO as a Qubit

**Hamiltonian:**

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega_o^2 \hat{x}^2$$



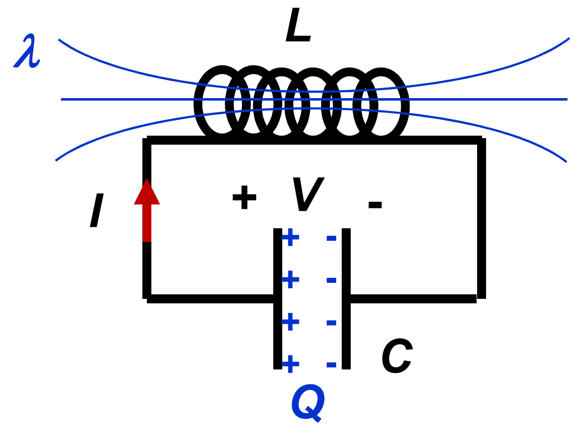

**Use the lowest two circuit states as your qubit !!**

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

**Problems:**

- All states have equal energy spacings
- This makes qubit operations impossible (one can accidentally put the circuit in one of the higher energy states during computation)
- Need to have just two energy levels (unless multilevel qubit logic is desired)
- How do we perform single qubit logic operations?

# A Lossless Superconducting *LC* Circuit




$$\left\{ \omega_o = \sqrt{\frac{1}{LC}} \right.$$

$$H = \frac{Q^2}{2C} + \frac{\lambda^2}{2L}$$

$$= \frac{Q^2}{2C} + \frac{1}{2} C\omega_o^2 \lambda^2$$

**Circuit equations:**

$$\frac{dQ}{dt} = -C\omega_o^2 \lambda$$

$$C\frac{d\lambda}{dt} = Q$$

**Compare with SHO:**

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega_o^2 x^2$$

$$\frac{dp}{dt} = -\frac{\partial V}{\partial x} = -m\omega_o^2 x$$

$$m\frac{dx}{dt} = p$$

$$Q \leftrightarrow p$$

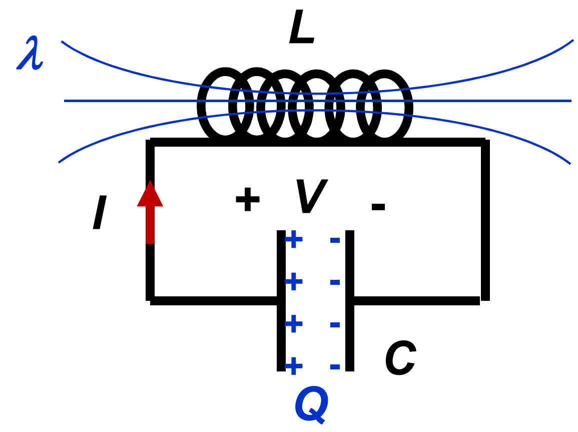
$$\lambda \leftrightarrow x$$

$$C \leftrightarrow m$$

$$\omega_o \leftrightarrow \omega_o$$

# A Quantum Lossless Superconducting *LC* Circuit

**The macroscopic quantum state of the circuit is described by a vector $|\psi(t)\rangle$ in a Hilbert space**




$$\omega_o = \sqrt{\frac{1}{LC}}$$

**Charge and flux, and voltage and current, are observables and the corresponding operators are:**

$$\hat{Q} = C\hat{V} \qquad \hat{\lambda} = L\hat{I}$$

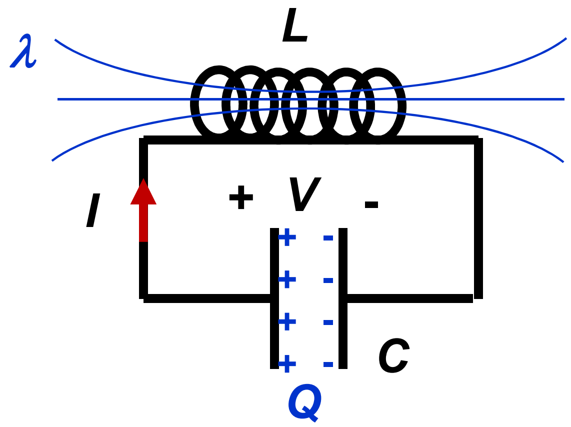
**A circuit has many measurable physical degrees of freedom:**

- **There are billions and billions of electrons and atoms in the wires, capacitor plates, etc. Each electron/atom has position, momentum, spin, etc, as observables. These are the microscopic degrees of freedom of the circuit.**

- **We are interested here in only the macroscopic electrical degrees of freedom of the circuit and will make a quantum description of only these degrees of freedom. That such a macroscopic quantum description is possible, without taking into account all the microscopic degrees of freedom, is quite remarkable.**

# A Quantum Lossless Superconducting *LC* Circuit

**The quantum state of the circuit is described by a vector $|\psi(t)\rangle$ in a Hilbert space**




$$\omega_o = \sqrt{\frac{1}{LC}}$$

**Charge and flux, and voltage and current, are observables and the corresponding operators are:**

$$\hat{Q} = C\hat{V} \qquad \hat{\lambda} = L\hat{I}$$

**The energy becomes the Hamiltonian operator :**

$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{1}{2}C\omega_o^2\hat{\lambda}^2$$

**Compare with:**

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_o^2\hat{x}^2$$

$$\langle x|p\rangle = \frac{e^{i\frac{p}{\hbar}x}}{\sqrt{\hbar}}$$

$$[\hat{x},\hat{p}] = i\hbar$$

**Postulate the following commutation relation:**

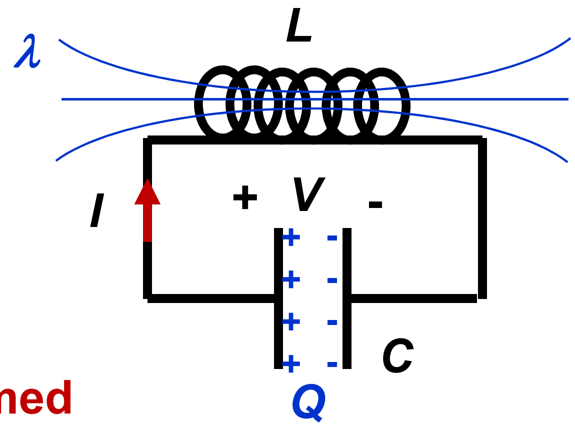
$$[\hat{\lambda},\hat{Q}] = i\hbar$$

$$\hat{Q} \leftrightarrow \hat{p}$$
$$\hat{\lambda} \leftrightarrow \hat{x}$$
$$C \leftrightarrow m$$
$$\omega_o \leftrightarrow \omega_o$$

# A Quantum Lossless Superconducting *LC* Circuit

Charge and flux, and voltage and current, being all observables, become operators

$$\hat{Q} = C\hat{V} \qquad \hat{\lambda} = L\hat{I}$$




$$\omega_o = \sqrt{\frac{1}{LC}}$$

Postulate complete basis states formed by charge and flux eigenstates:

$$\hat{\lambda}\,|\lambda\rangle = \lambda\,|\lambda\rangle \qquad \hat{Q}\,|Q\rangle = Q\,|Q\rangle$$

$$\int_{-\infty}^{+\infty} d\lambda\,|\lambda\rangle\langle\lambda| = \hat{1} \qquad \int_{-\infty}^{+\infty} \frac{dQ}{2\pi}|Q\rangle\langle Q| = \hat{1}$$

$$\left[\hat{\lambda},\hat{Q}\right] = i\hbar \implies \langle\lambda|Q\rangle = \frac{e^{i\frac{Q}{\hbar}\lambda}}{\sqrt{\hbar}}$$

Compare with:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_o^2\hat{x}^2$$

$$\langle x|p\rangle = \frac{e^{i\frac{p}{\hbar}x}}{\sqrt{\hbar}}$$

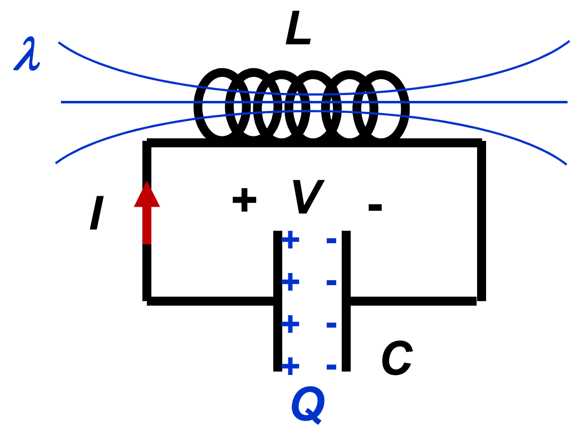
$$\left[\hat{x},\hat{p}\right] = i\hbar$$

$$\hat{Q} \leftrightarrow \hat{p}$$
$$\hat{\lambda} \leftrightarrow \hat{x}$$
$$C \leftrightarrow m$$
$$\omega_o \leftrightarrow \omega_o$$

Example:

If $|\psi\rangle = |\lambda\rangle$ then the inductor flux is certain but the capacitor charge is very uncertain (because the quantum state $|\psi\rangle$ is a superposition of different capacitor charge states)

# A Quantum Lossless Superconducting *LC* Circuit




$$\omega_o = \sqrt{\frac{1}{LC}}$$

**The energy becomes the Hamiltonian operator :**

$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{1}{2} C \omega_o^2 \hat{\lambda}^2$$

$$\left[ \hat{\lambda}, \hat{Q} \right] = i\hbar$$

**Compare with SHO:**

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega_o^2 \hat{x}^2$$

$$\left[ \hat{x}, \hat{p} \right] = i\hbar$$

$$\hat{a} = i\sqrt{\frac{1}{2m\hbar\omega_o}}\,\hat{p} + \sqrt{\frac{m\omega_o}{2\hbar}}\,\hat{x}$$

$$\hat{a}^\dagger = -i\sqrt{\frac{1}{2m\hbar\omega_o}}\,\hat{p} + \sqrt{\frac{m\omega_o}{2\hbar}}\,\hat{x}$$

**Define:**

$$\hat{a} = i\sqrt{\frac{1}{2C\hbar\omega_o}}\,\hat{Q} + \sqrt{\frac{C\omega_o}{2\hbar}}\,\hat{\lambda}$$

$$\hat{a}^\dagger = -i\sqrt{\frac{1}{2C\hbar\omega_o}}\,\hat{Q} + \sqrt{\frac{C\omega_o}{2\hbar}}\,\hat{\lambda}$$

$$\Rightarrow \left[ \hat{a}, \hat{a}^\dagger \right] = 1$$

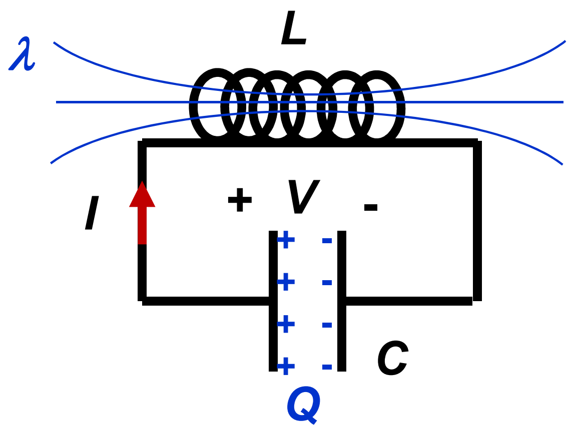
$$\hat{Q} \leftrightarrow \hat{p}$$

$$\hat{\lambda} \leftrightarrow \hat{x}$$

$$C \leftrightarrow m$$

$$\omega_o \leftrightarrow \omega_o$$

# A Quantum Lossless Superconducting *LC* Circuit




$$\left\{ \omega_o = \sqrt{\frac{1}{LC}} \right.$$

The Hamiltonian operator is:

$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{1}{2} C\omega_o^2 \hat{\lambda}^2$$

The Hamiltonian operator becomes:

$$\hat{H} = \hbar\omega_o\left(\hat{n} + \frac{1}{2}\right) = \hbar\omega_o\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

$$\left\{ \Rightarrow \left[\hat{a}, \hat{a}^\dagger\right] = 1 \right.$$

The eigenstates and eigenvalues are:

$$\hat{H}|n\rangle = \hbar\omega_o\left(\hat{n} + \frac{1}{2}\right)|n\rangle = \hbar\omega_o\left(n + \frac{1}{2}\right)|n\rangle = E_n|n\rangle \longrightarrow$$

$$\{ n = 0, 1, 2, 3 \ldots\ldots.$$

$$\vdots$$

$$E_3 = \left(3 + \frac{1}{2}\right)\hbar\omega_o$$

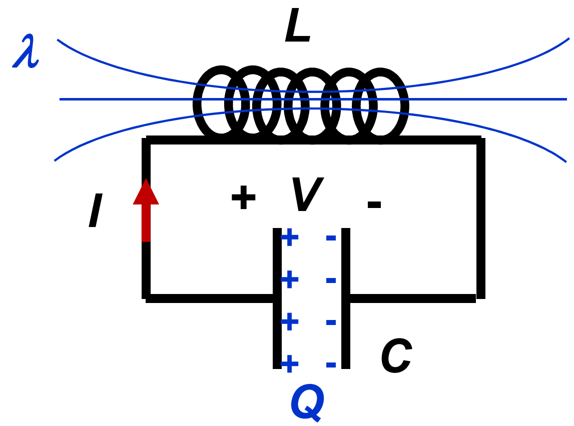
$$E_2 = \left(2 + \frac{1}{2}\right)\hbar\omega_o$$

$$E_1 = \left(1 + \frac{1}{2}\right)\hbar\omega_o$$

$$E_0 = \frac{1}{2}\hbar\omega_o$$

The circuit has been quantized !!!

# A Lossless Superconducting *LC* Circuit




**Suppose the circuit is in its lowest energy state:**

$$|\psi\rangle = |0\rangle$$

$$\hat{H}|0\rangle = \frac{1}{2}\hbar\omega_o|0\rangle$$

**Compare with SHO:**

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_o^2\hat{x}^2$$

$$\hat{Q} \leftrightarrow \hat{p}$$

$$\hat{\lambda} \leftrightarrow \hat{x}$$

$$C \leftrightarrow m$$

$$\omega_o \leftrightarrow \omega_o$$

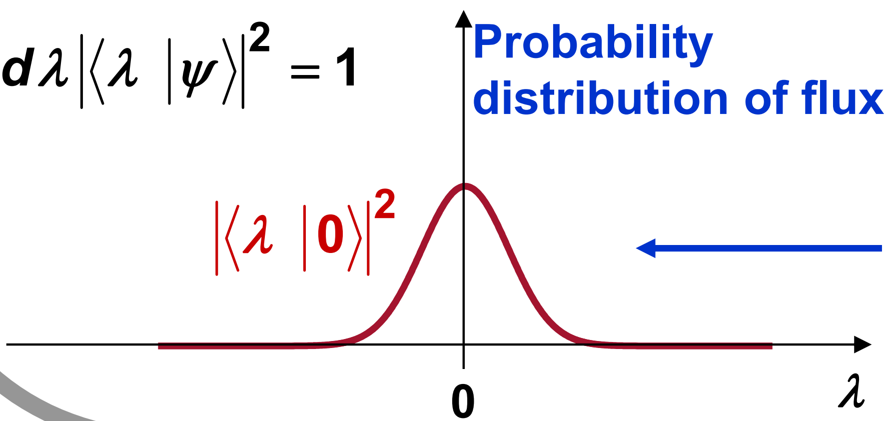
$$|\langle x|0\rangle|^2 = \left(\frac{m\omega_o}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega_o}{2\hbar}x^2}$$

**The inductor flux is measured in the circuit. What is the a-priori probability of finding the result "$\lambda$" :**

$$|\langle\lambda\,|\psi\rangle|^2 = |\langle\lambda\,|0\rangle|^2 = |\phi_0(\lambda\,)|^2 = \left(\frac{C\omega_o}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\frac{C\omega_o}{\hbar}\lambda^2}$$

$$\int d\lambda\,|\langle\lambda\,|\psi\rangle|^2 = 1$$

**Probability distribution of flux**



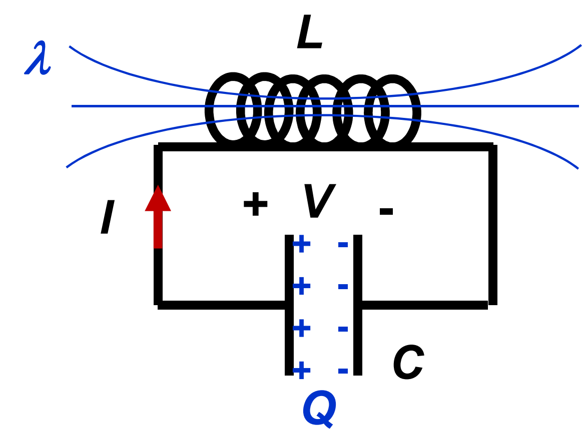

**Zero-point quantum fluctuations in flux**

# A Lossless Superconducting *LC* Circuit

**Suppose the circuit is in its lowest energy state:**

$$|\psi\rangle = |0\rangle$$

$$\hat{H}|0\rangle = \frac{1}{2}\hbar\omega_o|0\rangle$$




**The inductor current is measured in the circuit. What is the a-priori probability of finding the result "*I*" :**

**Since the inductor current and inductor flux operators are related by a constant:** $\hat{\lambda} = L\hat{I}$

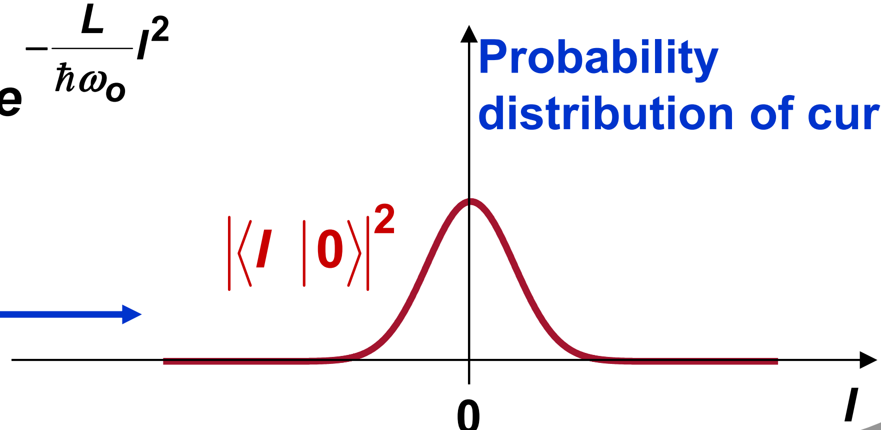
$$|I\rangle = \sqrt{L}\,|\lambda = LI\rangle$$

**The probability distribution function for the current will be:**

$$|\langle I\,|\psi\rangle|^2 = L|\langle\lambda\,|\psi\rangle|^2\Big|_{\lambda = LI} = \left(\frac{L}{\pi\hbar\omega_o}\right)^{\frac{1}{2}} e^{-\frac{L}{\hbar\omega_o}I^2}$$

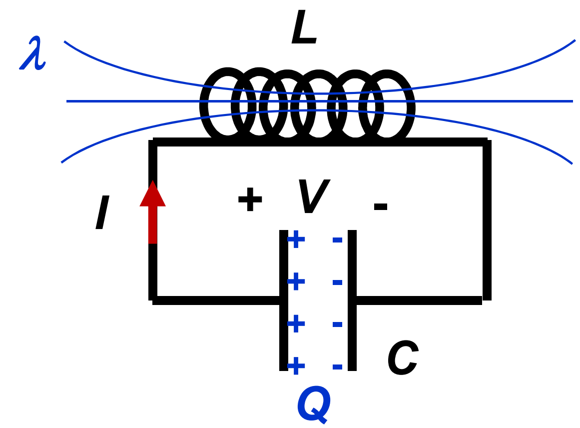
$$\int dI\,|\langle I\,|\psi\rangle|^2 = 1$$

**Zero-point quantum fluctuations in current**



# A Lossless Superconducting *LC* Circuit

**Suppose the circuit is in its lowest energy state:**




$$|\psi\rangle = |0\rangle$$

$$\hat{H}|0\rangle = \frac{1}{2}\hbar\omega_o|0\rangle$$

**The capacitor voltage is measured in the circuit. What is the a-priori probability of finding the result "*V*" :**

**Since the voltage and the charge operators are related by a constant:** $\hat{Q} = C\hat{V}$

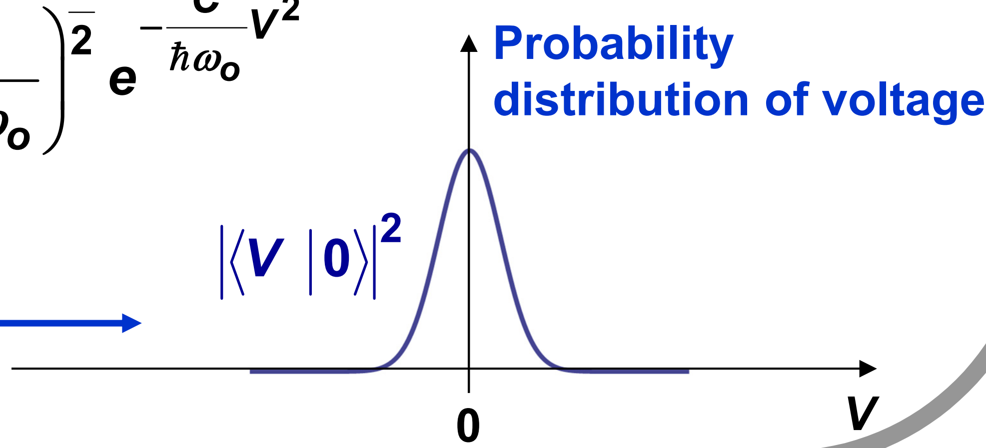
$$|V\rangle = \sqrt{C}\,|Q = CV\rangle$$

**The probability distribution function for the voltage will be:**

$$|\langle V|\psi\rangle|^2 = C\,|\langle Q|\psi\rangle|^2\Big|_{Q=CV} = 2\pi\left(\frac{C}{\pi\hbar\omega_o}\right)^{\frac{1}{2}} e^{-\frac{C}{\hbar\omega_o}V^2}$$

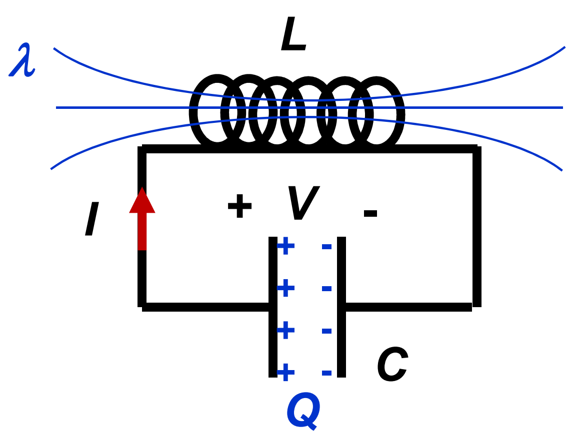
$$\int\frac{dV}{2\pi}|\langle V|\psi\rangle|^2 = 1$$

**Zero-point quantum fluctuations in voltage**

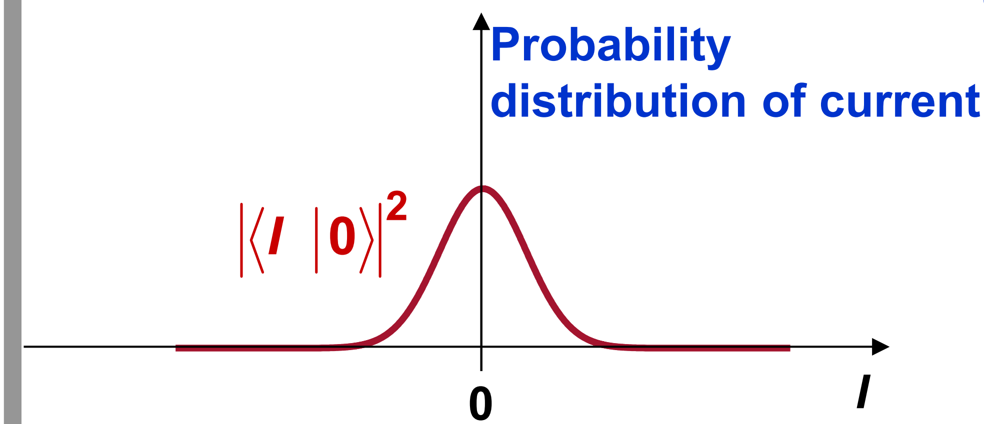


# A Lossless Superconducting *LC* Circuit

Suppose the circuit is in its lowest energy state:

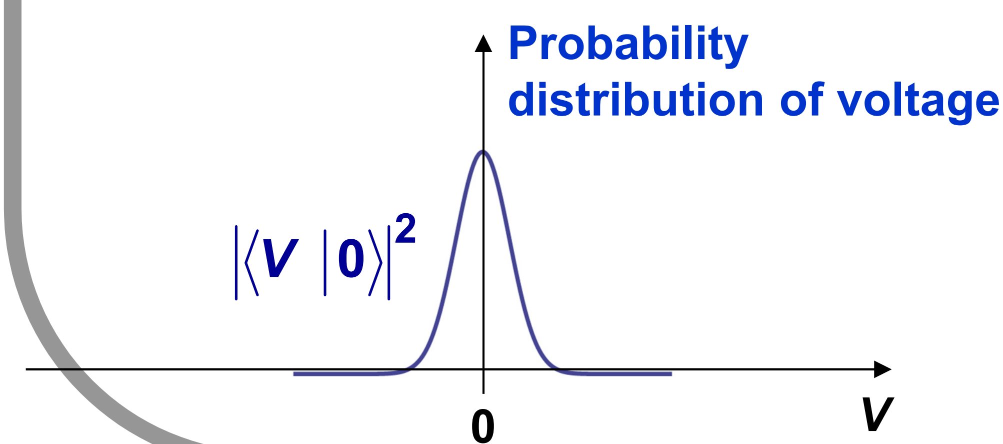



$$\omega_o = \sqrt{\frac{1}{LC}} = 10\ \text{GHz}$$
$$= 2\pi 10^{10}\ \text{rad/s}$$
$$L = 0.25\ \text{nH}$$
$$C = 1\ \text{pF}$$




RMS zero-point current quantum **fluctuations**?:

$$\sigma_I = \sqrt{\langle \Delta \hat{I}^2 \rangle} = \sqrt{\frac{\hbar\omega_o}{2L}} \sim 0.12\ \mu A$$

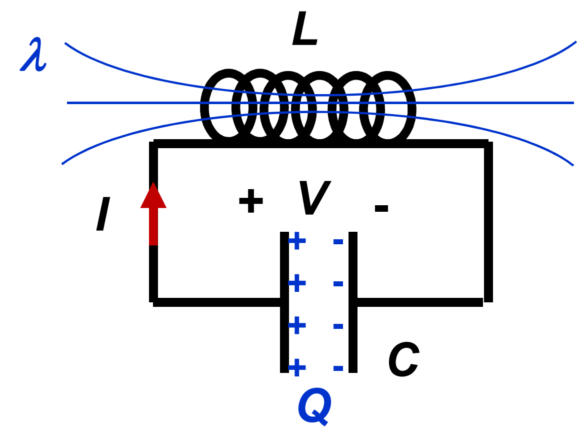



RMS zero-point voltage quantum **fluctuations**?:

$$\sigma_V = \sqrt{\langle \Delta \hat{V}^2 \rangle} = \sqrt{\frac{\hbar\omega_o}{2C}} \sim 1.8\ \mu V$$

# A Lossless Superconducting *LC* Circuit: Commutation Relations

**Voltage and current are non-commuting operators:**




$$\hat{Q} = C\hat{V} \qquad \hat{\lambda} = L\hat{I}$$

$$\left[\hat{\lambda}, \hat{Q}\right] = i\hbar$$

$$\Rightarrow \left[\hat{I}, \hat{V}\right] = \frac{i\hbar}{LC}$$

**Accurate simultaneous measurement of both the charge and flux is not possible**

**Accurate simultaneous measurement of both the current and voltage is not possible**

$$\Rightarrow \left[\hat{I}, \hat{V}\right] = \frac{i\hbar}{LC}$$

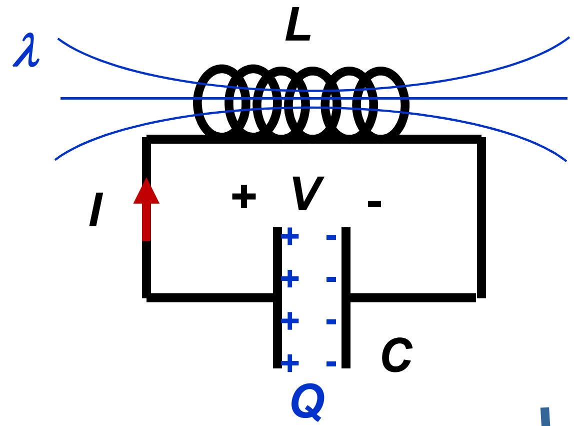
$$\left\langle \Delta \hat{I}^2 \right\rangle \left\langle \Delta \hat{V}^2 \right\rangle \geq \frac{\hbar^2}{4L^2C^2} = \frac{\hbar^2 \omega_o^2}{4LC} = \frac{\hbar^2 \omega_o^4}{4}$$

**For the ground state:**

$$\left\langle \Delta \hat{I}^2 \right\rangle = \frac{\hbar \omega_o}{2L}$$

$$\left\langle \Delta \hat{V}^2 \right\rangle = \frac{\hbar \omega_o}{2C}$$

# A Lossless Superconducting *LC* Circuit as a Qubit

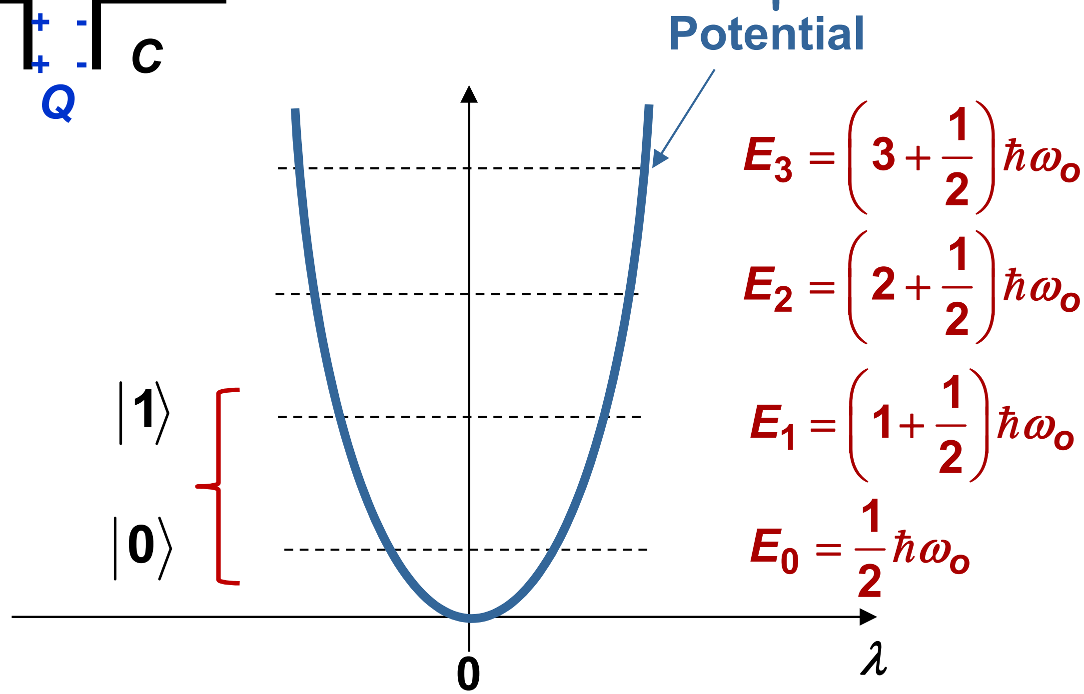



**Circuit Hamiltonian:**

$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{1}{2} C \omega_o^2 \hat{\lambda}^2$$

$$\hat{H} = \hbar\omega_o \left( \hat{n} + \frac{1}{2} \right) = \hbar\omega_o \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right)$$

$$\hat{H} = \frac{\hat{Q}^2}{2C} + \underbrace{\frac{1}{2} C \omega_o^2 \hat{\lambda}^2}_{\text{Potential}}$$



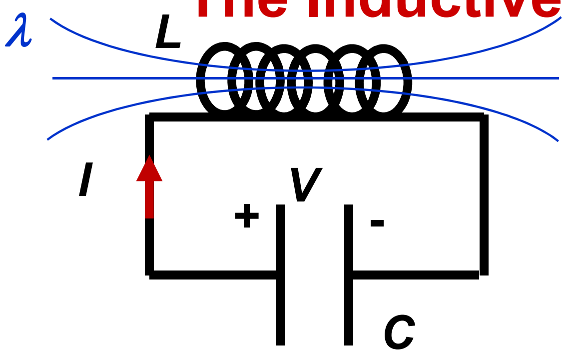

**Use the lowest two circuit states as your qubit !!**

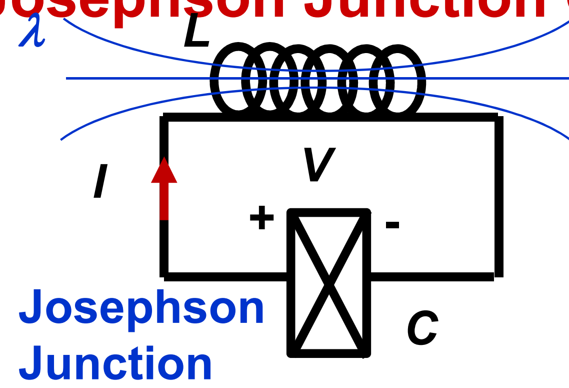
$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

**Problem:**

- All states have equal energy spacings
- This makes qubit operations impossible (one can accidentally put the circuit in one of the higher energy states during computation)
- Need to have just two energy levels (unless multilevel qubit logic is desired)
- How do we perform single qubit logic operations?

# The Inductively Shunted Josephson Junction Qubit



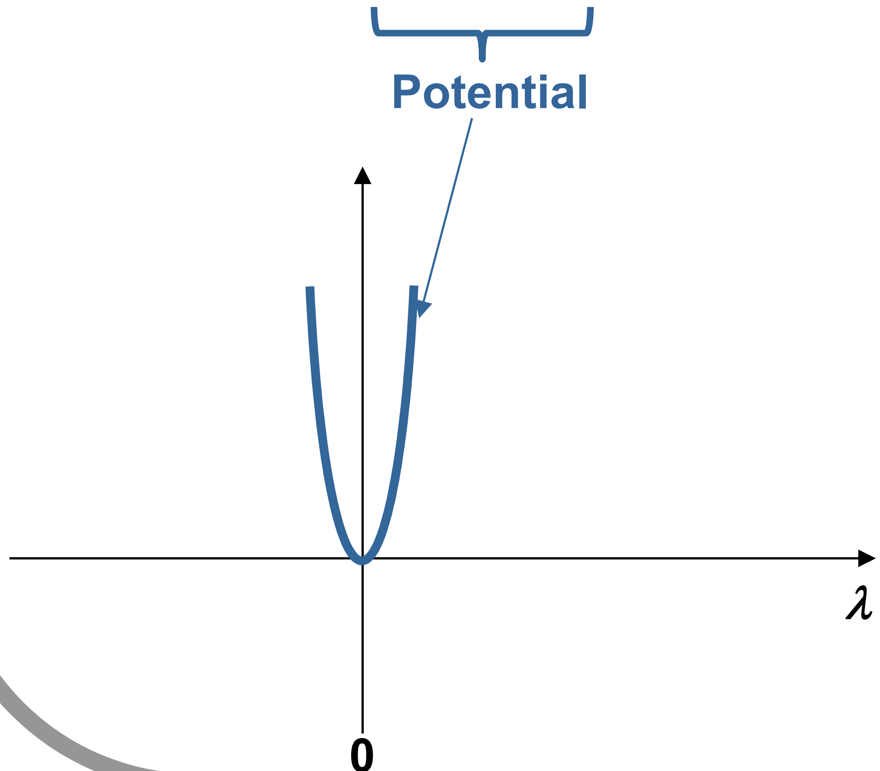

**Flux quantum**

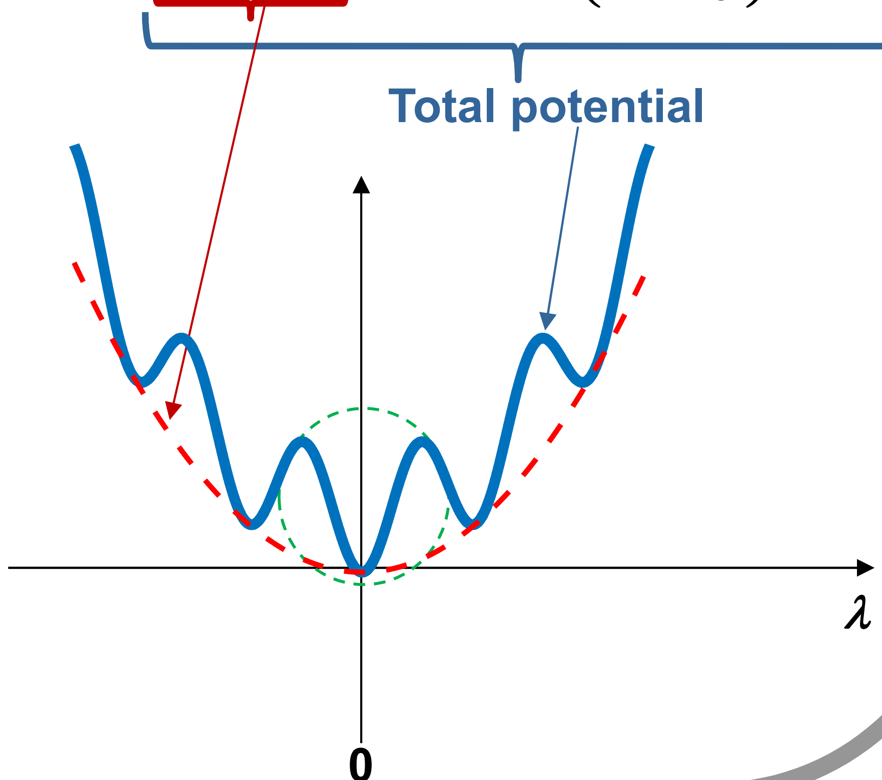
$$\lambda_o = \frac{\pi\hbar}{e}$$

**Circuit Hamiltonian:**

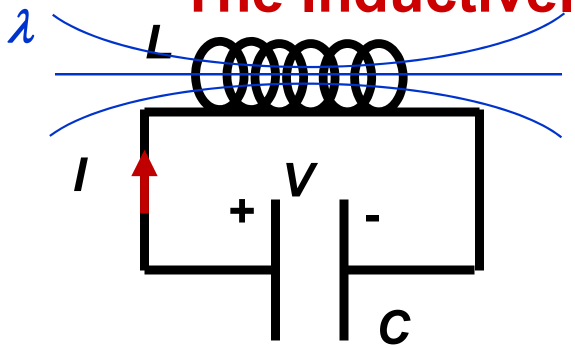
$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{1}{2}C\omega_o^2\hat{\lambda}^2$$

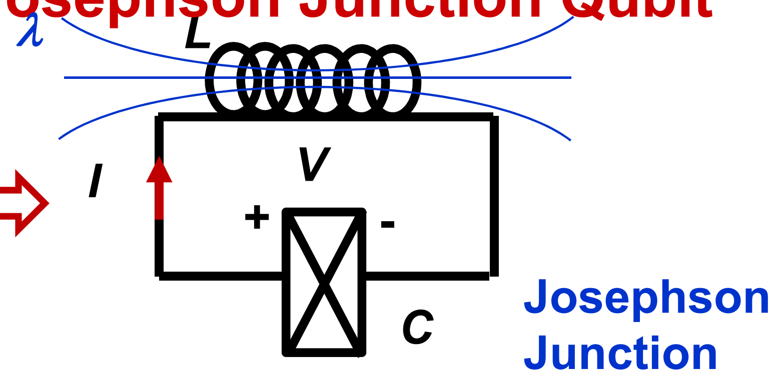
$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{1}{2}C\omega_o^2\hat{\lambda}^2 - E_J \cos\left(2\pi\frac{\hat{\lambda}}{\lambda_o}\right) + E_J$$



# The Inductively Shunted Josephson Junction Qubit

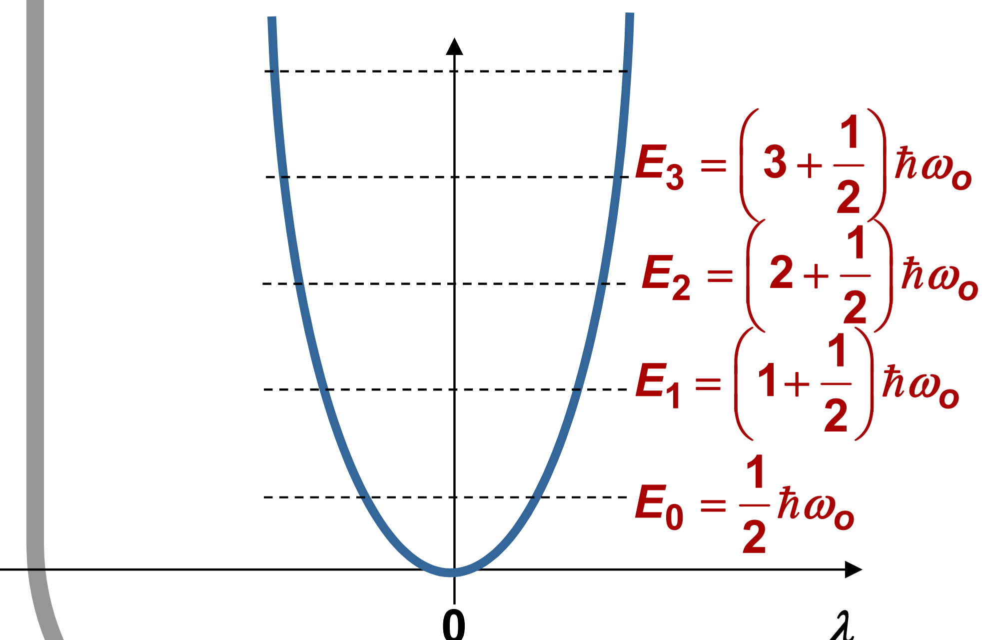


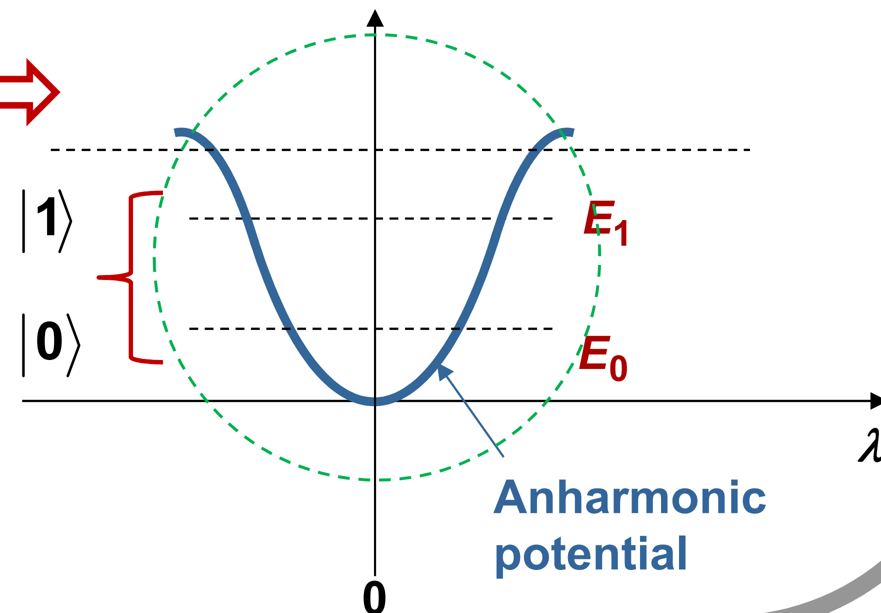

**Circuit Hamiltonian:**

$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{1}{2} C\omega_o^2 \hat{\lambda}^2$$

$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{1}{2} C\omega_o^2 \hat{\lambda}^2 - E_J \cos\left(2\pi \frac{\hat{\lambda}}{\lambda_o}\right) + E_J$$



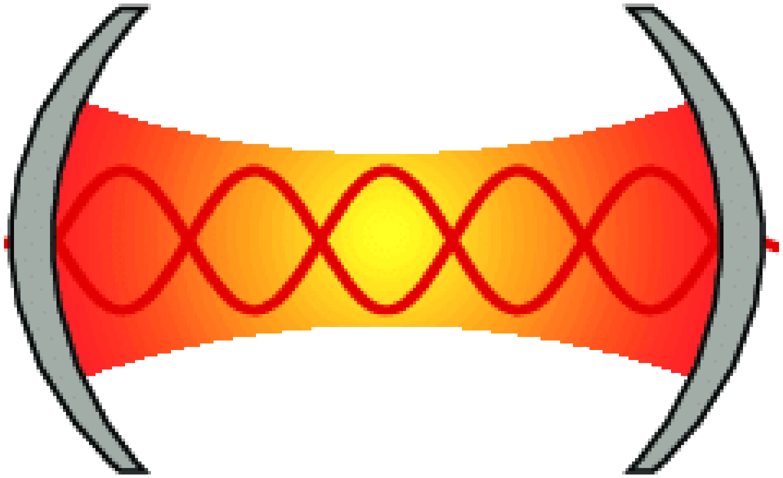

**Only two confined energy levels near the potential minimum. Problem solved !**



# An Electromagnetic Mode in a Photonic Cavity

$$\vec{E}(\vec{r},t) = \frac{q_E(t)}{\sqrt{\mu_o \varepsilon_o}} \vec{U}(\vec{r})$$

$$\vec{H}(\vec{r},t) = -\frac{q_H(t)}{\sqrt{\mu_o \varepsilon_o}} \nabla \times \vec{U}(\vec{r})$$



**Wave equation:**

$$\nabla \times \nabla \times \vec{U}(\vec{r}) = \frac{\omega_o^2}{c^2} \vec{U}(\vec{r})$$

$$H = \frac{q_E^2(t)}{2\mu_o} + \frac{1}{2}\mu_o \omega_o^2 q_H^2(t)$$

$$\frac{dq_E(t)}{dt} = -\mu_o \omega_o^2 q_H(t)$$

$$\mu_o \frac{dq_H(t)}{dt} = q_E(t)$$

**Compare with:**

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega_o^2 x^2$$

$$\frac{dp}{dt} = -\frac{\partial V}{\partial x} = -m\omega_o^2 x$$

$$m \frac{dx}{dt} = p$$

$$q_E \leftrightarrow p$$

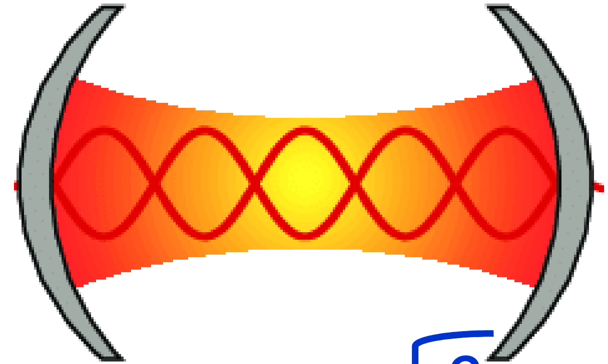
$$q_H \leftrightarrow x$$

$$\mu_o \leftrightarrow m$$

$$\omega_o \leftrightarrow \omega_o$$

# A Quantized Electromagnetic Mode in a Photonic Cavity

The quantum state of the electromagnetic mode is described by a vector $\left|\psi(t)\right\rangle$ in a Hilbert space



Electric and magnetic field of the mode are observables and the corresponding operators are:

$$\hat{\vec{E}}(\vec{r}) = \frac{\hat{q}_E}{\sqrt{\mu_o \varepsilon_o}} \vec{U}(\vec{r})$$

$$\hat{\vec{H}}(\vec{r}) = -\frac{\hat{q}_H}{\sqrt{\mu_o \varepsilon_o}} \nabla \times \vec{U}(\vec{r})$$

The energy becomes the Hamiltonian operator :

$$\hat{H} = \frac{\hat{q}_E^2}{2\mu_o} + \frac{1}{2}\mu_o \omega_o^2 \hat{q}_H^2$$

Postulate the following commutation relation:

$$\left[\hat{q}_H, \hat{q}_E\right] = i\hbar$$

Compare with:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega_o^2 \hat{x}^2$$

$$\left[\hat{x}, \hat{p}\right] = i\hbar$$

$$\left\langle x \middle| p \right\rangle = \frac{e^{i\frac{p}{\hbar}x}}{\sqrt{\hbar}}$$

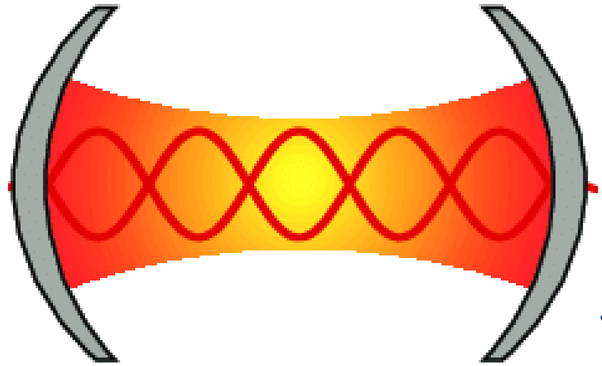
$$\hat{q}_E \leftrightarrow \hat{p}$$
$$\hat{q}_H \leftrightarrow \hat{x}$$
$$\mu_o \leftrightarrow m$$
$$\omega_o \leftrightarrow \omega_o$$

# A Quantized Electromagnetic Mode in a Photonic Cavity

**Fields become operators and so do $q_E$ and $q_H$:**

$$\hat{\vec{E}}(\vec{r}) = \frac{\hat{q}_E}{\sqrt{\mu_o \varepsilon_o}} \vec{U}(\vec{r})$$

$$\hat{\vec{H}}(\vec{r}) = -\frac{\hat{q}_H}{\sqrt{\mu_o \varepsilon_o}} \nabla \times \vec{U}(\vec{r})$$



**Compare with:**

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega_o^2 \hat{x}^2$$

$$\left[\hat{x}, \hat{p}\right] = i\hbar$$

$$\langle x | p \rangle = \frac{e^{i\frac{p}{\hbar}x}}{\sqrt{\hbar}}$$

**Postulate complete basis states formed by $q_E$ and $q_H$ eigenstates:**

$$\hat{q}_H |q_H\rangle = q_H |q_H\rangle \qquad \hat{q}_E |q_E\rangle = q_E |q_E\rangle$$

$$\int_{-\infty}^{+\infty} dq_H |q_H\rangle\langle q_H| = \hat{1} \qquad \int_{-\infty}^{+\infty} \frac{dq_E}{2\pi} |q_E\rangle\langle q_E| = \hat{1}$$

$$\left[\hat{q}_H, \hat{q}_E\right] = i\hbar \implies \langle q_H | q_E \rangle = \frac{e^{i\frac{q_E}{\hbar}q_H}}{\sqrt{\hbar}}$$

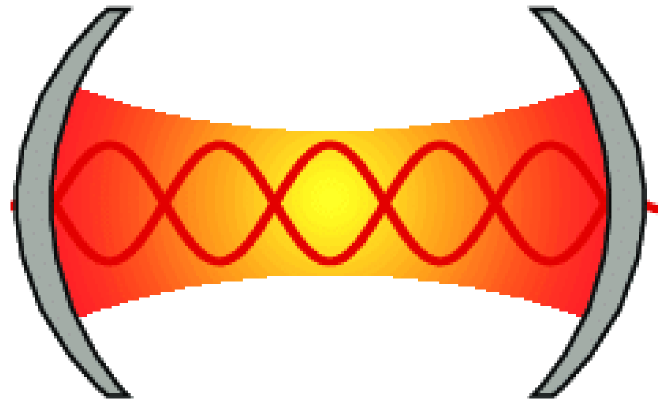
$$\hat{q}_E \leftrightarrow \hat{p}$$
$$\hat{q}_H \leftrightarrow \hat{x}$$
$$\mu_o \leftrightarrow m$$
$$\omega_o \leftrightarrow \omega_o$$

# A Quantized Electromagnetic Mode in a Photonic Cavity

**Fields become operators:**

$$\hat{\vec{E}}(\vec{r}) = \frac{\hat{q}_E}{\sqrt{\mu_o \varepsilon_o}} \vec{U}(\vec{r})$$

$$\hat{\vec{H}}(\vec{r}) = -\frac{\hat{q}_H}{\sqrt{\mu_o \varepsilon_o}} \nabla \times \vec{U}(\vec{r})$$



**The energy becomes the Hamiltonian operator :**

$$\hat{H} = \frac{\hat{q}_E^2}{2\mu_o} + \frac{1}{2}\mu_o \omega_o^2 \hat{q}_H^2$$

**Define:**

$$\hat{a} = i\sqrt{\frac{1}{2\mu_o \hbar \omega_o}}\hat{q}_E + \sqrt{\frac{\mu_o \omega_o}{2\hbar}}\;\hat{q}_H$$

$$\hat{a}^\dagger = -i\sqrt{\frac{1}{2\mu_o \hbar \omega_o}}\hat{q}_E + \sqrt{\frac{\mu_o \omega_o}{2\hbar}}\;\hat{q}_H$$

$$\Rightarrow \left[\hat{a}, \hat{a}^\dagger\right] = 1$$

**Compare with SHO:**

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega_o^2 \hat{x}^2$$

$$\left[\hat{x}, \hat{p}\right] = i\hbar$$

$$\hat{a} = i\sqrt{\frac{1}{2m\hbar\omega_o}}\hat{p} + \sqrt{\frac{m\omega_o}{2\hbar}}\;\hat{x}$$

$$\hat{a}^\dagger = -i\sqrt{\frac{1}{2m\hbar\omega_o}}\hat{p} + \sqrt{\frac{m\omega_o}{2\hbar}}\;\hat{x}$$

$$\hat{q}_E \leftrightarrow \hat{p}$$

$$\hat{q}_H \leftrightarrow \hat{x}$$

$$\mu_o \leftrightarrow m$$

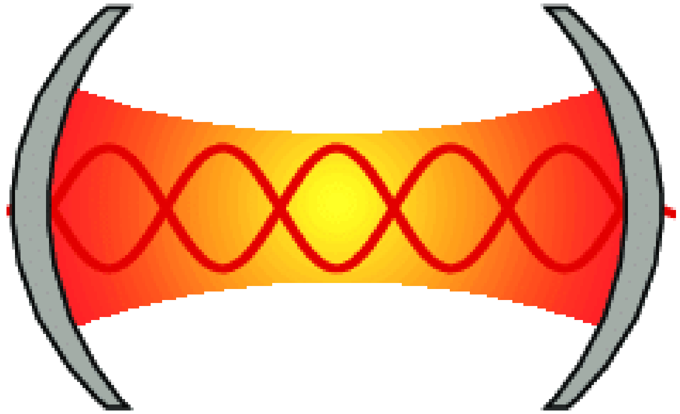
$$\omega_o \leftrightarrow \omega_o$$

# A Quantized Electromagnetic Mode in a Photonic Cavity

**Fields become operators:**

$$\hat{\vec{E}}(\vec{r}) = \frac{\hat{q}_E}{\sqrt{\mu_o \varepsilon_o}} \vec{U}(\vec{r})$$

$$\hat{\vec{H}}(\vec{r}) = -\frac{\hat{q}_H}{\sqrt{\mu_o \varepsilon_o}} \nabla \times \vec{U}(\vec{r})$$



**Compare with SHO:**

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega_o^2 \hat{x}^2$$

$$[\hat{x}, \hat{p}] = i\hbar$$

$$\hat{a} = i\sqrt{\frac{1}{2m\hbar\omega_o}}\,\hat{p} + \sqrt{\frac{m\omega_o}{2\hbar}}\,\hat{x}$$

$$\hat{a}^\dagger = -i\sqrt{\frac{1}{2m\hbar\omega_o}}\,\hat{p} + \sqrt{\frac{m\omega_o}{2\hbar}}\,\hat{x}$$

**The energy becomes the Hamiltonian operator :**

$$\hat{H} = \frac{\hat{q}_E^2}{2\mu_o} + \frac{1}{2}\mu_o \omega_o^2 \hat{q}_H^2$$

**The Hamiltonian operator becomes:**

$$\hat{H} = \hbar\omega_o\left(\hat{n} + \frac{1}{2}\right) = \hbar\omega_o\left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right)$$

$$\hat{q}_E \leftrightarrow \hat{p}$$
$$\hat{q}_H \leftrightarrow \hat{x}$$
$$\mu_o \leftrightarrow m$$
$$\omega_o \leftrightarrow \omega_o$$

**The eigenstates and eigenvalues are:**

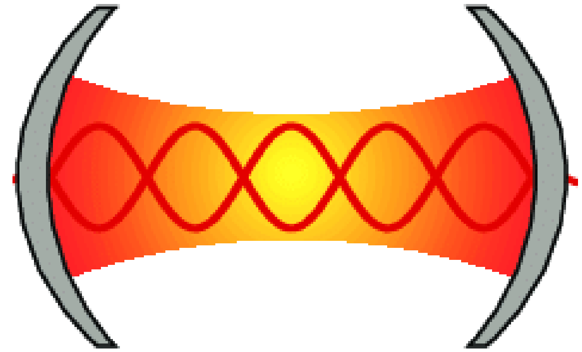
$$\hat{H}|n\rangle = \hbar\omega_o\left(\hat{n} + \frac{1}{2}\right)|n\rangle = \hbar\omega_o\left(n + \frac{1}{2}\right)|n\rangle$$

$$\{n = 0, 1, 2, 3........$$

**The electromagnetic mode has quantized energies!!!**

# A Quantized Electromagnetic Mode in a Photonic Cavity

**The eigenstates and eigenvalues are:**



$$\hat{H}|n\rangle = \hbar\omega_o\left(\hat{n}+\frac{1}{2}\right)|n\rangle = \hbar\omega_o\left(n+\frac{1}{2}\right)|n\rangle$$

$|0\rangle \Longrightarrow \frac{1}{2}\hbar\omega_o$ — **State with no photons (vacuum!!)**

$\Rightarrow |1\rangle = \hat{a}^\dagger|0\rangle \Longrightarrow \left(\frac{1}{2}+1\right)\hbar\omega_o$ — **State with one photon**

$\Rightarrow |2\rangle = \frac{\left(\hat{a}^\dagger\right)^2}{\sqrt{2!}}|0\rangle \Longrightarrow \left(\frac{1}{2}+2\right)\hbar\omega_o$ — **State with two photons**

⋮

$\Rightarrow |n\rangle = \frac{\left(\hat{a}^\dagger\right)^n}{\sqrt{n!}}|0\rangle \Longrightarrow \left(\frac{1}{2}+n\right)\hbar\omega_o$ — **State with *n* photon**

**Why does a state with no photons – the vacuum - has energy $\frac{1}{2}\hbar\omega_o$ ?**

# The Field Operators; Quantum Field Theory

**Fields become operators:**

$$\hat{\vec{E}}(\vec{r}) = \frac{\hat{q}_E}{\sqrt{\mu_o \varepsilon_o}} \vec{U}(\vec{r}) \qquad \hat{\vec{H}}(\vec{r}) = -\frac{\hat{q}_H}{\sqrt{\mu_o \varepsilon_o}} \nabla \times \vec{U}(\vec{r})$$

**Where:**

$$\hat{\vec{E}}(\vec{r}) = \sqrt{\frac{\hbar \omega_o}{2\varepsilon_o}} \left( \frac{\hat{a} - \hat{a}^\dagger}{i} \right) \vec{U}(\vec{r})$$

$$\hat{\vec{H}}(\vec{r}) = -\frac{1}{\mu_o} \sqrt{\frac{\hbar}{2\varepsilon_o \omega_o}} \left( \hat{a} + \hat{a}^\dagger \right) \nabla \times \vec{U}(\vec{r})$$

$$\left\{ \begin{aligned} \hat{a} &= i\sqrt{\frac{1}{2\mu_o \hbar \omega_o}} \hat{q}_E + \sqrt{\frac{\mu_o \omega_o}{2\hbar}}\, \hat{q}_H \\ \hat{a}^\dagger &= -i\sqrt{\frac{1}{2\mu_o \hbar \omega_o}} \hat{q}_E + \sqrt{\frac{\mu_o \omega_o}{2\hbar}}\, \hat{q}_H \end{aligned} \right.$$

**Mean value of the fields in the zero photon number state:**

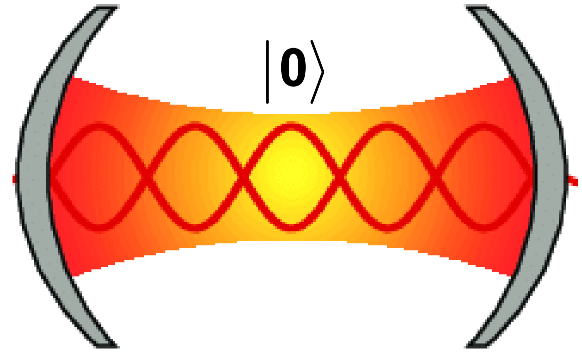
$$\langle 0 | \hat{\vec{E}}(\vec{r}) | 0 \rangle = \langle 0 | \hat{\vec{H}}(\vec{r}) | 0 \rangle = 0$$

**Mean value of the fields in any photon number state:**

$$\langle n | \hat{\vec{E}}(\vec{r}) | n \rangle = \langle n | \hat{\vec{H}}(\vec{r}) | n \rangle = 0$$

# A Quantized Electromagnetic Mode: Vacuum Fluctuations

$|0\rangle$ : Why does a state with no photons has energy $\frac{1}{2}\hbar\omega_o$?




The Hamiltonian operator is:

$$\hat{H} = \int d^3\vec{r}\left[\frac{1}{2}\varepsilon_o\hat{\vec{E}}(\vec{r}).\hat{\vec{E}}(\vec{r}) + \frac{1}{2}\mu_o\hat{\vec{H}}(\vec{r}).\hat{\vec{H}}(\vec{r})\right]$$

$$= \frac{\hat{q}_E^2}{2\mu_o} + \frac{1}{2}\mu_o\omega_o^2\hat{q}_H^2 = \hbar\omega_o\left(\hat{n} + \frac{1}{2}\right)$$

$$\langle 0|\hat{H}|0\rangle = \langle 0|\int d^3\vec{r}\left[\frac{1}{2}\varepsilon_o\hat{\vec{E}}(\vec{r}).\hat{\vec{E}}(\vec{r}) + \frac{1}{2}\mu_o\hat{\vec{H}}(\vec{r}).\hat{\vec{H}}(\vec{r})\right]|0\rangle$$

$$= \langle 0|\frac{\hat{q}_E^2}{2\mu_o} + \frac{1}{2}\mu_o\omega_o^2\hat{q}_H^2|0\rangle = \langle 0|\hbar\omega_o\left(\hat{n} + \frac{1}{2}\right)|0\rangle = \frac{1}{2}\hbar\omega_o$$

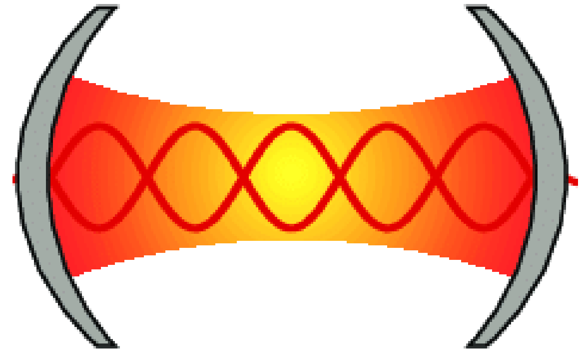
The vacuum is not exactly a vacuum!!

It has fluctuating electric and magnetic fields!!

# A Quantized Electromagnetic Mode: Vacuum Fluctuations

$$\hat{H} = \int d^3\vec{r}\left[\frac{1}{2}\varepsilon_o\hat{\vec{E}}(\vec{r}).\hat{\vec{E}}(\vec{r}) + \frac{1}{2}\mu_o\hat{\vec{H}}(\vec{r}).\hat{\vec{H}}(\vec{r})\right]$$

$$= \frac{\hat{q}_E^2}{2\mu_o} + \frac{1}{2}\mu_o\omega_o^2\hat{q}_H^2 = \hbar\omega_o\left(\hat{n} + \frac{1}{2}\right)$$



$$\hat{\vec{E}}(\vec{r}) = \frac{\hat{q}_E}{\sqrt{\mu_o\varepsilon_o}}\vec{U}(\vec{r})$$

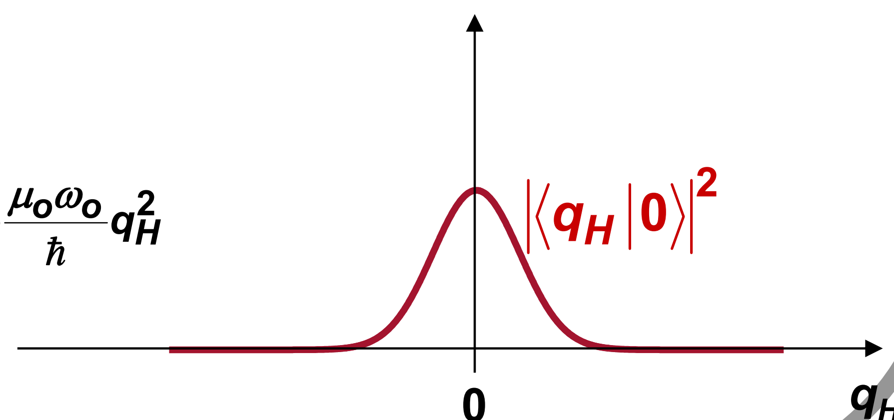
$$\hat{\vec{H}}(\vec{r}) = -\frac{\hat{q}_H}{\sqrt{\mu_o\varepsilon_o}}\nabla\times\vec{U}(\vec{r})$$

**Suppose the quantum state of the mode was $|0\rangle$**

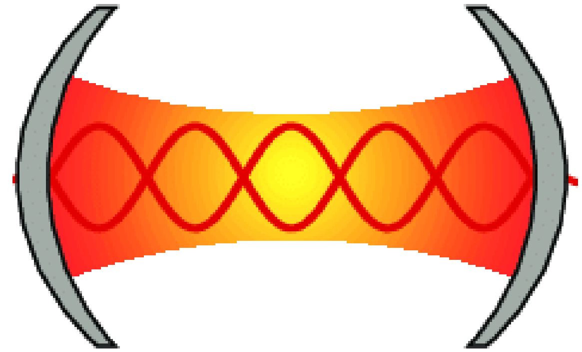
**If a measurement is made of the magnetic field amplitude what is the a-priori probability of measuring $q_H$ ??**

**Answer:**

$$\left|\langle q_H|\psi\rangle\right|^2 = \left|\langle q_H|0\rangle\right|^2 = \left|\phi_0(q_H)\right|^2 = \left(\frac{\mu_o\omega_o}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\frac{\mu_o\omega_o}{\hbar}q_H^2}$$



# A Quantized Electromagnetic Mode: Commutation Relations



The commutation relation between the field amplitudes is:

$$\left[\hat{q}_H, \hat{q}_E\right] = i\hbar$$

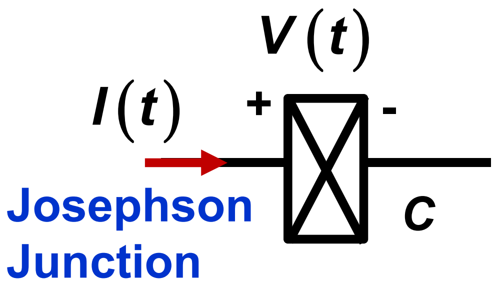
$$\begin{cases} \hat{\vec{E}}(\vec{r}) = \dfrac{\hat{q}_E}{\sqrt{\mu_o \varepsilon_o}} \vec{U}(\vec{r}) \\ \hat{\vec{H}}(\vec{r}) = -\dfrac{\hat{q}_H}{\sqrt{\mu_o \varepsilon_o}} \nabla \times \vec{U}(\vec{r}) \end{cases}$$

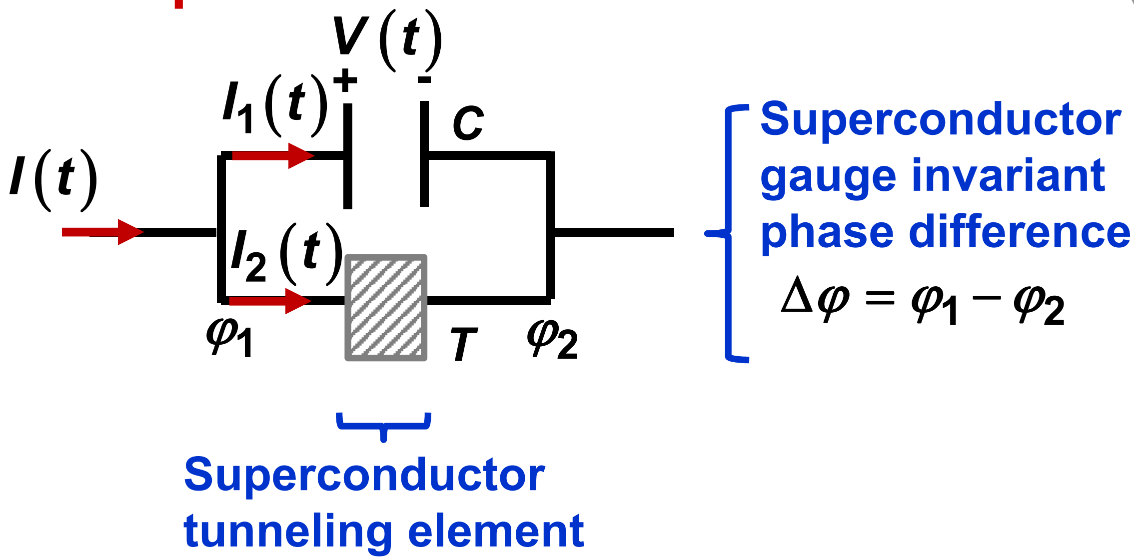
**Accurate simultaneous measurement of both the electric and the magnetic field amplitudes is not impossible**

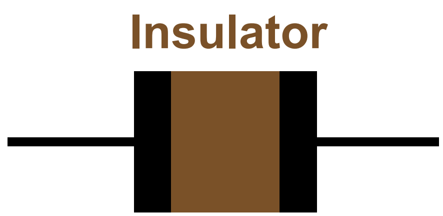
$$\Rightarrow \left[\hat{q}_H, \hat{q}_E\right] = i\hbar$$

$$\left\langle \Delta \hat{q}_H^2 \right\rangle \left\langle \Delta \hat{q}_E^2 \right\rangle \geq \frac{\hbar^2}{4}$$

# Appendix: The Josephson Junction - I



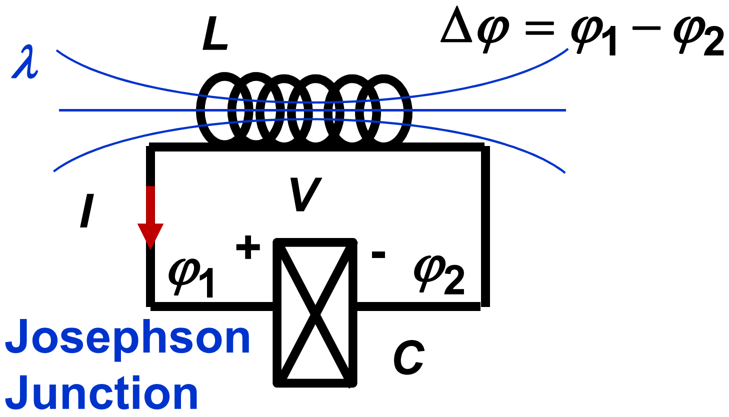

**The Josephson junction relations:**

1) $$I(t) = I_1(t) + I_2(t) = C\frac{dV(t)}{dt} + \frac{2\pi}{\lambda_o} E_J \sin\left(\Delta\varphi(t)\right)$$

Superconductor tunneling current

2) $$\frac{d\Delta\varphi(t)}{dt} = \frac{2\pi}{\lambda_o} V(t)$$

# Appendix: The Josephson Junction - II




$$I(t) = C\frac{dV(t)}{dt} + \frac{2\pi}{\lambda_o}E_J \sin(\Delta\varphi(t))$$

3) $$\Delta\varphi(t) = 2\pi\frac{\lambda(t)}{\lambda_o} + 2\pi n$$

The superconductor phase difference can be related to the trapped flux

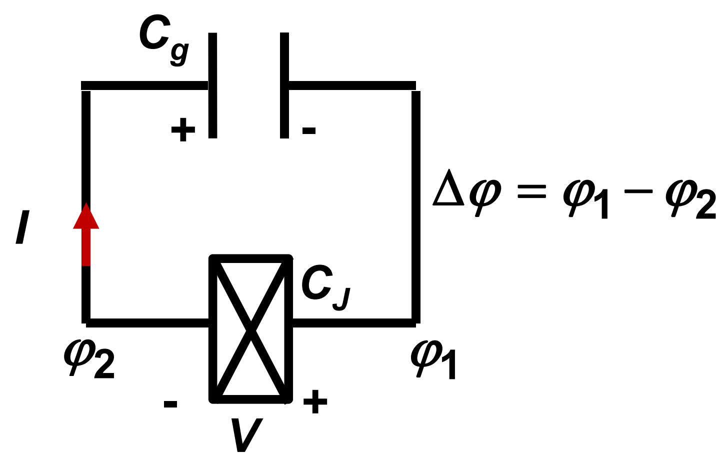
$$\Rightarrow I(t) = C\frac{dV(t)}{dt} + \frac{2\pi}{\lambda_o}E_J \sin\left(2\pi\frac{\lambda(t)}{\lambda_o}\right)$$

**Energy stored in the junction:**

$$\Rightarrow \int_0^T VI\,dt = \int_0^T dt\left[CV(t)\frac{dV(t)}{dt} + \frac{2\pi}{\lambda_o}E_J V(t)\sin\left(2\pi\frac{\lambda(t)}{\lambda_o}\right)\right] = \int_0^T dt\left[CV(t)\frac{dV(t)}{dt} + \frac{2\pi}{\lambda_o}E_J \frac{d\lambda(t)}{dt}\sin\left(2\pi\frac{\lambda(t)}{\lambda_o}\right)\right]$$

$$= \int_0^{V(t)} CV(t)\,dV(t) + \int_0^{\lambda(t)} \frac{2\pi}{\lambda_o}E_J \sin\left(2\pi\frac{\lambda(t)}{\lambda_o}\right)d\lambda(t) = \frac{1}{2}CV^2(t) - E_J\cos\left(2\pi\frac{\lambda(t)}{\lambda_o}\right) + E_J = \frac{Q^2(t)}{2C} - E_J\cos\left(2\pi\frac{\lambda(t)}{\lambda_o}\right) + E_J$$

# Appendix: The Capacitively Shunted Josephson Junction Qubit: Transmon



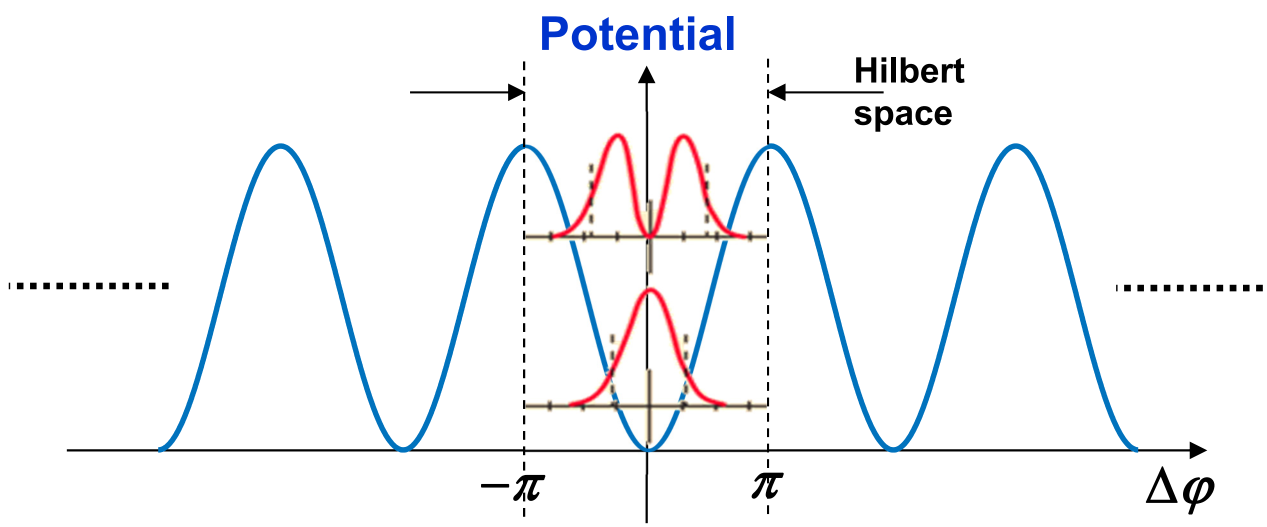

$$\hat{Q} = \hat{Q}_J - \hat{Q}_g = C_J\hat{V} - C_g\hat{V}$$

$$\hat{H} = \frac{\hat{Q}^2}{2\left[C_J + C_g\right]} - E_J\cos\left(\Delta\hat{\varphi}\right) + E_J$$

$$\left[\Delta\hat{\varphi}\ ,\hat{Q}\ \right] = 2|e|\,i$$

**Transmon limit:**

$$E_C = \frac{e^2}{2\left(C_J + C_g\right)} << E_J$$




$$\Rightarrow\ \hat{H} = \frac{\hat{Q}^2}{2\left[C_J + C_g\right]} - E_J\cos\left(\Delta\hat{\varphi}\right) + E_J \approx \frac{\hat{Q}^2}{2\left[C_J + C_g\right]} + \frac{E_J}{2}\Delta\hat{\varphi}^2$$