Lecture 12

The Finite Potential Well: A Quantum Well

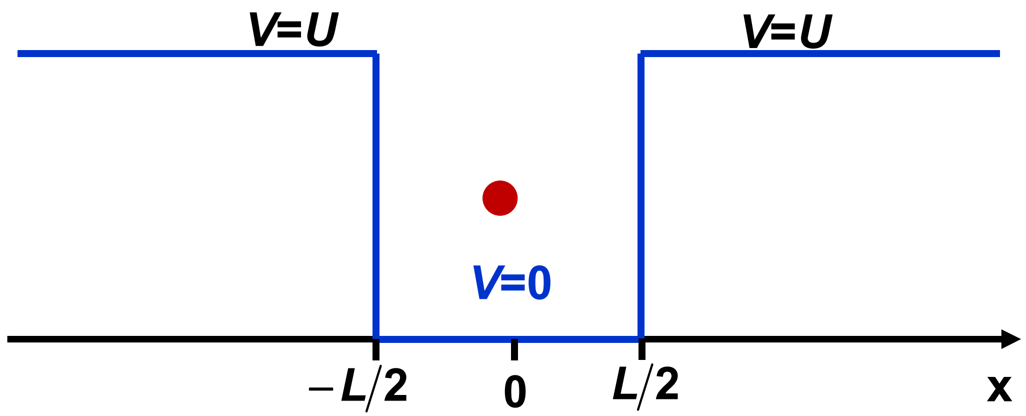
In this lecture you will learn:

- **Particle in a finite potential well**
- **Bound and unbound states in quantum physics**

The Finite Potential Well Problem in 1D

Consider a particle placed inside a 1D potential box

Inside the box the potential energy *V*(*x*) is 0
Outside the box the potential energy *V*(*x*) is *U*



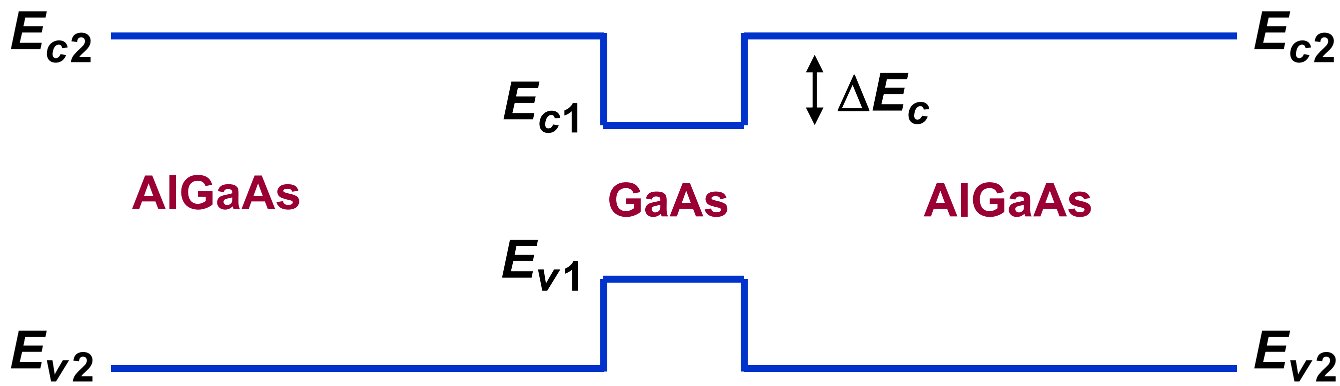
We assume:

$$\psi(x,t) = \phi(x) e^{-i\frac{E}{\hbar}t}$$

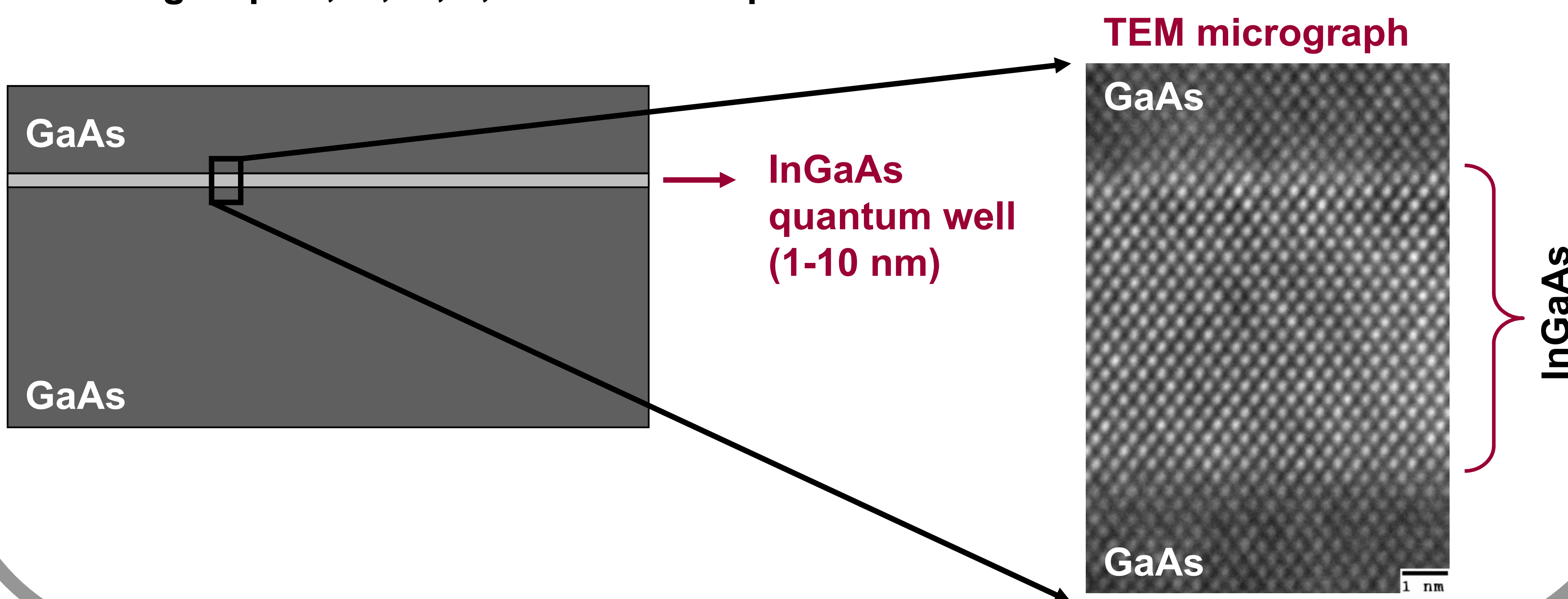
$$-\frac{\hbar^2}{2m}\frac{\partial^2 \phi(x)}{\partial x^2} + V(x)\phi(x) = E\phi(x)$$

Semiconductor Quantum Wells

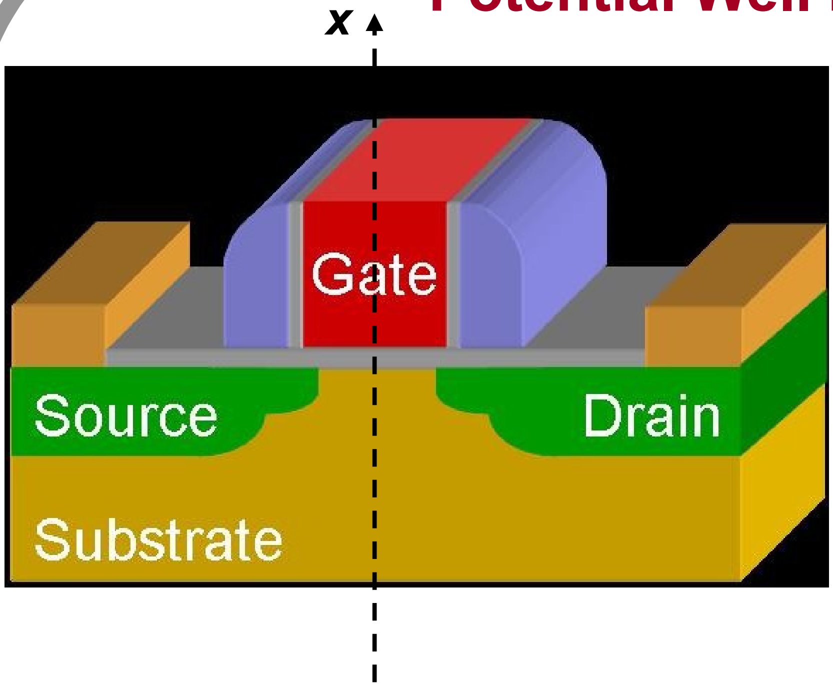
A thin (~1-10 nm) narrow bandgap material sandwiched between two wide bandgap materials

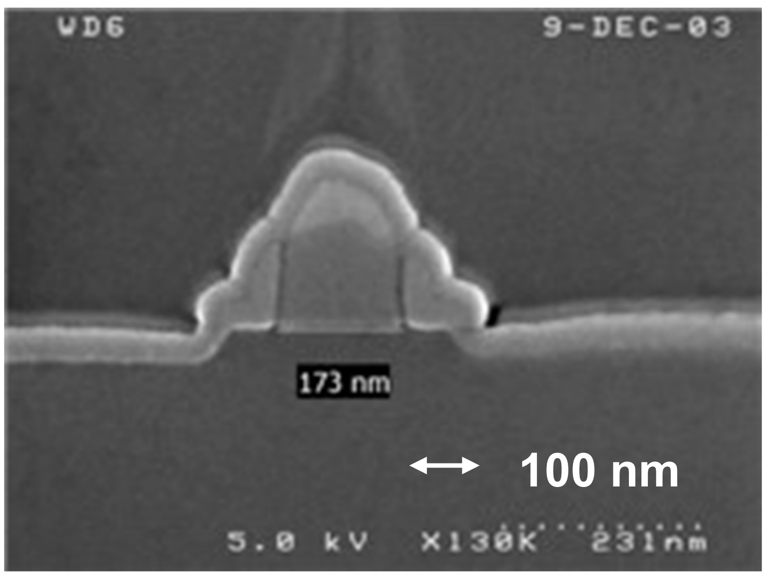


Semiconductor quantum wells can be composed of pretty much any semiconductor from the groups II, III, IV, V, and VI of the periodic table

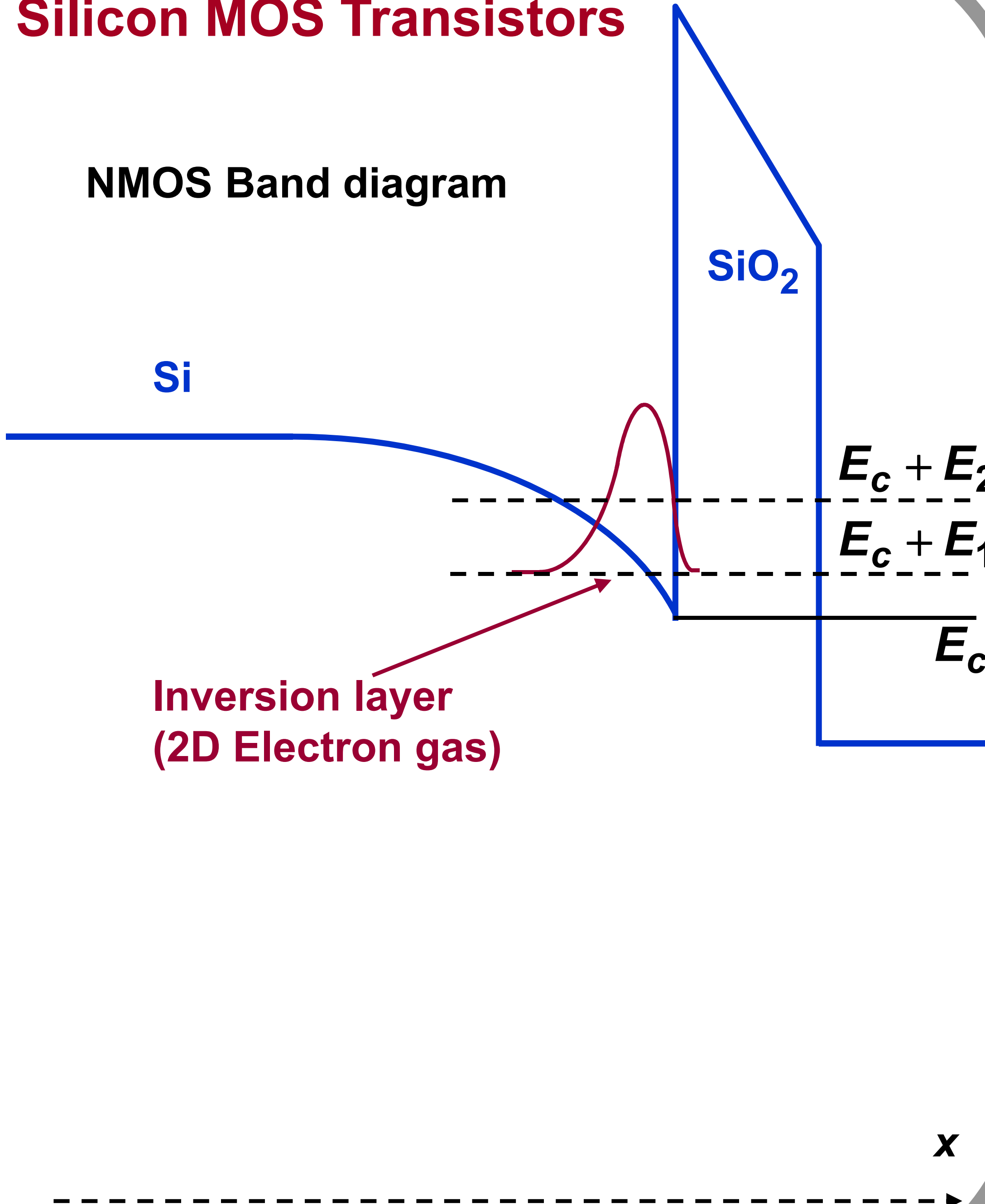


Potential Well in Silicon MOS Transistors

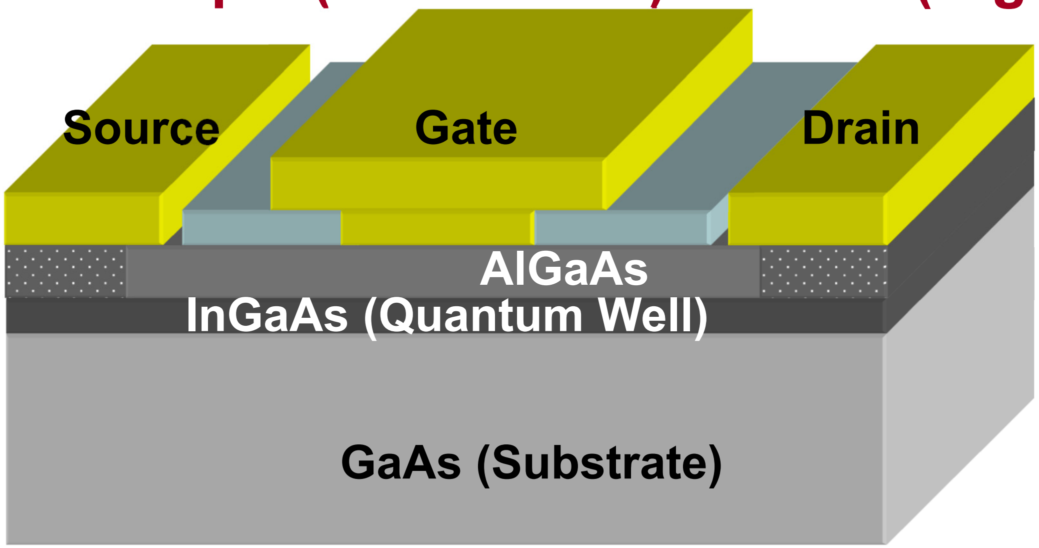




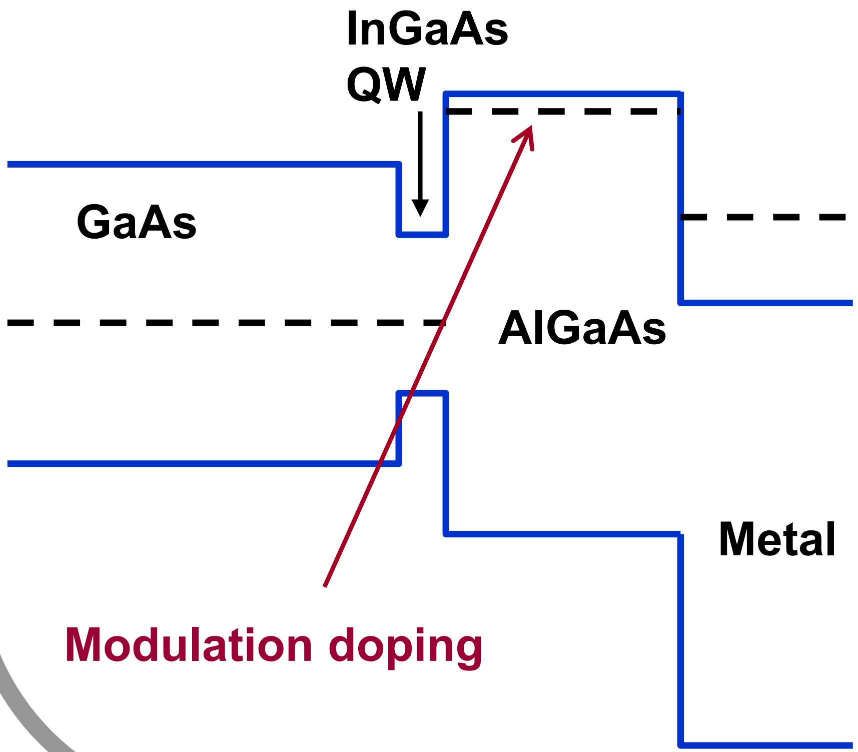
A 50 nm gate MOS transistor (INTEL)

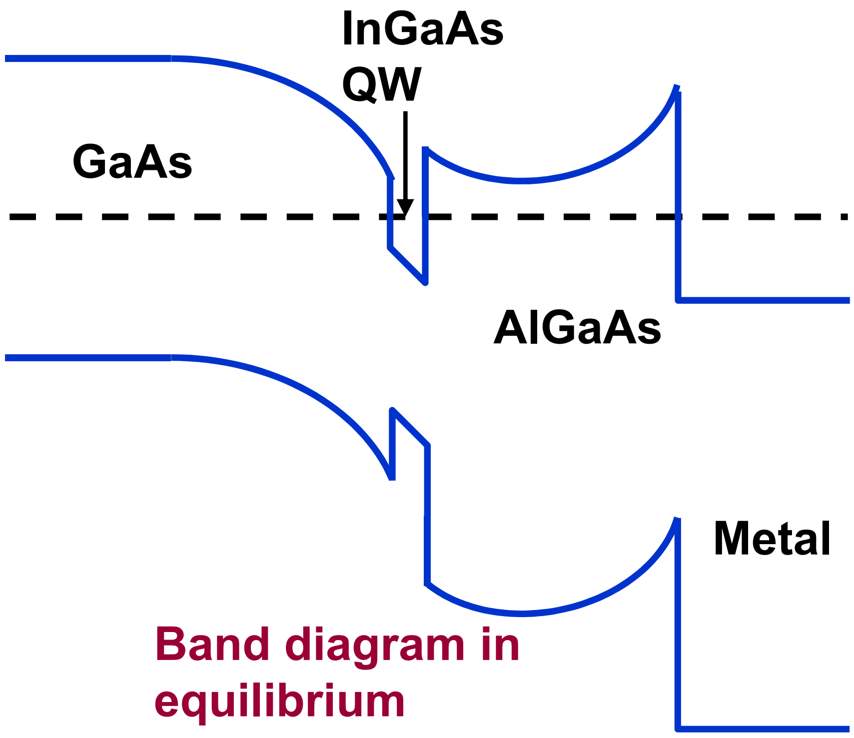


Example (Electronics): HEMTs (High Electron Mobility Transistors)



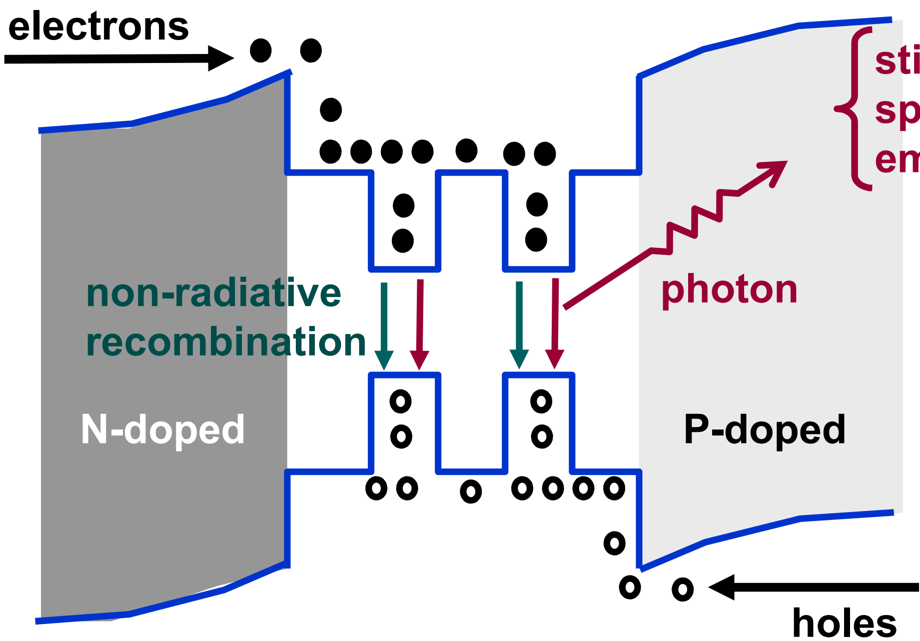
The HEMT operates like a MOS transistor:
The application of a positive or negative bias on the gate can increase or decrease the electron density in the quantum well channel thereby changing the current density



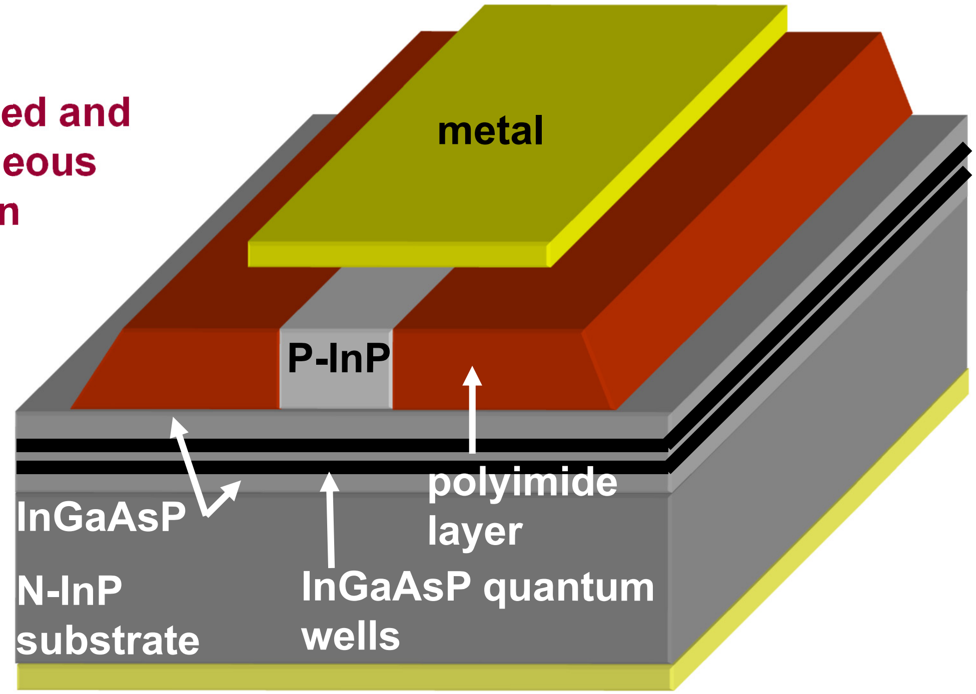


Example (Photonics): Semiconductor Quantum Well Lasers

A quantum well laser (band diagram)

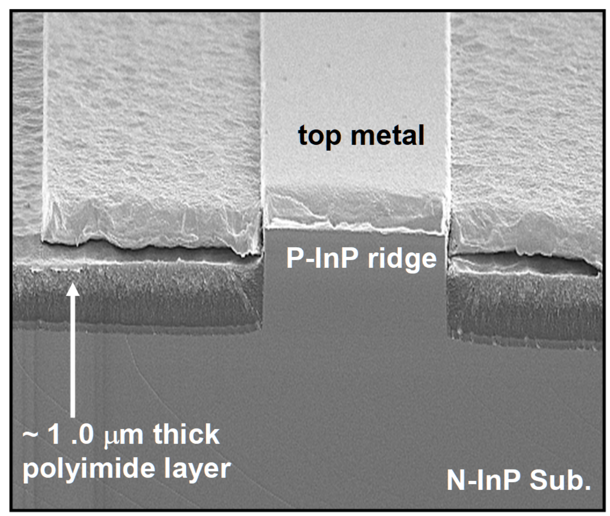


A ridge waveguide laser structure



Some advantages of quantum wells for laser applications:

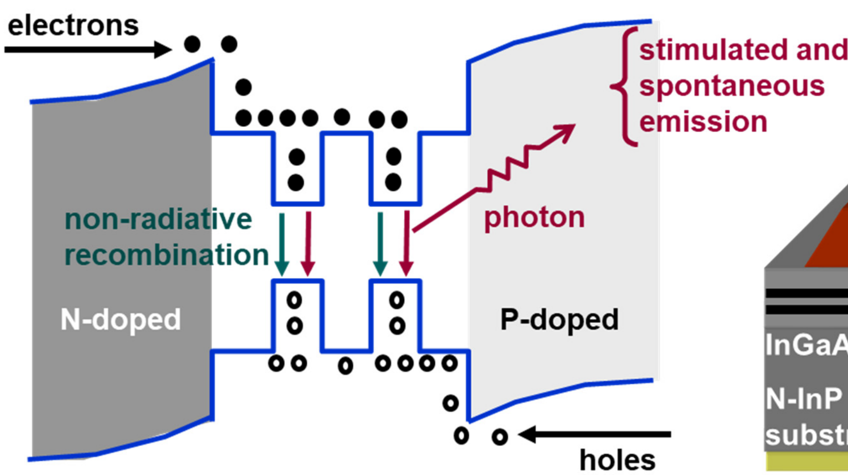
- Low laser threshold currents due to reduced density of states
- High speed laser current modulation due to large differential gain
- Ability to control emission wavelength via quantum size effect



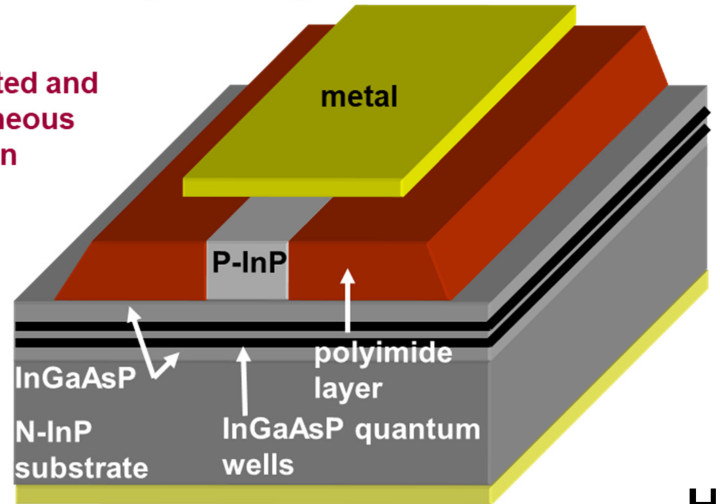
All lasers used in fiber optical communication systems are semiconductor quantum well lasers

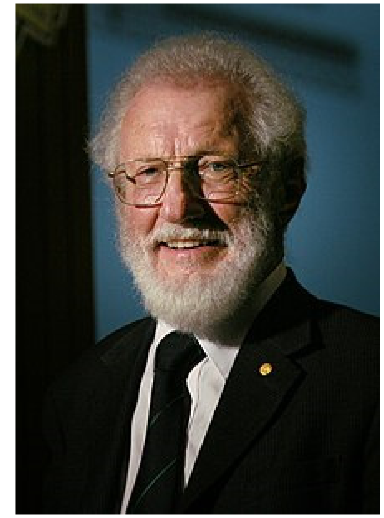
Nobel Prize: Semiconductor Quantum Well Lasers

A quantum well laser (band diagram)



A ridge waveguide laser structure

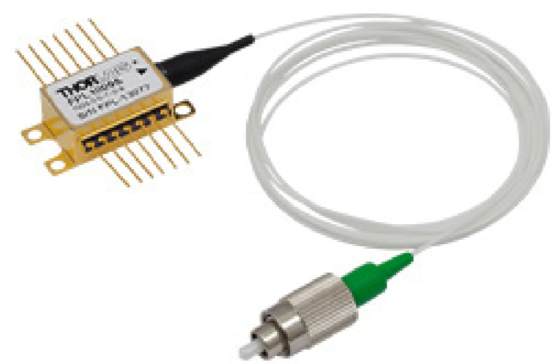


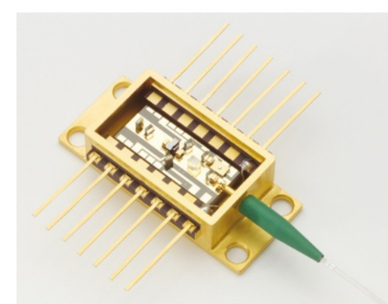


Herbert Kroemer (1928-)
Nobel Prize (2000)

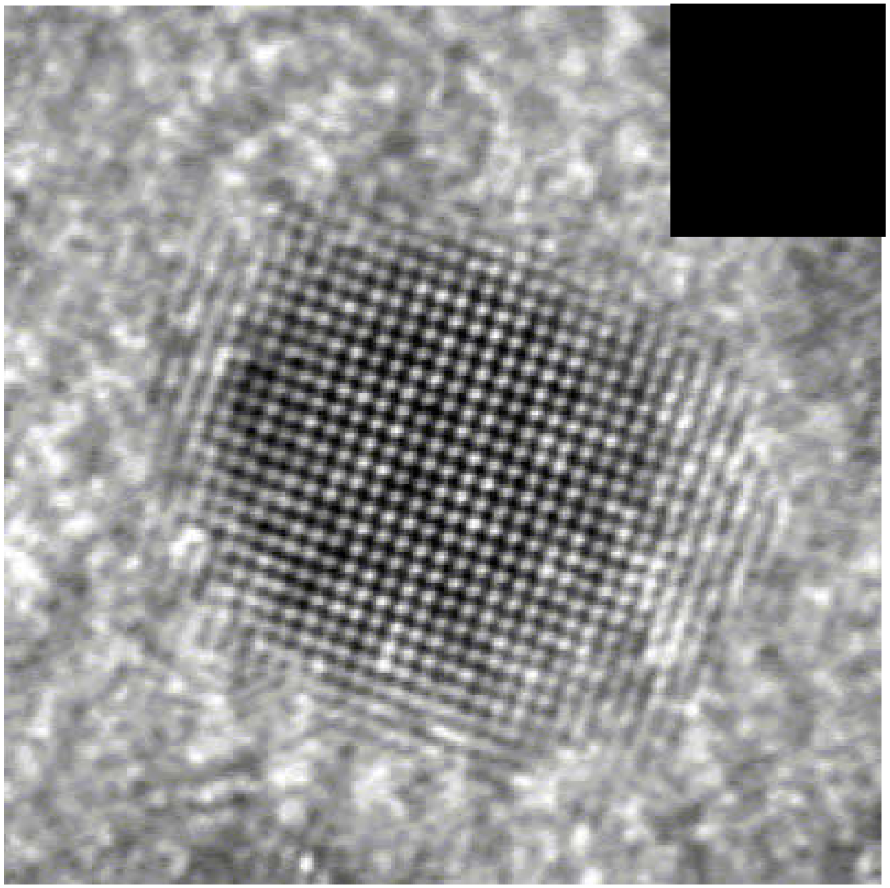
Fiber Optics and the Internet

Quantum well lasers: powering up the internet



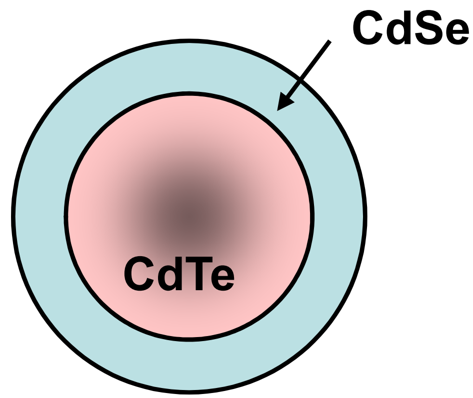


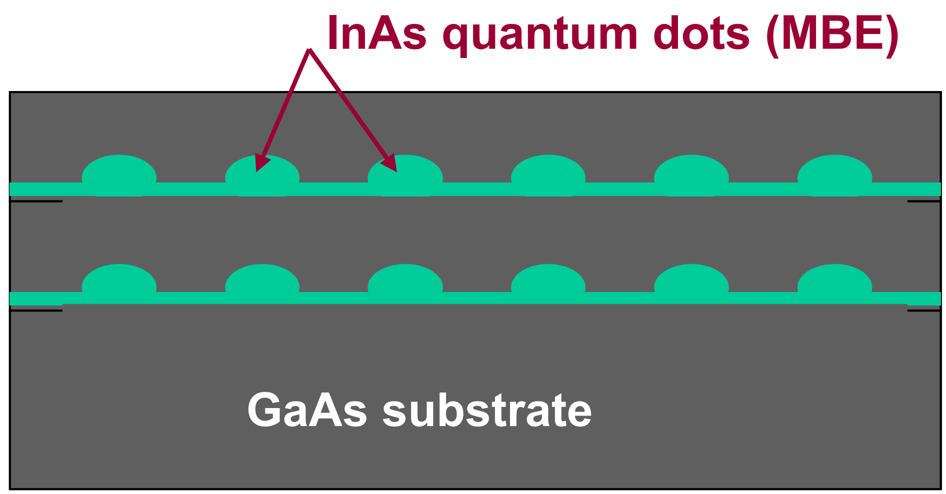
Semiconductor Quantum Dots: A 3D Finite Potential Well



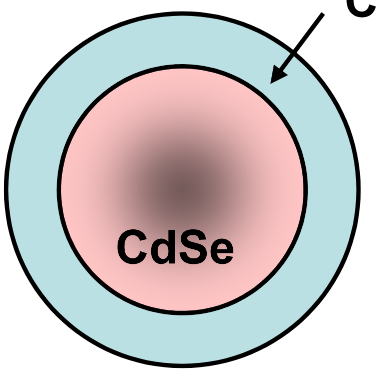
TEM of a PbS quantum dot

Core-shell colloidal quantum dots
(Mostly II-VI semiconductors)



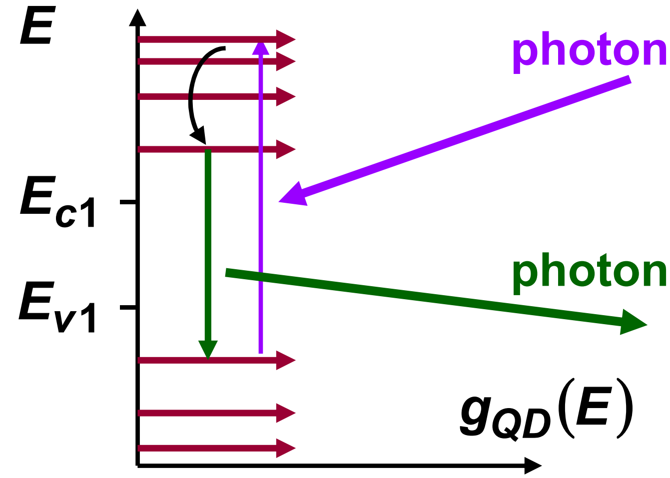


Semiconductor Quantum Dots: Wonders of Quantum Size Effect

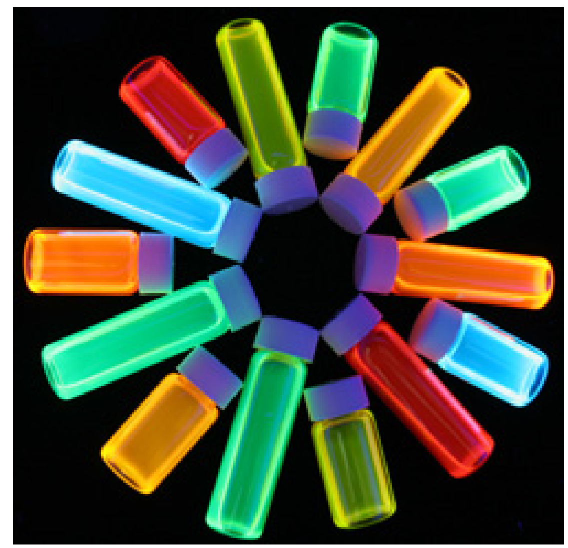


$$E_c(1) \approx E_c + \frac{\hbar^2}{2m_e}\left(\frac{\pi}{R}\right)^2$$

$$E_v(1) \approx E_v - \frac{\hbar^2}{2m_h}\left(\frac{\pi}{R}\right)$$

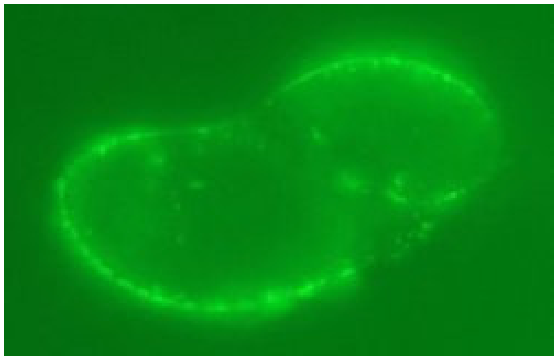


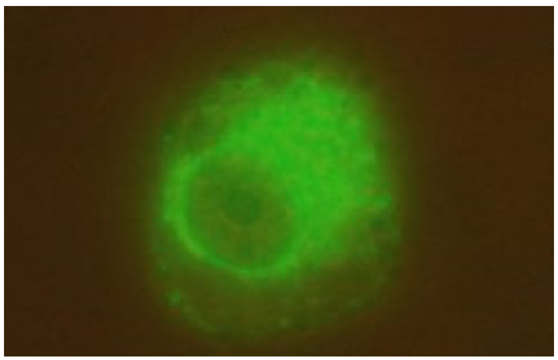
Photoluminescence from CdSe/ZnS (core-shell colloidal) quantum dots of different sizes (~2-6 nm) pumped with the same laser



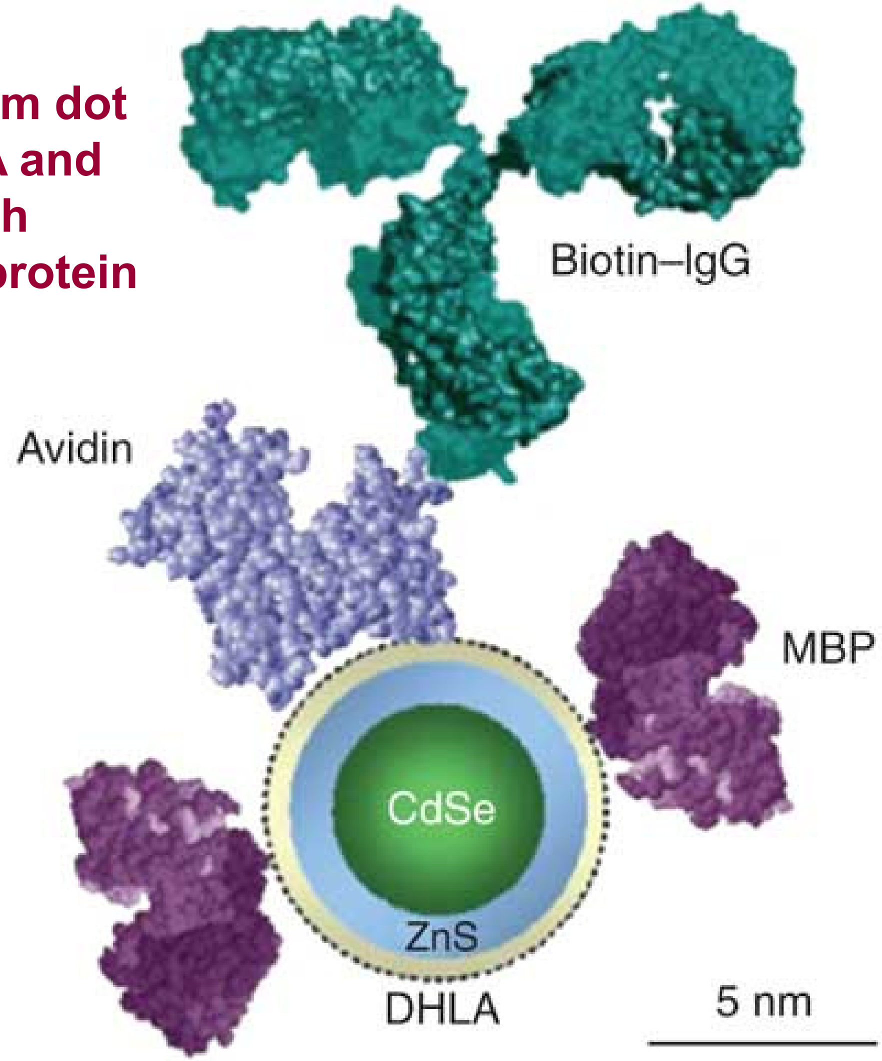
Photoluminescence from CdTe/CdSe (core-shell colloidal) quantum dots of different sizes

Quantum Dots: Biology Applications





CdSe/ZnS quqntum dot coated with DHLA and functionalized with maltose binding protein (MBP) and Avidin

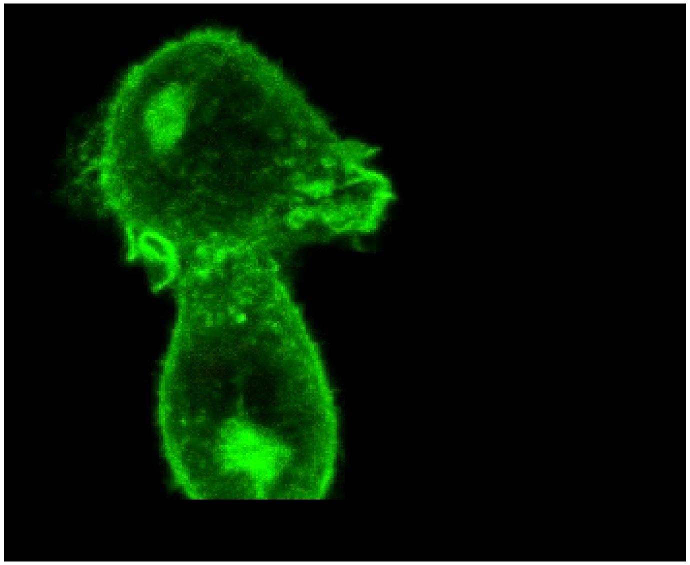


Motion of quantum-dot-attached-RNA into cells monitored by the luminescence (the quantum dots used are CdSe (core) and ZnS (shell)

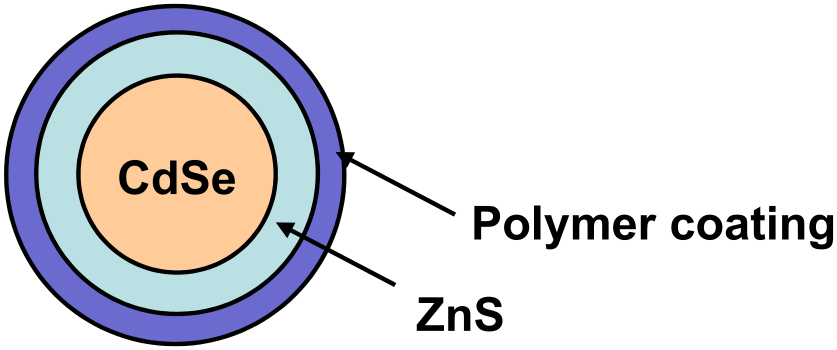
Quantum Dots: Biology Applications

Invitro microscopy of the binding of EGF to erbB1

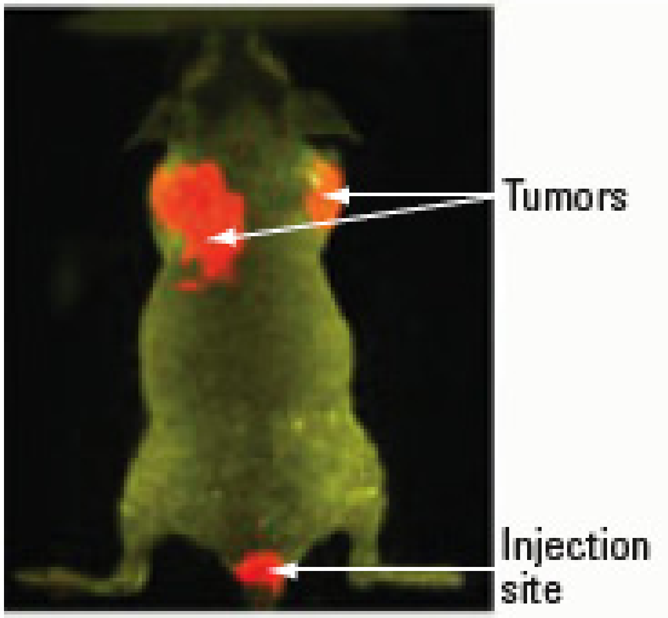
erB1 bound to eGFP (enhanced green fluorescent protein)
EGF (epidermal growth factor) bound to quantum dot
Movie shows binding of EGF tagged with fluorescent quantum dots to erB1 tagged with the green fluorescent protein



Nat. Biotechnol., 22, 198-203 (2004)

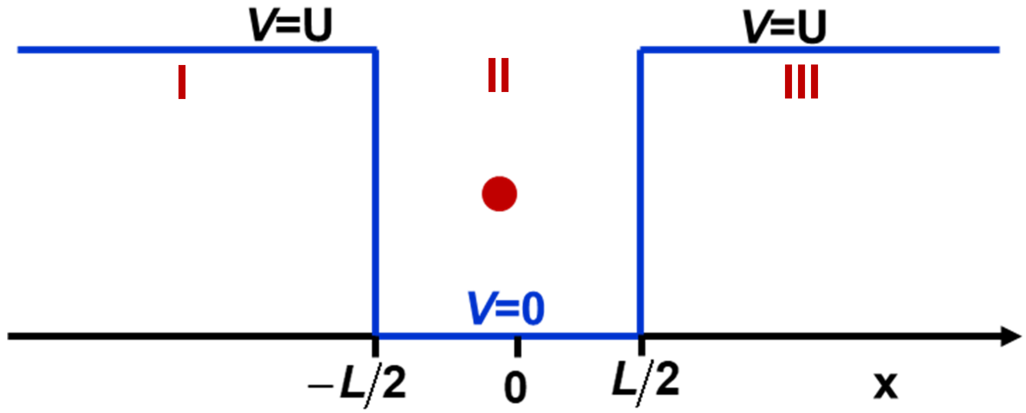


Imaging of antibody (PSMA) coated quantum dots targeting cancer tumors cells



Nat. Biotechnol., 22, 969 (2004)

The Finite Potential Well Problem in 1D



1) Inside the potential well we have (region II):

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \phi(x)}{\partial x^2} = E\phi(x)$$

V(*x*)=0 inside

2) Outside the potential well we have (regions I and III):

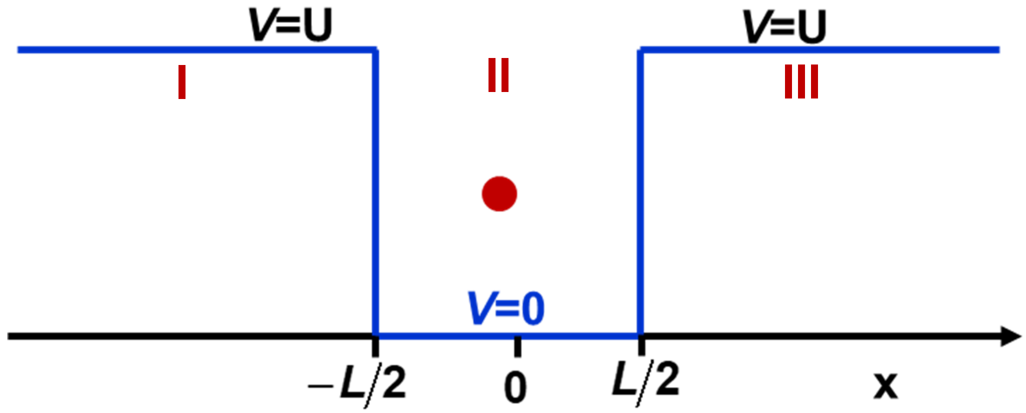
$$-\frac{\hbar^2}{2m}\frac{\partial^2 \phi(x)}{\partial x^2} + U\phi(x) = E\phi(x)$$

V(*x*)=*U* outside

Boundary conditions:

The wavefunction and its derivative are continuous at the boundaries between the regions

The Finite Potential Well Problem in 1D: Parity of the Potential



Notice that:

$$V(-x) = V(x)$$ (Even parity, inversion symmetry)

This means that wavefunctions can be chosen to have a definite parity:

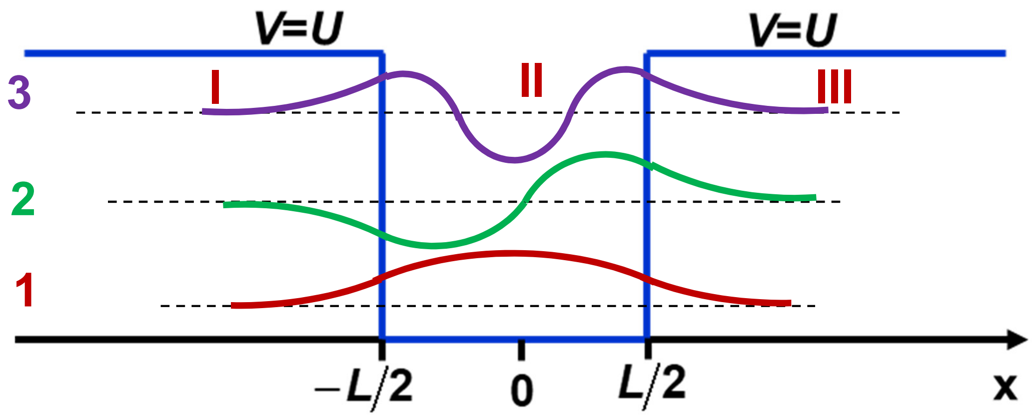
$$\phi(-x) = \phi(x)$$

(Even parity)

$$\phi(-x) = -\phi(x)$$

(Odd parity)

The Finite Potential Well Problem in 1D: Bound Solutions



Inside the potential well we have (region II):

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi(x)}{\partial x^2} = E\phi(x)$$

$V(x)$=0 inside

Solutions with even parity:

$$\phi(x) = A\cos(kx) \quad \Rightarrow \quad E = \frac{\hbar^2 k^2}{2m}$$

Solutions with odd parity:

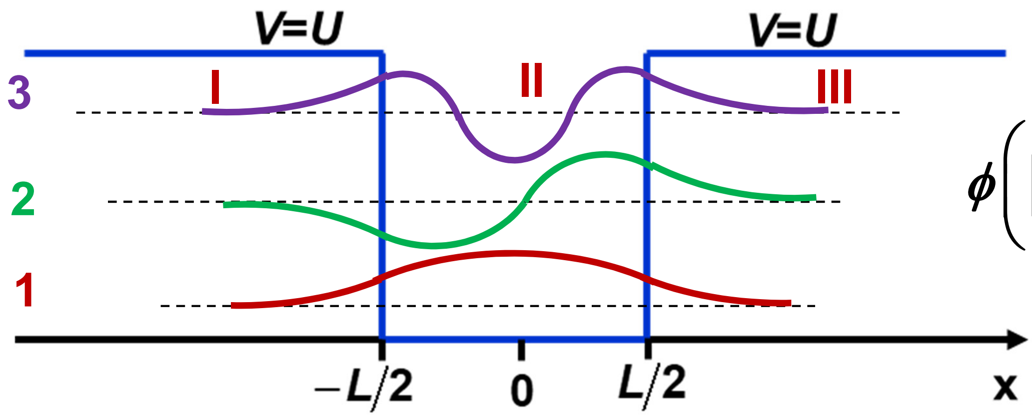
$$\phi(x) = A\sin(kx) \quad \Rightarrow \quad E = \frac{\hbar^2 k^2}{2m}$$

k=??
E=??

We write them both together for convenience as:

$$\phi\left(|x| < \frac{L}{2}\right) = \begin{cases} A\cos(kx) & \longrightarrow \text{Even solutions} \\ A\sin(kx) & \longrightarrow \text{Odd solutions} \end{cases}$$

The Finite Potential Well Problem in 1D: Bound Solutions



$$\phi\left(|x|<\frac{L}{2}\right)=\begin{cases}A\cos(kx)\\A\sin(kx)\end{cases}$$

Outside the potential well we have (region I and III):

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi(x)}{\partial x^2}+U\phi(x)=E\phi(x)$$

V(*x*)=0 inside

We write them both together for convenience as:

$$\phi\left(x>\frac{L}{2}\right)=\begin{cases}Be^{-\gamma(x-L/2)}\\Be^{-\gamma(x-L/2)}\end{cases}\qquad \phi\left(x<\frac{L}{2}\right)=\begin{cases}Be^{+\gamma(x+L/2)}\\-Be^{+\gamma(x+L/2)}\end{cases}$$

γ=??
E=??

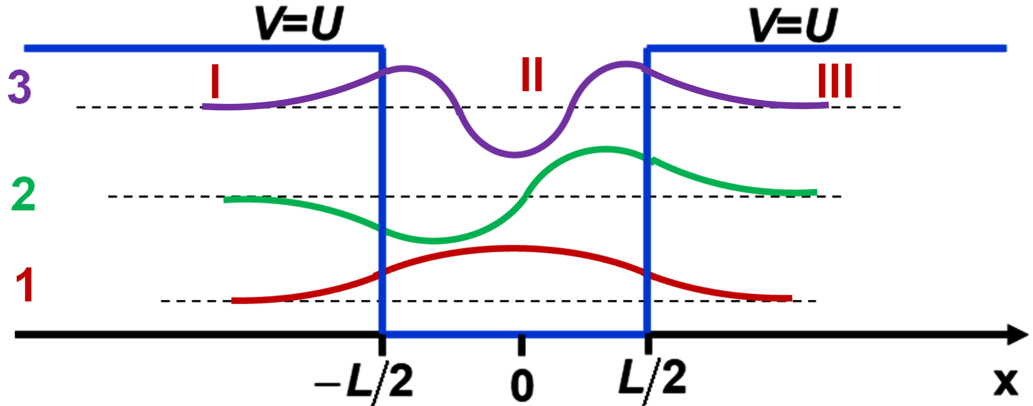
Plugging these solutions into the Schrodinger equation gives:

$$\Rightarrow \quad E=-\frac{\hbar^2\gamma^2}{2m}+U$$

$$\Rightarrow \gamma=\sqrt{\frac{2m}{\hbar^2}(U-E)}=\sqrt{\frac{2m}{\hbar^2}U-k^2}$$

For bound solutions to exist: *E*<*U*

The Finite Potential Well Problem in 1D: Bound Solutions



$$\phi\left(x<\frac{L}{2}\right)=\begin{cases}Be^{+\gamma(x+L/2)}\\-Be^{+\gamma(x+L/2)}\end{cases}$$

$$\phi\left(|x|<\frac{L}{2}\right)=\begin{cases}A\cos(kx)\\A\sin(kx)\end{cases}$$

$$\phi\left(x>\frac{L}{2}\right)=\begin{cases}Be^{-\gamma(x-L/2)}\\Be^{-\gamma(x-L/2)}\end{cases}$$

Wave function and its derivative must be continuous at *x=L/2* :

$$\begin{cases}A\cos(kL/2)=B\\A\sin(kL/2)=B\end{cases}\longrightarrow \text{(1)}$$

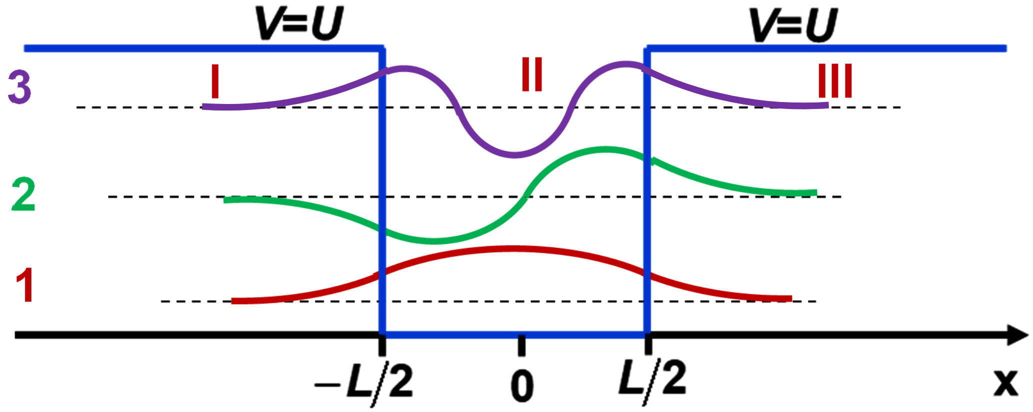
$$\begin{cases}-Ak\sin(kL/2)=-\gamma B\\Ak\cos(kL/2)=-\gamma B\end{cases}\longrightarrow \text{(2)}$$

Divide (2) by (1) gives:

Transcendental equation in k →

$$\begin{cases}\tan\left(k\dfrac{L}{2}\right)=\dfrac{\gamma}{k}=\dfrac{\sqrt{\dfrac{2m}{\hbar^2}U-k^2}}{k} & \longrightarrow \text{Even solutions}\\[2ex]-\cot\left(k\dfrac{L}{2}\right)=\dfrac{\gamma}{k}=\dfrac{\sqrt{\dfrac{2m}{\hbar^2}U-k^2}}{k} & \longrightarrow \text{Odd solutions}\end{cases}$$

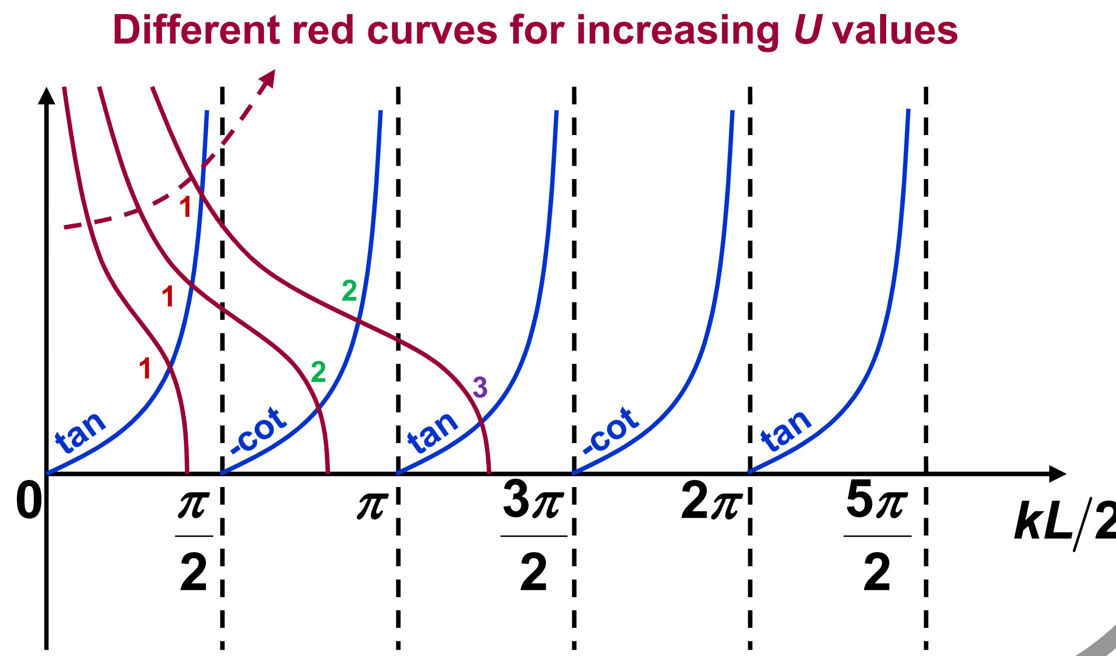
The Finite Potential Well Problem in 1D: Bound Solutions



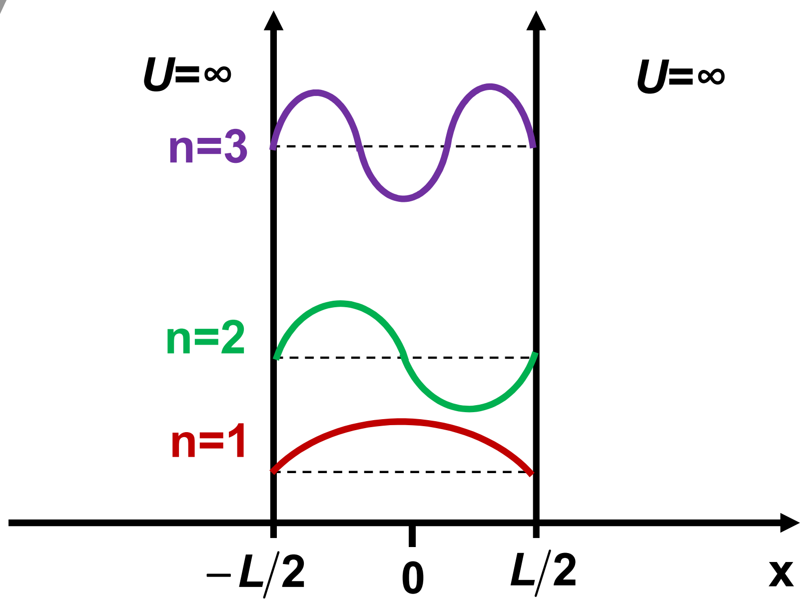
Plot the LHS and RHS as a function of *k*
Intersection points will give you the eigenvalues

$$\begin{cases} \tan\left(k\frac{L}{2}\right) = \frac{\gamma}{k} = \frac{\sqrt{\frac{2m}{\hbar^2}U - k^2}}{k} \\ -\cot\left(k\frac{L}{2}\right) = \frac{\gamma}{k} = \frac{\sqrt{\frac{2m}{\hbar^2}U - k^2}}{k} \end{cases}$$

LHS vs RHS curves

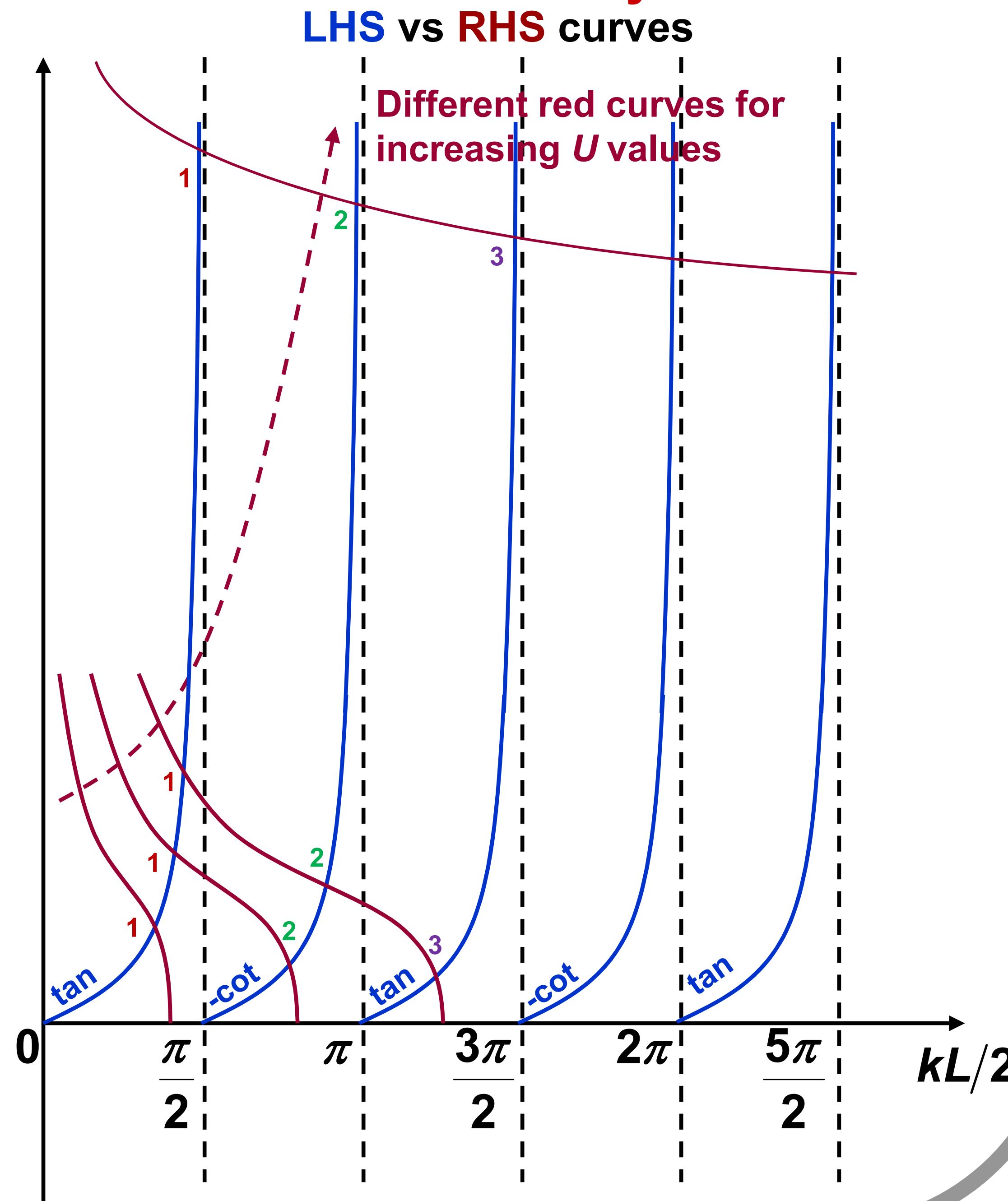


The Infinite Potential Well Problem in 1D: Sanity Check

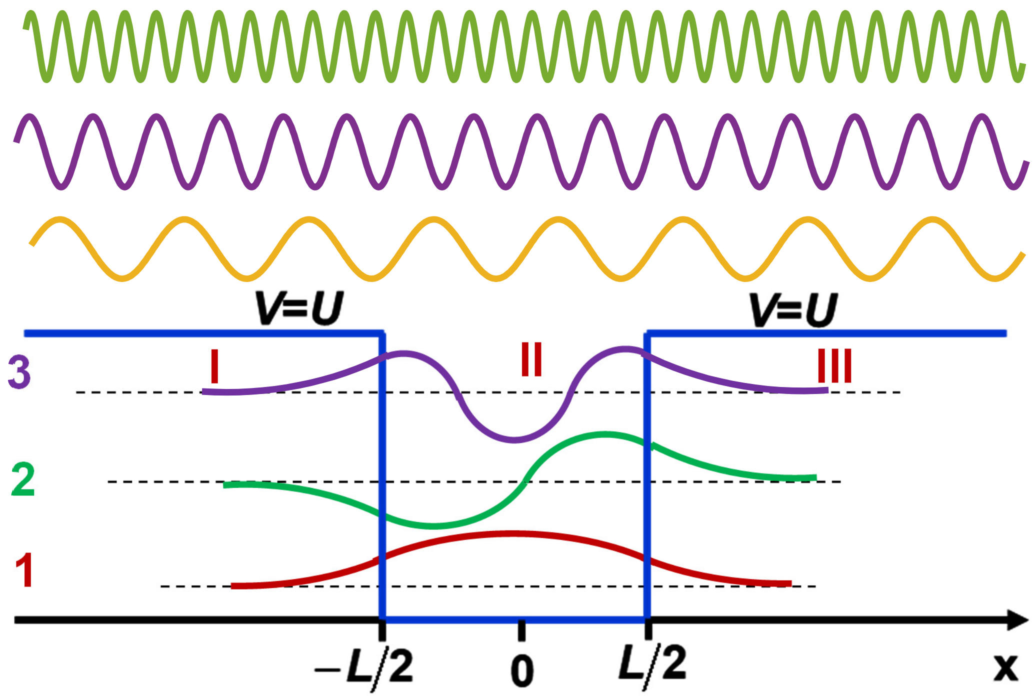


In the limit $U \to \infty$ we get for the *n*-th solution:

$$k \to n\frac{\pi}{L}$$



The Finite Potential Well Problem in 1D: Unbound Solutions



A finite potential well has a continuum of higher energy solutions that are not bound inside the well

These higher energy solutions behave more or less like free-particle plane wave solutions