Lecture 10

Commutation Relations, Measurements, Disturbances, and Heisenberg Uncertainty Relations

In this lecture you will learn:

- **Heisenberg uncertainty relations in quantum physics**
- **Operator commutations and Heisenberg uncertainty relations**

Werner Karl Heisenberg
(1901 – 1976)
Nobel Prize 1932

Mean Values, Standard Deviation, and Uncertainty

Consider an observable O (could be position, energy, momentum, spin, etc)

The mean value of the observable O with respect to a quantum state $|\psi\rangle$ is:

$$\langle O \rangle = \langle \psi | \hat{O} | \psi \rangle$$

Sometimes the same mean value is also written as:

$$\langle \hat{O} \rangle = \langle \psi | \hat{O} | \psi \rangle$$

Note the carrot

The standard deviation in the value of O is given by:

$$\sigma_O^2 = \langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2 = \left\langle \left(\hat{O} - \langle \hat{O} \rangle \right)^2 \right\rangle = \langle \psi | \left(\hat{O} - \langle \hat{O} \rangle \right)^2 | \psi \rangle$$

$$= \langle \psi | \hat{O}^2 | \psi \rangle - \langle \psi | \hat{O} | \psi \rangle^2$$

The a-priori uncertainty ΔO in the value of O is given by:

$$\Delta O = \sigma_O = \sqrt{\langle \psi | \hat{O}^2 | \psi \rangle - \langle \psi | \hat{O} | \psi \rangle^2}$$

Eigenstates and Uncertainty - I

Consider an observable O (could be position, energy, momentum, spin, etc) of a particle

The corresponding operator $\hat{O}$ has the following complete set of eigenstates:

$$\hat{O}\left|v_j\right\rangle = \lambda_j \left|v_j\right\rangle$$

Suppose the quantum state $\left|\psi\right\rangle$ of the particle is an eigenstate of the operator $\hat{O}$:

$$\left|\psi\right\rangle = \left|v_m\right\rangle$$

Mean value of O:

$$\left\langle \hat{O} \right\rangle = \left\langle \psi \middle| \hat{O} \middle| \psi \right\rangle = \left\langle v_m \middle| \hat{O} \middle| v_m \right\rangle = \lambda_m$$

Standard deviation or the a-priori uncertainty in the value of O:

$$\sigma_O^2 = \left\langle \psi \middle| \hat{O}^2 \middle| \psi \right\rangle - \left\langle \psi \middle| \hat{O} \middle| \psi \right\rangle^2 = \left\langle v_m \middle| \hat{O}^2 \middle| v_m \right\rangle - \left\langle v_m \middle| \hat{O} \middle| v_m \right\rangle^2 = \lambda_m^2 - \lambda_m^2 = 0$$

Lesson I:
If the quantum state is an eigenstate of an observable O then the a-priori uncertainty in the value of O is zero and the value of O obtained if a measurement is performed will be the eigenvalue corresponding to this eigenstate

Eigenstates and Uncertainty - II

Consider an observable O of a particle

The corresponding operator $\hat{O}$ has the following complete set of eigenstates:

$$\hat{O}\left|v_j\right\rangle = \lambda_j \left|v_j\right\rangle$$

Suppose the quantum state $\left|\psi\right\rangle$ of the particle is:

$$\left|\psi\right\rangle = \sum_j a_j \left|v_j\right\rangle$$

Mean value of O:

$$\left\langle \hat{O} \right\rangle = \left\langle \psi \right| \hat{O} \left| \psi \right\rangle = \sum_j \lambda_j \left|a_j\right|^2$$

Standard deviation or the a-priori uncertainty in the value of O:

$$\sigma_O^2 = \left\langle \psi \right| \hat{O}^2 \left| \psi \right\rangle - \left\langle \psi \right| \hat{O} \left| \psi \right\rangle^2 = \sum_j \lambda_j^2 \left|a_j\right|^2 - \left(\sum_j \lambda_j \left|a_j\right|^2 \right)^2 \geq 0$$

Lesson II:
A-priori uncertainty in the value of an observable O arises when the quantum state is in a superposition of the eigenstates of the observable O

Properties of Wavefunctions: Position Uncertainty

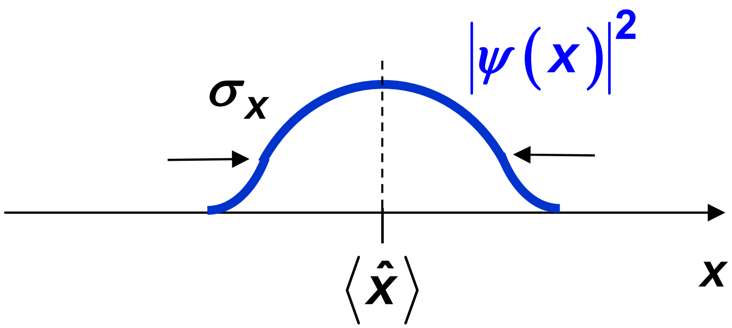
Suppose the wavefunction of a particle is $\psi(x)$

$$|\psi\rangle = \int_{-\infty}^{\infty} dx\, \psi(x)\,|x\rangle$$

The mean position is:

$$\langle \hat{x} \rangle = \langle \psi | \hat{x} | \psi \rangle = \int dx\; x\, \psi^*(x)\psi(x)$$

The standard deviation for position is:



Define: $\Delta\hat{x} = \hat{x} - \langle \hat{x} \rangle$

$$\sigma_x^2 = \langle \Delta\hat{x}^2 \rangle = \left\langle \left(\hat{x} - \langle \hat{x} \rangle\right)^2 \right\rangle = \langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2$$

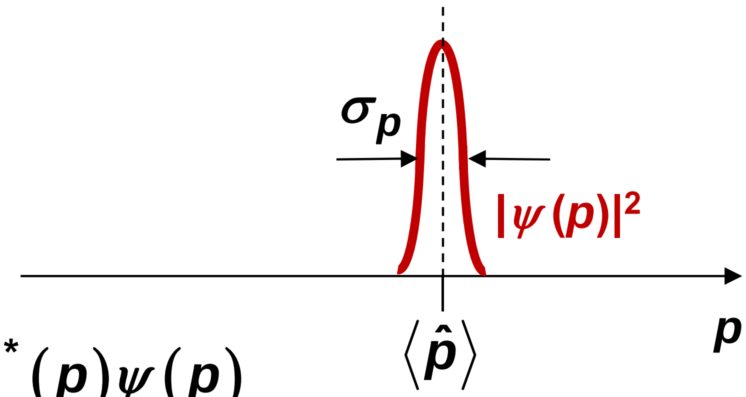
The a-priori uncertainty Δx in the position is defined as:

$$\Delta x = \sigma_x = \sqrt{\langle \Delta\hat{x}^2 \rangle}$$

Properties of Wavefunctions: Momentum Uncertainty

The momentum space wavefunction is:

$$\psi(p) = \langle p|\psi\rangle = \int dx \frac{e^{-i\frac{p}{\hbar}x}}{\sqrt{\hbar}} \psi(x)$$



The mean momentum value is:

$$\langle \hat{p} \rangle = \int dx\, \psi^*(x) \left[\frac{\hbar}{i}\frac{\partial}{\partial x}\right] \psi(x) = \int \frac{dp}{2\pi}\, p\, \psi^*(p)\psi(p)$$

Define: $\Delta\hat{p} = \hat{p} - \langle \hat{p} \rangle$

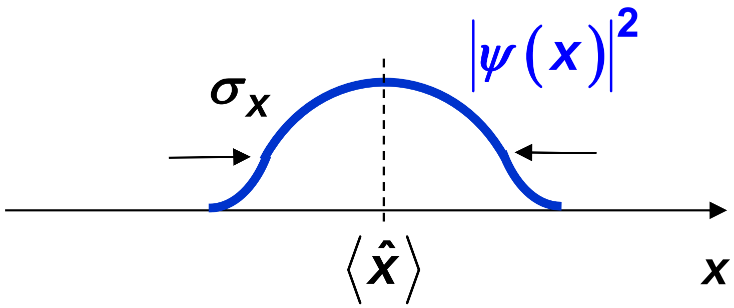
The standard deviation in momentum is:

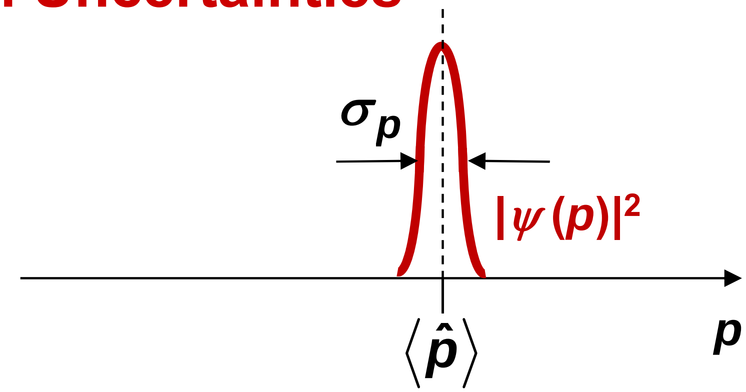
$$\sigma_p^2 = \langle \Delta\hat{p}^2 \rangle = \left\langle (p - \langle p \rangle)^2 \right\rangle = \langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2$$

The a-priori uncertainty Δp in the momentum is defined as:

$$\Delta p = \sigma_p = \sqrt{\langle \Delta\hat{p}^2 \rangle}$$

Position and Momentum Uncertainties





The a-priori uncertainty in the position is defined as: $\Delta x = \sigma_x$

The a-priori uncertainty in the momentum is defined as: $\Delta p = \sigma_p$

Question:

What can we say about $\Delta x = \sigma_x$ and $\Delta p = \sigma_p$?

Commutation Relations in Quantum Physics

Operators in quantum physics don't always commute:

$$\hat{A}\hat{B} \neq \hat{B}\hat{A}$$

This property is usually expressed in terms of the commutation relation or the commutator:

$$\left[\hat{A},\hat{B}\right] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

Lets find the commutation relation between position and momentum:

$$\left[\hat{x},\hat{p}\right] = \hat{x}\hat{p} - \hat{p}\hat{x} = ?$$

Position-Momentum Commutation Relation

Let:

$$[\hat{x},\hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = \hat{c}$$ → **We need to find $\hat{c}$**

We sandwich the above between $\langle x|$ and $|\psi\rangle$ and see what happens:

$$\langle x|[\hat{x},\hat{p}]|\psi\rangle = \langle x|\hat{x}\hat{p} - \hat{p}\hat{x}|\psi\rangle = \langle x|\hat{c}|\psi\rangle$$

1) $$\langle x|\hat{x}\hat{p}|\psi\rangle = x\langle x|\hat{p}|\psi\rangle = x\frac{\hbar}{i}\frac{\partial\psi(x)}{\partial x}$$

2) $$\langle x|\hat{p}\hat{x}|\psi\rangle = \langle x|\hat{p}|\theta\rangle = \frac{\hbar}{i}\frac{\partial\theta(x)}{\partial x}$$

$$= \frac{\hbar}{i}\psi(x) + x\frac{\hbar}{i}\frac{\partial\psi(x)}{\partial x}$$

$$|\theta\rangle = \hat{x}|\psi\rangle$$
$$\theta(x) = \langle x|\theta\rangle = \langle x|\hat{x}|\psi\rangle = x\psi(x)$$

3) $$\langle x|[\hat{x},\hat{p}]|\psi\rangle = \langle x|\hat{x}\hat{p} - \hat{p}\hat{x}|\psi\rangle = i\hbar\psi(x) = \langle x|\hat{c}|\psi\rangle$$

4) **Since the above is true for any arbitrary $|\psi\rangle$ it must be that:** $\hat{c} = i\hbar\hat{1}$

$$[\hat{x},\hat{p}] = i\hbar\,\hat{1} \equiv i\hbar$$

Position-Momentum Commutation Relations in 3D

Different components of position and momentum commute!

If:

$$\hat{\vec{r}} = \hat{x}e_x + \hat{y}e_y + \hat{z}e_z = \hat{r}_x e_x + \hat{r}_y e_y + \hat{r}_z e_z$$

$$\hat{\vec{p}} = \hat{p}_x e_x + \hat{p}_y e_y + \hat{p}_z e_z$$

e_x , e_y , and e_z are the Cartesian unit vectors

Then:

$$\left[\hat{r}_j , \hat{p}_k\right] = i\hbar\,\delta_{jk}$$

$$\left[\hat{r}_j , \hat{r}_k\right] = 0$$

$$\left[\hat{p}_j , \hat{p}_k\right] = 0$$

{ where: j,k = x,y,z

Commutation Relations and the Uncertainty Principle

Suppose two Hermitian operators have the following commutation relations:

$$\left[\hat{A},\hat{B}\right] = \hat{A}\hat{B} - \hat{B}\hat{A} = i\hat{C}$$

→ And $\hat{C}$ is Hermitian too

Consider an arbitrary quantum state: $\left|\psi\right\rangle$

Consider the mean values of the operators and the standard deviations wrt to this quantum state:

$$\left\langle \hat{A} \right\rangle = \left\langle \psi \middle| \hat{A} \middle| \psi \right\rangle \quad \rightarrow \quad \Delta\hat{A} = \hat{A} - \left\langle \hat{A} \right\rangle$$

$$\sigma_A^2 = \left\langle \Delta\hat{A}^2 \right\rangle = \left\langle \left(\hat{A} - \left\langle \hat{A} \right\rangle \right)^2 \right\rangle = \left\langle \psi \middle| \left(\hat{A} - \left\langle \hat{A} \right\rangle \right)^2 \middle| \psi \right\rangle$$

$$\left\langle \hat{B} \right\rangle = \left\langle \psi \middle| \hat{B} \middle| \psi \right\rangle \quad \rightarrow \quad \Delta\hat{B} = \hat{B} - \left\langle \hat{B} \right\rangle$$

$$\sigma_B^2 = \left\langle \Delta\hat{B}^2 \right\rangle = \left\langle \left(\hat{B} - \left\langle \hat{B} \right\rangle \right)^2 \right\rangle = \left\langle \psi \middle| \left(\hat{B} - \left\langle \hat{B} \right\rangle \right)^2 \middle| \psi \right\rangle$$

$$\left[\Delta\hat{A}, \Delta\hat{B}\right] = i\hat{C}$$

Question: what can we say about the uncertainty product from the commutation relation:

$$\sigma_A^2 \sigma_B^2 = \left\langle \Delta\hat{A}^2 \right\rangle \left\langle \Delta\hat{B}^2 \right\rangle = ?$$

Commutation Relation and Uncertainty Principle

$$\left[\hat{A},\hat{B}\right]=\hat{A}\hat{B}-\hat{B}\hat{A}=i\hat{C}$$ → **And $\hat{C}$ is Hermitian too**

Consider an arbitrary quantum state: $|\psi\rangle$

Now consider the state: $|\phi\rangle=\left(\Delta\hat{A}+i\lambda\Delta\hat{B}\right)|\psi\rangle$

λ **is real**

$$\langle\phi|\phi\rangle\geq 0$$

$$\Rightarrow\langle\psi|\left(\Delta\hat{A}-i\lambda\Delta\hat{B}\right)\left(\Delta\hat{A}+i\lambda\Delta\hat{B}\right)|\psi\rangle\geq 0$$

$$\Rightarrow\langle\psi|\left(\Delta\hat{A}^2+\lambda^2\Delta\hat{B}^2+i\lambda\left[\Delta\hat{A},\Delta\hat{B}\right]\right)|\psi\rangle\geq 0$$

$$\Rightarrow\langle\psi|\left(\Delta\hat{A}^2+\lambda^2\Delta\hat{B}^2-\lambda\hat{C}\right)|\psi\rangle\geq 0$$

$$\Rightarrow\lambda^2\left\langle\Delta\hat{B}^2\right\rangle-\lambda\langle\psi|\hat{C}|\psi\rangle+\left\langle\Delta\hat{A}^2\right\rangle\geq 0$$

$$\Rightarrow\left[\langle\psi|\hat{C}|\psi\rangle\right]^2-4\left\langle\Delta\hat{A}^2\right\rangle\left\langle\Delta\hat{B}^2\right\rangle\leq 0$$

$$\Rightarrow\left\langle\Delta\hat{A}^2\right\rangle\left\langle\Delta\hat{B}^2\right\rangle\geq\frac{\left[\langle\psi|\hat{C}|\psi\rangle\right]^2}{4}$$

$$\Rightarrow\sigma_A^2\sigma_B^2\geq\frac{\left[\langle\psi|\hat{C}|\psi\rangle\right]^2}{4}$$

$$\lambda^2 a-\lambda b+c\geq 0$$

This can only happen if:

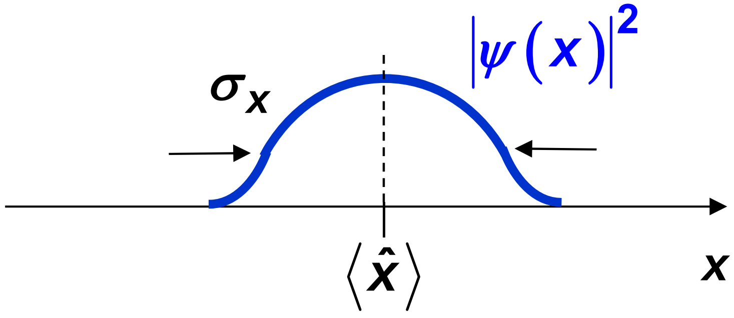
$$b^2-4ac\leq 0$$

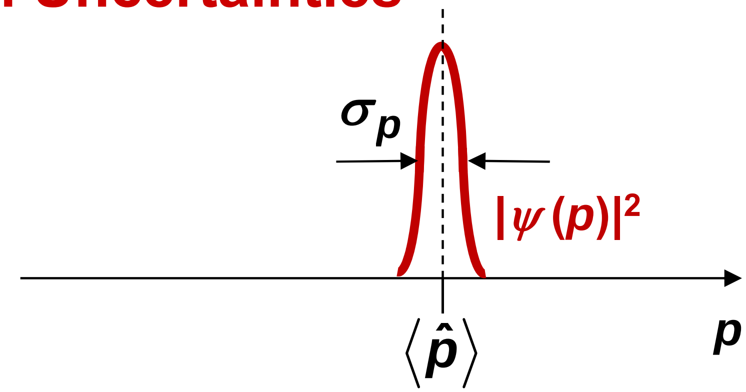
Special case: $\left[\hat{x},\hat{p}\right]=i\hbar$

$$\Rightarrow\left\langle\Delta\hat{x}^2\right\rangle\left\langle\Delta\hat{p}^2\right\rangle\geq\frac{\hbar^2}{4}$$

$$\Rightarrow\sigma_x^2\sigma_p^2\geq\frac{\hbar^2}{4}$$

Position and Momentum Uncertainties





The a-priori uncertainty in the position is defined as: $\Delta x = \sigma_x$

The a-priori uncertainty in the momentum is defined as: $\Delta p = \sigma_p$

What can we say about $\Delta x = \sigma_x$ and $\Delta p = \sigma_p$?

Answer:

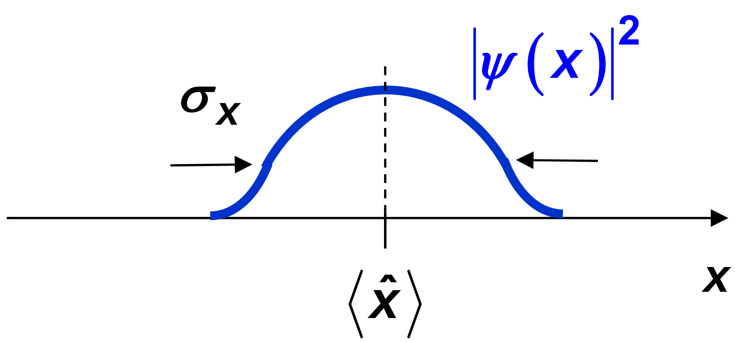
$$\sigma_x \, \sigma_p \geq \frac{\hbar}{2}$$

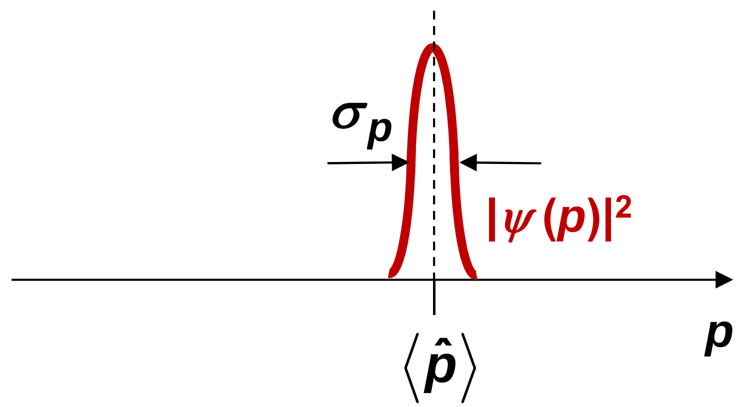
Heisenberg Uncertainty Relation: Statement #1

For any quantum state $|\psi\rangle$, the product of the position and momentum uncertainties satisfy:

$$\sigma_x \, \sigma_p \geq \frac{\hbar}{2}$$

The Heisenberg principle stated as above is a statement about the a-priori uncertainties in the position and momentum of a particle (i.e. before any measurement is actually made)





Commutation Relation and Uncertainty Principle

Suppose two Hermitian operators do commute:

$$\left[\hat{A},\hat{B}\right]=\hat{A}\hat{B}-\hat{B}\hat{A}=0$$

Then they can have a common set of eigenvectors:

$$\hat{A}\left|v_j\right\rangle=a_j\left|v_j\right\rangle$$

$$\hat{B}\left|v_j\right\rangle=b_j\left|v_j\right\rangle$$

Suppose we take:

$$\left|\psi\right\rangle=\left|v_m\right\rangle$$

Then:

$$\left\langle\hat{A}\right\rangle=\left\langle\psi\right|\hat{A}\left|\psi\right\rangle=a_m \quad\rightarrow\quad \Delta\hat{A}=\hat{A}-\left\langle\hat{A}\right\rangle$$

$$\sigma_A^2=\left\langle\Delta\hat{A}^2\right\rangle=\left\langle\left(\hat{A}-\left\langle\hat{A}\right\rangle\right)^2\right\rangle=\left\langle\psi\right|\left(\hat{A}-\left\langle\hat{A}\right\rangle\right)^2\left|\psi\right\rangle=0$$

$$\left\langle\hat{B}\right\rangle=\left\langle\psi\right|\hat{B}\left|\psi\right\rangle=b_m \quad\rightarrow\quad \Delta\hat{B}=\hat{B}-\left\langle\hat{B}\right\rangle$$

$$\sigma_B^2=\left\langle\Delta\hat{B}^2\right\rangle=\left\langle\left(\hat{B}-\left\langle\hat{B}\right\rangle\right)^2\right\rangle=\left\langle\psi\right|\left(\hat{B}-\left\langle\hat{B}\right\rangle\right)^2\left|\psi\right\rangle=0$$

$$\longrightarrow \quad \sigma_A^2\sigma_B^2=0$$

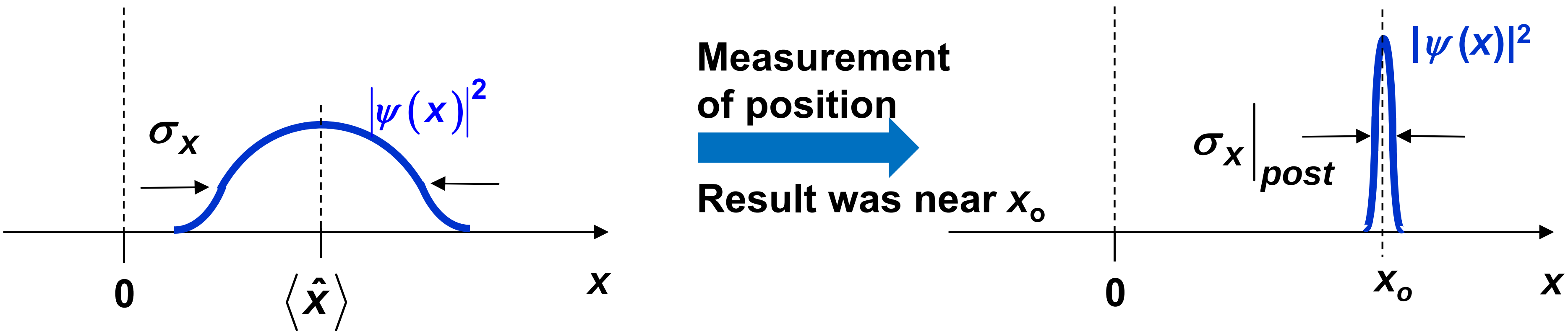
Heisenberg Uncertainty Relation and Measurement Induced Disturbances

If a measurement is made to determine the position of a particle with an accuracy of Δ, then after the measurement the uncertainty $\sigma_x|_{post}$ in the position of the particle in the post-measurement quantum state must satisfy:

$$\sigma_x\big|_{post} \leq \Delta$$

Before measurement

After measurement: collapsed wavefunction



After the measurement the uncertainty $\sigma_p|_{post}$ in the momentum of the particle in the post-measurement quantum state must satisfy:

$$\sigma_p\big|_{post} \geq \frac{\hbar}{2\,\sigma_x\big|_{post}} \geq \frac{\hbar}{2\Delta}$$

Heisenberg Uncertainty Relation: Statement #2

If a measurement is made to determine the position of a particle with an accuracy of $\Delta x = \sigma_x|_{post}$, then after the measurement the uncertainty $\sigma_p|_{post}$ in the momentum of the particle in the post-measurement quantum state must satisfy:

$$\sigma_p\Big|_{post} \geq \frac{\hbar}{2\,\sigma_x\Big|_{post}}$$

If a measurement is made to determine the momentum of a particle with an accuracy of $\Delta p = \sigma_p|_{post}$, then after the measurement the uncertainty $\sigma_x|_{post}$ in the position of the particle in the post-measurement quantum state must satisfy:

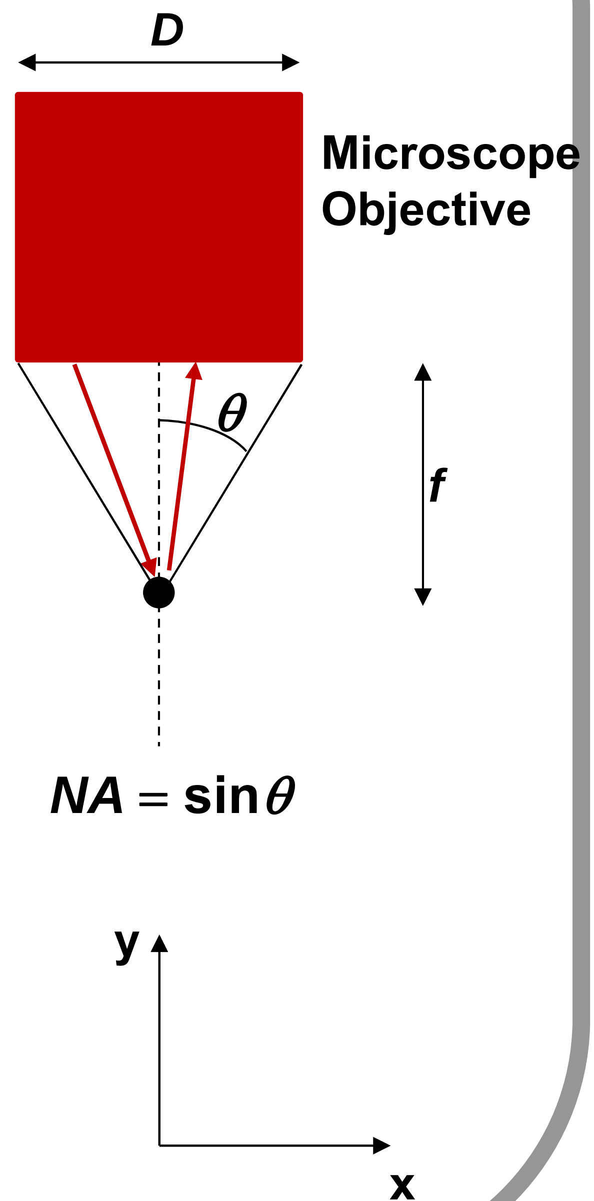
$$\sigma_x\Big|_{post} \geq \frac{\hbar}{2\,\sigma_p\Big|_{post}}$$

Written this way, Heisenberg relations are not statements about a-priori uncertainties in wavefunctions. They tell us that for two non-commuting observables, more the accuracy of the measurement of one observable, less the certainty in the value of the other observable post-measurement

In other words, measurements cause unavoidable disturbances !!!

The Heisenberg Microscope and the Uncertainty Relation

Werner Heisenberg considered the following thought experiment to measure the location of an electron with an optical microscope



The spatial resolution of a microscope is given by:

$$\frac{\lambda}{2NA} = \frac{\lambda}{2(D/2f)} = \frac{\lambda}{2\sin\theta}$$

So after the electron position has been measured, the remaining uncertainty Δx in its position <u>post-measurement</u> must be:

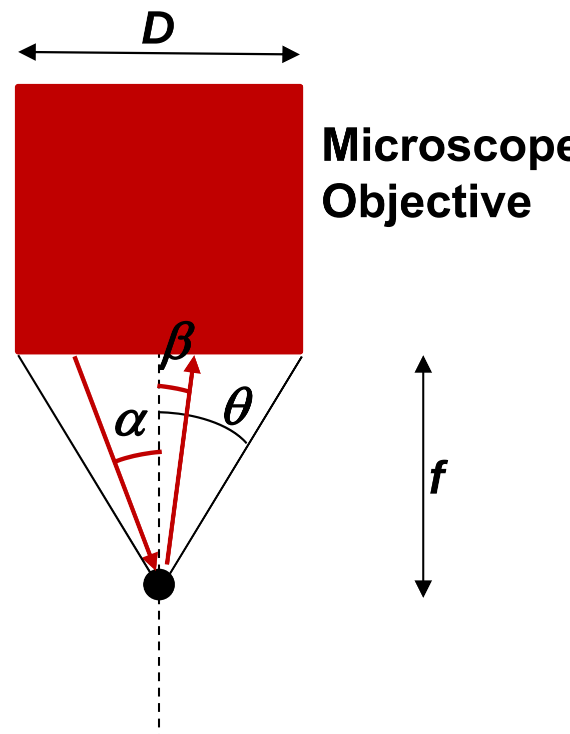
$$\Delta x \sim \frac{\lambda}{2\sin\theta}$$

Next, we find the uncertainty Δp_x in the x-component of the momentum of the electron <u>post-measurement</u>

The Heisenberg Microscope and the Uncertainty Relation

Suppose the electron was initially sitting at rest

Consider the light ray shown, that comes from the microscope at an angle α, bounces off the electron, and goes back into the microscope with an angle β:



Photon momentum: $p = \frac{2\pi\hbar}{\lambda}$

Incident photon momentum y-component: $p_y^{inc} = \frac{2\pi\hbar}{\lambda}\cos\alpha$

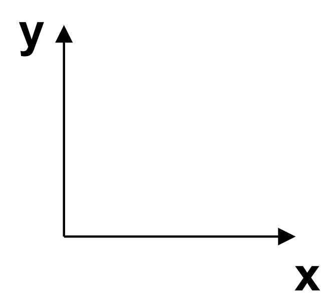
Incident photon momentum x-component: $p_x^{inc} = \frac{2\pi\hbar}{\lambda}\sin\alpha$

$NA = \sin\theta$

Once the photon has bounced off the electron, momentum conservation requires:

$$p_x^{inc} = m\Delta v_x + p_x^{ref}$$

$$p_y^{inc} = m\Delta v_y + p_y^{ref}$$



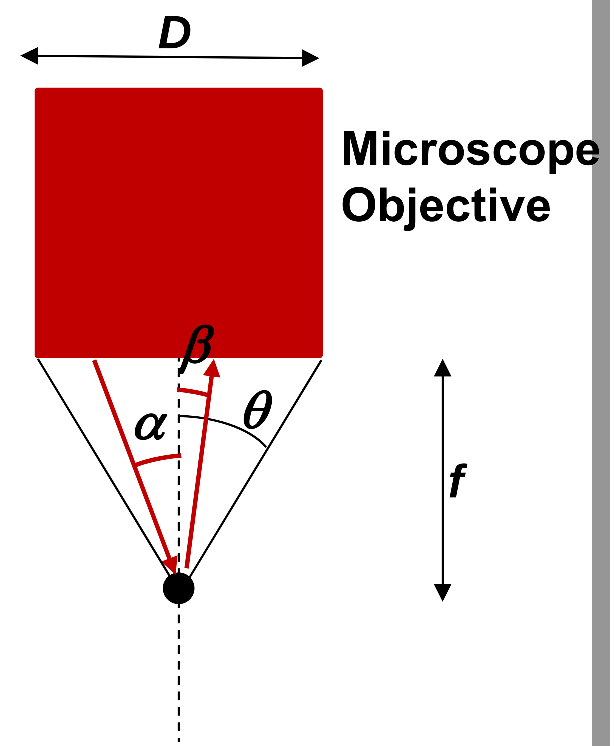
This implies:

$$m\Delta v_x = p_x^{inc} - p_x^{ref} = \frac{2\pi\hbar}{\lambda}[\sin\alpha - \sin\beta]$$

The Heisenberg Microscope and the Uncertainty Relation

Suppose the electron was initially sitting at rest

Consider the light ray shown, that comes from the microscope at an angle α, bounces off the electron, and goes back into the microscope with an angle β:



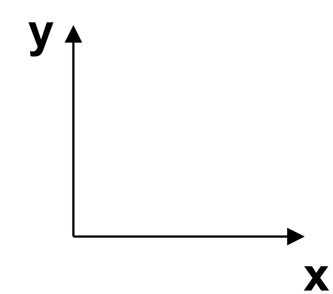
The momentum kick to the particle is then :

$$m\Delta v_x = p_x^{inc} - p_x^{ref} = \frac{2\pi\hbar}{\lambda}\left[\sin\alpha - \sin\beta\right]$$

$$NA = \sin\theta$$

The mean square average momentum kick to the particle is of the order:

$$\Delta p_x \sim \sqrt{\left\langle \left(m\Delta v_x\right)^2 \right\rangle} \sim \frac{2\pi\hbar}{\lambda}\left[\frac{\sin\theta}{\pi}\right]$$



So the product of the <u>post-measurement</u> uncertainty in the position and momentum of the particle is:

$$\Delta x\, \Delta p_x \sim \frac{\lambda}{2\sin\theta}\,\frac{2\pi\hbar}{\lambda}\,\frac{\sin\theta}{\pi} \sim \hbar$$

The Heisenberg Microscope and the Uncertainty Relation

Heisenberg's uncertainty principle is not a about the problems or difficulties in making accurate measurements

To say that the particle really has some well defined values of both momentum and position, and we just can't measure both accurately, would be completely wrong

What would be correct is that neither the particle has well defined values of both momentum and position, and nor can we ever measure them both accurately.

Furthermore, Bohr's Copenhagen interpretation emphasizes that if one cannot ever simultaneously measure two things accurately, in a very fundamental way, then one should not ascribe simultaneous physical reality to them in the first place!

Measurements, Disturbances, and Reality

"We can no longer speak of the behaviour of the particle independently of the process of observation. As a final consequence, the natural laws formulated mathematically in quantum theory no longer deal with the elementary particles themselves but with our knowledge of them. Nor is it any longer possible to ask whether or not these particles exist in space and time objectively ... When we speak of the picture of nature in the exact science of our age, we do not mean a picture of nature so much as a picture of our relationships with nature. ...Science no longer confronts nature as an objective observer, but sees itself as an actor in this interplay between man and nature. The scientific method of analyzing, explaining and classifying has become conscious of its limitations, which arise out of the fact that by its intervention science alters and refashions the object of investigation. In other words, method and object can no longer be separated."

Werner Heisenberg in "The Physicist's Conception of Nature"

Measurements, Disturbances, and Commutation Relations

Suppose two Hermitian operators, corresponding to two observables, have the following commutation relations:

$$\left[\hat{A},\hat{B}\right] \neq 0$$

Also suppose that these operators have the following eigenvectors and eigenstates:

$$\hat{A}\left|a_j\right\rangle = \alpha_j \left|a_j\right\rangle \qquad \hat{B}\left|b_j\right\rangle = \beta_j \left|b_j\right\rangle$$

Also suppose that the eigenvalues are all different

Consider an arbitrary quantum state: $\left|\psi\right\rangle$

We are going to make simultaneous measurement of observables A and B on the state $\left|\psi\right\rangle$

Simultaneous measurements mean that the measurements are made in succession, one after the other, so fast that there is NO significant time-evolution in between two successive measurements and so we can forget about time-evolution in what follows

Measurements, Disturbances, and Commutation Relations

1) Suppose we first make a measurement of the observable A:

State just before the measurement: $|\psi\rangle$

Result of the measurement: α_m —A-priori probability for this particular result→ $|\langle a_m|\psi\rangle|^2$

State just after the measurement (after collapse):

$$\left(|a_m\rangle\langle a_m|\right)|\psi\rangle = \langle a_m|\psi\rangle\,|a_m\rangle \xrightarrow{\text{Normalize}} |a_m\rangle$$

2) Suppose we make a second measurement of the observable A right after the first one:

State just before the measurement: $|a_m\rangle$ → Eigenstate of observable A

Result of the measurement: α_m —A-priori probability for this particular result→ 1

State just after the measurement (after collapse):

$$\left(|a_m\rangle\langle a_m|\right)|a_m\rangle = |a_m\rangle$$

Repeated measurements of observable A does not change the result

Measurements, Disturbances, and Commutation Relations

Lets try again:

1) Suppose we first make a measurement of the observable A:

State just before the measurement: $|\psi\rangle$

Result of the measurement: α_m —A-priori probability for this particular result→ $|\langle a_m|\psi\rangle|^2$

State just after the measurement (after collapse):

$$\left(|a_m\rangle\langle a_m|\right)|\psi\rangle = \langle a_m|\psi\rangle\,|a_m\rangle \xrightarrow{\text{Normalize}} |a_m\rangle$$

2) Suppose we then make a measurement of the observable B:

State just before the measurement: $|a_m\rangle$ → Eigenstate of observable A

Result of the measurement: β_n —A-priori probability for this particular result→ $|\langle b_n|a_m\rangle|^2$

State just after the measurement (after collapse):

$$\left(|b_n\rangle\langle b_n|\right)|a_m\rangle = \langle b_n|a_m\rangle\,|b_n\rangle \xrightarrow{\text{Normalize}} |b_n\rangle$$

Measurements, Disturbances, and Commutation Relations

3) Suppose we then again make a measurement of the observable A:

State just before the measurement: $\left|b_n\right\rangle$ → Eigenstate of observable B

Result of the measurement: α_j —— A-priori probability for this particular result → $\left|\left\langle a_j \middle| b_n\right\rangle\right|^2$

State just after the measurement (after collapse):

$$\left(\left|a_j\right\rangle\left\langle a_j\right|\right)\left|\psi\right\rangle = \left\langle a_j \middle| b_n\right\rangle\left|a_j\right\rangle \xrightarrow{\text{Normalize}} \left|a_j\right\rangle$$

The measurement of observable B has disturbed the quantum state so that the subsequent measurement of observable A again does not give the same result as it did the first time!

We say that:

- A and B are incompatible observables
- They cannot BOTH be measured simultaneously with arbitrary accuracy
- Measurement of one of them disturbs the value of the other one

So when are two observables, A and B, compatible observables??

Measurements, Disturbances, and Commutation Relations

Two Hermitian operators, corresponding to two observables, are compatible if they commute:

$$\left[\hat{A},\hat{B}\right]=0$$

If two operators, corresponding to two observables, commute then the corresponding observables are compatible and can both be measured simultaneously with arbitrary good accuracy

The man idea:

If two operators commute, they can have a common set of eigenvectors:

$$\hat{A}\left|a_j\right\rangle=\alpha_j\left|a_j\right\rangle \qquad \hat{B}\left|a_j\right\rangle=\beta_j\left|a_j\right\rangle$$

Also suppose that the eigenvalues are all different …………PTO

Measurements, Disturbances, and Commutation Relations

Lets try again, now for compatible observables:

1) Suppose we first make a measurement of the observable A:

State just before the measurement: $|\psi\rangle$

Result of the measurement: α_m —A-priori probability for this particular result→ $|\langle a_m|\psi\rangle|^2$

State just after the measurement (after collapse):

$$\left(|a_m\rangle\langle a_m|\right)|\psi\rangle = \langle a_m|\psi\rangle\,|a_m\rangle \xrightarrow{\text{Normalize}} |a_m\rangle$$

2) Suppose we then make a measurement of the observable B:

State just before the measurement: $|a_m\rangle$ → Eigenstate of observables A and B

Result of the measurement: β_m —A-priori probability for this particular result→ 1

State just after the measurement:

$$\left(|a_m\rangle\langle a_m|\right)|a_m\rangle = |a_m\rangle$$

→ Remains the same as the one before the measurement of B

Measurements, Disturbances, and Commutation Relations

3) Suppose we then again make a measurement of the observable A:

State just before the measurement: $|a_m\rangle$ → Eigenstate of observables A and B

Result of the measurement: α_m → A-priori probability for this particular result: 1

State just after the measurement:

$$\left(|a_m\rangle\langle a_m|\right)|a_m\rangle = |a_m\rangle$$

→ Remains the same as the one before the measurement of *A*

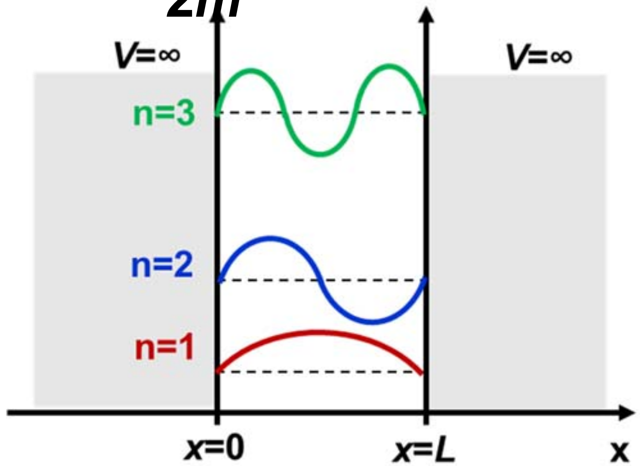
The measurement of observable B has not disturbed the quantum state so that the subsequent measurement of observable A again gives the same result as it did the first time!

We say that:

- A and B are compatible observables
- They can BOTH be measured simultaneously with arbitrary accuracy
- Measurement of one of them does not disturb the value of the other one

Uncertainty Relations and Confinement

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$$



Consider the problem of a particle in an infinite potential well

Question: **Why is the lowest energy not just equal to zero?**

Suppose the lowest energy were zero and the particle then is at rest somewhere inside the well. **But then since we know that the particle is at rest somewhere inside the well, the standard deviation in its position must be:**

$$\sigma_x \sim \frac{L}{2} \qquad \sigma_x^2 \sim \frac{L^2}{4}$$

From Heisenberg uncertainty relation:

$$\sigma_p^2 \geq \frac{\hbar^2}{4\sigma_x^2} = \frac{\hbar^2}{L^2}$$

The average energy of the particle is then:

$$\langle \hat{H} \rangle = \left\langle \frac{\hat{p}^2}{2m} \right\rangle = \frac{\sigma_p^2}{2m} \geq \frac{\hbar^2}{2m}\left(\frac{1}{L^2}\right)$$

⇨ **There is no way the particle can have zero energy when it is CONFINED**

Commutation Relation with Hamiltonian and Time Evolution

Consider an observable O and the corresponding operator $\hat{O}$

The mean value of the observable at time *t* is given by:

$$\langle \hat{O} \rangle (t) = \langle \psi(t) | \hat{O} | \psi(t) \rangle$$

where:

$$|\psi(t)\rangle = e^{-i\frac{\hat{H}}{\hbar}t} |\psi(t=0)\rangle$$

Now suppose the observable commutes with the Hamiltonian:

$$\left[\hat{O}, \hat{H}\right] = 0$$

Then:

$$\langle \hat{O} \rangle (t) = \langle \psi(t) | \hat{O} | \psi(t) \rangle = \langle \psi(t=0) | e^{+i\frac{\hat{H}}{\hbar}t} \hat{O} e^{-i\frac{\hat{H}}{\hbar}t} | \psi(t=0) \rangle$$

$$= \langle \psi(t=0) | e^{+i\frac{\hat{H}}{\hbar}t} e^{-i\frac{\hat{H}}{\hbar}t} \hat{O} | \psi(t=0) \rangle$$

→ **Since:** $\left[\hat{O}, \hat{H}\right] = 0$

$$= \langle \psi(t=0) | \hat{O} | \psi(t=0) \rangle$$

$$= \langle \hat{O} \rangle (t=0)$$

→ **When an operator commutes with the Hamiltonian, its mean value is time-independent !!**