Department of Electrical and Computer Engineering, Cornell University

ECE 4060: Quantum Physics and Engineering

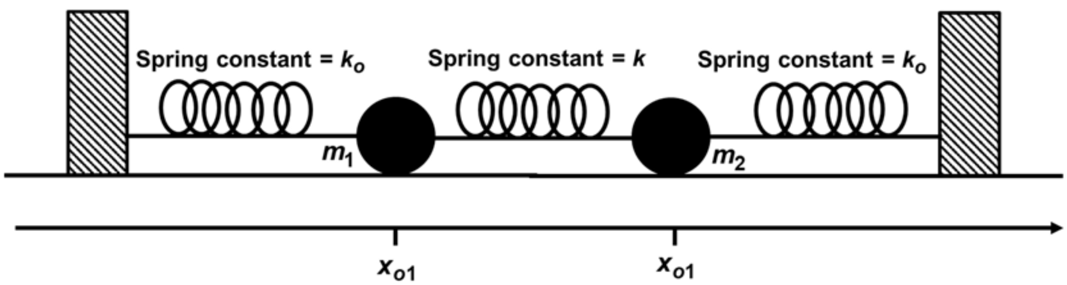
Fall 2020

Homework 9 **Due on Dec. 02, 2020 by 5:00 PM (via email)**

Problem 9.1: (Coupled Simple Harmonic Oscillators)

You might be surprised to know that in the standard model of Physics almost the entire universe and all matter, forces, and energy in it can be described as one big set of coupled simple harmonic oscillators. This is how quantum field theory describes this universe. And the model works for everything ……..well, almost everything except gravity. Here we will consider a much simple problem of two coupled simple harmonic oscillators. But the importance such models hold for physics cannot be overestimated.

Consider two coupled simple harmonic oscillators:



The Hamiltonian is,

$$\hat{H} = \left[\frac{\hat{p}_1^2}{2m_1} + \frac{1}{2}k_o\left(\hat{x}_1 - x_{o1}\right)^2\right] + \left[\frac{\hat{p}_2^2}{2m_2} + \frac{1}{2}k_o\left(\hat{x}_2 - x_{o2}\right)^2\right] + \frac{1}{2}k\left[\left(\hat{x}_1 - x_{o1}\right) - \left(\hat{x}_2 - x_{o2}\right)\right]^2$$

The coupling is introduced by the coupling spring in between the two masses. One can define new position operators (to get rid of the offsets),

$$\hat{y}_1 = \hat{x}_1 - x_{o1}$$

$$\hat{y}_2 = \hat{x}_2 - x_{o2}$$

and write the Hamiltonian as,

$$\hat{H} = \left[\frac{\hat{p}_1^2}{2m_1} + \frac{1}{2}k_o\hat{y}_1^2\right] + \left[\frac{\hat{p}_2^2}{2m_2} + \frac{1}{2}k_o\hat{y}_2^2\right] + \frac{1}{2}k\left(\hat{y}_1 - \hat{y}_2\right)^2$$

Note that the Hamiltonian is really describing a composite system of two different masses, 1 and 2, so if one really was trying to be very particular about the tensor notation used to describe composite systems, one should write the Hamiltonian as follows,

$$\hat{H} = \left[\frac{\hat{p}_1^2}{2m_1} + \frac{1}{2}k_o\hat{y}_1^2\right] + \left[\frac{\hat{p}_2^2}{2m_2} + \frac{1}{2}k_o\hat{y}_2^2\right] + \frac{1}{2}k\left(\hat{y}_1 - \hat{y}_2\right)^2$$

$$= \left[\frac{\hat{p}_1^2}{2m_1} + \frac{1}{2}(k_o + k)\hat{y}_1^2\right] + \left[\frac{\hat{p}_2^2}{2m_2} + \frac{1}{2}(k_o + k)\hat{y}_2^2\right] - k\,\hat{y}_1\,\hat{y}_2$$

$$= \left[\frac{\hat{p}_1^2}{2m_1} + \frac{1}{2}(k_o + k)\hat{y}_1^2\right] \otimes \hat{1}_2 + \hat{1}_1 \otimes \left[\frac{\hat{p}_2^2}{2m_2} + \frac{1}{2}(k_o + k)\hat{y}_2^2\right] - k\,\hat{y}_1 \otimes \hat{y}_2$$

But literally nobody writes the Hamiltonian using the full tensor notation. So I will stick to the simpler notation,

$$\hat{H} = \left[\frac{\hat{p}_1^2}{2m_1} + \frac{1}{2}k_o\hat{y}_1^2\right] + \left[\frac{\hat{p}_2^2}{2m_2} + \frac{1}{2}k_o\hat{y}_2^2\right] + \frac{1}{2}k\left(\hat{y}_1 - \hat{y}_2\right)^2$$

Note that the only non-zero, commutation relations are,

$$\left[\hat{y}_1, \hat{p}_1\right] = i\hbar \qquad \left[\hat{y}_2, \hat{p}_2\right] = i\hbar$$

One can switch to creation and destructor operator notation by defining,

$$\hat{a}_1 = i\sqrt{\frac{1}{2m_1\hbar\omega_1}}\hat{p}_1 + \sqrt{\frac{m_1\omega_1}{2\hbar}}\,\hat{y}_1 \qquad \hat{a}_2 = i\sqrt{\frac{1}{2m_2\hbar\omega_2}}\hat{p}_2 + \sqrt{\frac{m_2\omega_2}{2\hbar}}\,\hat{y}_2$$

$$\hat{a}_1^\dagger = -i\sqrt{\frac{1}{2m_1\hbar\omega_1}}\hat{p}_1 + \sqrt{\frac{m_1\omega_1}{2\hbar}}\,\hat{y}_1 \qquad \hat{a}_2^\dagger = -i\sqrt{\frac{1}{2m_2\hbar\omega_2}}\hat{p}_2 + \sqrt{\frac{m_2\omega_2}{2\hbar}}\,\hat{y}_2$$

$$\omega_1 = \sqrt{\frac{k_o + k}{m_1}} \qquad \omega_2 = \sqrt{\frac{k_o + k}{m_2}}$$

The Hamiltonian becomes,

$$\hat{H} = \left[\frac{\hat{p}_1^2}{2m_1} + \frac{1}{2}(k_o + k)\hat{y}_1^2\right] + \left[\frac{\hat{p}_2^2}{2m_2} + \frac{1}{2}(k_o + k)\hat{y}_1^2\right] - k\,\hat{y}_1\,\hat{y}_2$$

$$= \hbar\omega_1\left(\hat{a}_1^\dagger\hat{a}_1 + \frac{1}{2}\right) + \hbar\omega_2\left(\hat{a}_2^\dagger\hat{a}_2 + \frac{1}{2}\right) - k\sqrt{\frac{\hbar}{2m_1\omega_1}}\sqrt{\frac{\hbar}{2m_2\omega_2}}\left(\hat{a}_1 + \hat{a}_1^\dagger\right)\left(\hat{a}_2 + \hat{a}_2^\dagger\right)$$

You can see the problem now. If you consider a tensor product of the standard number states, $|n\rangle_1 \otimes |m\rangle_2$, where,

$$\hat{a}_1^\dagger\hat{a}_1|n\rangle_1 \otimes |m\rangle_2 = n|n\rangle_1 \otimes |m\rangle_2$$

$$\hat{a}_2^\dagger\hat{a}_2|n\rangle_1 \otimes |m\rangle_2 = m|n\rangle_1 \otimes |m\rangle_2$$

then these states will be the eigenstates of the first two terms in the Hamiltonian but will not be eigenstates of the last coupling term in the Hamiltonian, and therefore these states will not be the eigenstates of the full Hamiltonian. What to do now?

Here one can employ a very standard technique in physics where one decomposes the problem into center-of-mass coordinates and relative coordinates. This works nicely if $m_1=m_2$. You will see that in these new coordinates, the two coupled simple harmonic oscillators will appear as two new uncoupled simple harmonic oscillators that can be solved independently using standard techniques and then a tensor product of their respective number states will be a valid eigenstate of the full Hamiltonian.

Define two new coordinate operators as follows,

$$\hat{R} = \frac{m_1 \hat{y}_1 + m_2 \hat{y}_2}{M} \qquad \{M = m_1 + m_2$$

$$\hat{r} = \hat{y}_1 - \hat{y}_2$$

$\hat{R}$ is the operator for the center-of-mass coordinate, and $\hat{r}$ is the operator for the relative coordinate. The corresponding momentum operators are defined as,

$$\hat{Q} = \hat{p}_1 + \hat{p}_2$$

$$\hat{p} = \frac{m_2 \hat{p}_1 - m_1 \hat{p}_2}{M}$$

Note that in classical physics: $\begin{cases} Q(t) = M \dfrac{dR(t)}{dt} = p_1(t) + p_2(t) \\ p(t) = \dfrac{m_1 m_2}{M} \dfrac{dr(t)}{dt} = \mu \dfrac{dr(t)}{dt} = \dfrac{m_2 p_1(t) - m_1 p_2(t)}{M} \end{cases}$

The mass,

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_1 m_2}{M}$$

is called the "reduced mass". The momentum $\hat{Q}$ is the total momentum of the two masses and $\hat{p}$ is called the relative motion momentum.

a) Show that the only non-zero commutators among $\hat{R}, \hat{Q}, \hat{r}$, and $\hat{p}$ are,

$$\left[\hat{R}, \hat{Q}\right] = i\hbar \qquad \left[\hat{r}, \hat{p}\right] = i\hbar$$

b) Assuming for simplicity that $m_1 = m_2 = m$, show that the in terms of the new coordinates and momenta, the Hamiltonian becomes,

$$\hat{H} = \left[\frac{\hat{p}_1^2}{2m_1} + \frac{1}{2} k_o \hat{y}_1^2\right] + \left[\frac{\hat{p}_2^2}{2m_2} + \frac{1}{2} k_o \hat{y}_2^2\right] + \frac{1}{2} k \left(\hat{y}_1 - \hat{y}_2\right)^2$$

$$= \left[\frac{\hat{Q}^2}{2M} + \frac{1}{2}(2k_o)\hat{R}^2\right] + \left[\frac{\hat{p}^2}{2\mu} + \frac{1}{2}\left(\frac{k_o}{2} + k\right)\hat{r}^2\right]$$

The Hamiltonian above now appears as an ***uncoupled*** Hamiltonian of two different and independent simple harmonic oscillators.

c) What is the frequency Ω of the center-of-mass motion simple harmonic oscillators and what is the frequency ω of the relative motion simple harmonic oscillator?

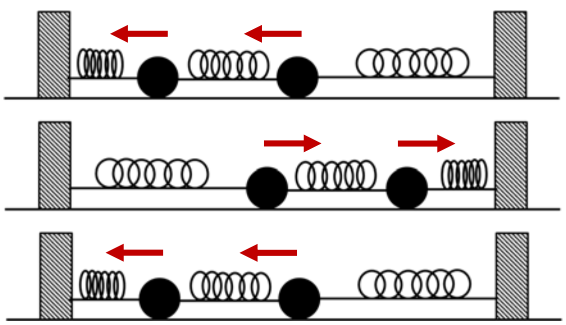
d) Define creation and destruction operators, $\hat{a}$ and $\hat{a}^\dagger$, for the center-of-mass motion simple harmonic oscillator and also define creation and destruction operators, $\hat{b}$ and $\hat{b}^\dagger$, for the relative motion simple harmonic oscillator such that the Hamiltonian becomes,

$$\hat{H} = \hbar\Omega\left[\hat{a}^\dagger \hat{a} + \frac{1}{2}\right] + \hbar\omega\left[\hat{b}^\dagger \hat{b} + \frac{1}{2}\right]$$

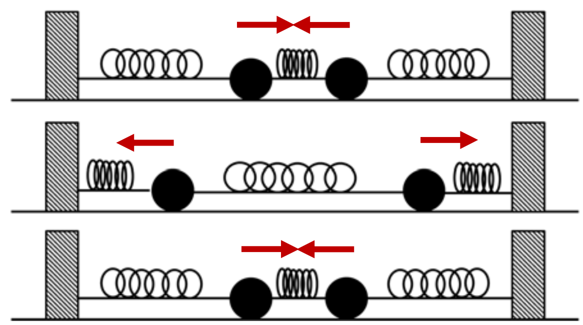
e) What are all the eigenstates and eigenvalues of the Hamiltonian?

What we did in this problem:

What we really did in this problem was to change the description of the problem of two coupled SHOs, which was originally expressed in terms of the individual coordinates of the two masses, and expressed the problem in terms of "collective coordinates" that better captured the underlying physics. These collective coordinates captured the two different modes of oscillation of the two coupled masses: a) the mode in which the two masses move together in-phase such that the center spring is never compressed or stretched, and b) the mode in which the two masses move out-of-phase and in opposite directions. These two modes of oscillation are depicted below. These modes are uncoupled. These uncoupled modes are sometimes also called the "normal" modes of the system. Therefore, the Hamiltonian, when expressed in terms of the new coordinates also consists of two uncoupled SHOs and each SHO corresponds to one of the two normal modes of oscillation.



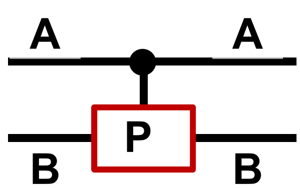
Oscillation of the center-of-mass motion SHO



Oscillation of the relative motion SHO

Problem 9.2: (Mystery Quantum Gate)

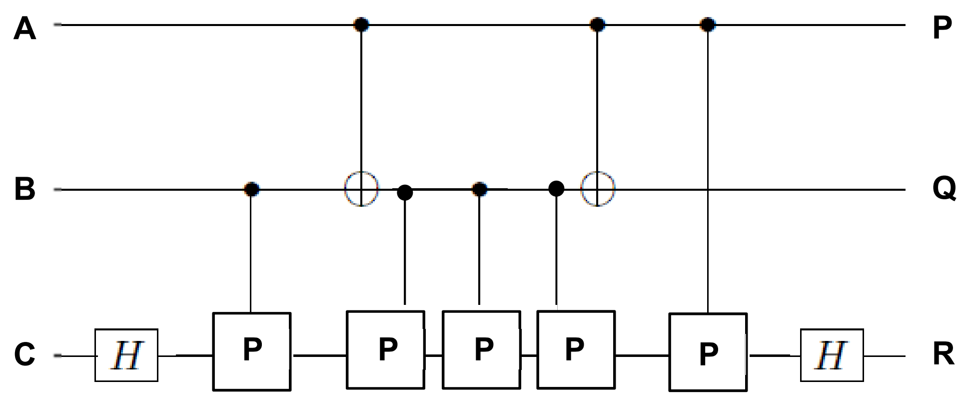
The control-P gate, shown below, has the following property:



The qubit B is acted upon by the quantum P gate if A qubit is $|1\rangle_A$ and nothing happens to qubit B if A qubit is $|0\rangle_A$. The control-P gate is just another one in the family of two-qubit control gates (which also includes, for example, control-X, control-Y, and control-Z gates). The P gate is a one-qubit gate that is represented by the following unitary matrix,

$$\hat{U} = \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix}$$

a) Now consider the quantum gate shown below that operates on three qubits (it's a three-qubit gate). It is made using the following two-qubit and one-qubit gates: Hadamard gate, control-X gate, and control-P gate.



As we said in the lectures, the operation of any quantum gate can be described by its action on the unentangled basis states. There are 8 unentangled basis states that describe the three-qubit input (as in the table below). Fill the following table below for the output states, and describe what the above three-qubit gate does. Do not assume that the output qubits are not entangled across P, Q, and R.

Input			Output
A	B	C	P, Q, and R
|0>	**|0>**	**|0>**	
|0>	**|0>**	**|1>**	
|0>	**|1>**	**|0>**	
|0>	**|1>**	**|1>**	
|1>	**|0>**	**|0>**	
|1>	**|0>**	**|1>**	
|1>	**|1>**	**|0>**	
|1>	**|1>**	**|1>**	