ECE 4060: Quantum Physics and Engineering

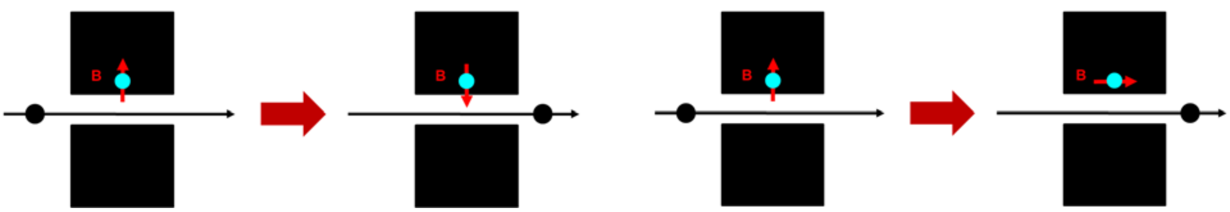
Fall 2020

Homework 7 **Due on Nov. 04, 2020 by 5:00 PM (via email)**

Problem 7.1: (The Double-Slit Experiment)

Consider the double-slit experiment, as discussed in the lecture handouts, where a spin-qubit was placed in slit 1 to detect the passage of the electron through slit 1. But, the engineer involved in designing the experiment miscalculated the interaction between the passing electron (system A) and the spin qubit (system B) such that when the spin qubit is prepared in the state $|z\uparrow\rangle$, and the electron passes through slit 1, the spin qubit goes into the state $|x\uparrow\rangle$ as shown below, instead of the state $|z\downarrow\rangle$. Consequently, the state of the composite system, after the electron has passed through the slits, is:

$$|\chi\rangle \approx \frac{tA}{\sqrt{2}}\left[|\phi_1\rangle_A \otimes |x\uparrow\rangle_B + |\phi_2\rangle_A \otimes |z\uparrow\rangle_B\right]$$



The correct design **The faulty design**

a) What is the probability of detecting an electron at location $\vec{r}$ beyond the slits?

b) Does your answer in part (a) show an interference pattern?

The interference pattern in a double slit experiment of is characterized by the "fringe contrast" which is defined as follows: the probability of locating the electron is computed and plotted as a function of the angle θ from the z-axis at a fixed distance r from the slits and for $\phi = 0$. The "fringe contrast" is defined as the ratio of the maximum to the minimum probability in this plot. The fringe contrast is infinite in a double slit experiment when there is no observation of any kind because in that case the minimum probability is zero for angles θ along which complete destructive interference takes place.

c) Calculate the fringe contrast for the probability computed in part (a).

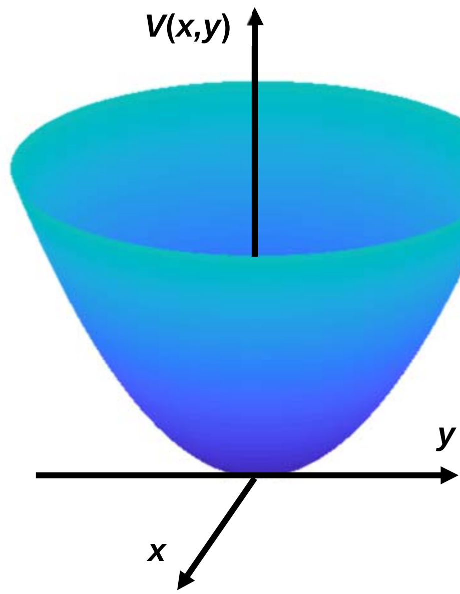
Problem 7.2: (The Two-Dimensional (2D) SHO: Solving a Complex Quantum Problem)

Consider a particle confined to two dimensions (as opposed to in one dimension) and sitting in a 2D quadratic potential well. The Hamiltonian is,

$$\hat{H} = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2m} + \frac{1}{2} m\omega_o^2 \left(\hat{x}^2 + \hat{y}^2\right)$$

Atoms in crystals can be modeled as 3D SHOs. Atoms in 2D materials (like graphene) can be modeled as 2D SHOs. As in the 1D case, the Hamiltonian is most easily solved algebraically (rather than solving partial differential equations).

From the point of view of classical physics, looking at the 2D potential profile, one wonders whether a particle placed in this potential well swings to and fro (as in the 1D case) or if the particle goes around in circles (i.e. has a net angular momentum). From the point of view of quantum mechanics, both things can happen depending on whether one is interested in states of definite angular momentum or not.



Noting that the only non-zero commutators are $\left[\hat{x}, \hat{p}_x\right] = i\hbar$ **and** $\left[\hat{y}, \hat{p}_y\right] = i\hbar$ **, one can define two different sets of creation and destruction operators – one for each dimension,**

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega_o}}\left(\hat{a} + \hat{a}^\dagger\right) \qquad \hat{y} = \sqrt{\frac{\hbar}{2m\omega_o}}\left(\hat{b} + \hat{b}^\dagger\right)$$

$$\hat{p}_x = -i\sqrt{\frac{\hbar m\omega_o}{2}}\left(\hat{a} - \hat{a}^\dagger\right) \qquad \hat{p}_y = -i\sqrt{\frac{\hbar m\omega_o}{2}}\left(\hat{b} - \hat{b}^\dagger\right)$$

a) Find the following commutators (hint: they all have the same value):

$$\left[\hat{a}, \hat{b}\right] = ? \qquad \left[\hat{a}, \hat{b}^\dagger\right] = ? \qquad \left[\hat{b}, \hat{a}^\dagger\right] = ? \qquad \left[\hat{a}^\dagger, \hat{b}^\dagger\right] = ?$$

b) Find the following commutators (hint: they all have the same value):

$$\left[\hat{a},\hat{a}^{\dagger}\right]=? \qquad \left[\hat{b},\hat{b}^{\dagger}\right]=?$$

c) Show that the Hamiltonian can be written as,

$$\hat{H}=\hbar\omega_o\left(\hat{a}^{\dagger}\hat{a}+\hat{b}^{\dagger}\hat{b}+1\right)$$

Consider eigenstates of the Hamiltonian of the form $|n,m\rangle$ (i.e. labeled by two integers instead of a single integer) and where the orthogonality relation is, $\langle n',m'|n,m\rangle=\delta_{n,n'}\delta_{m,m'}$ and we also have,

$$\hat{a}|n,m\rangle=\sqrt{n}\,|n-1,m\rangle \qquad \hat{a}^{\dagger}|n,m\rangle=\sqrt{n+1}\,|n+1,m\rangle$$

$$\hat{b}|n,m\rangle=\sqrt{m}\,|n,m-1\rangle \qquad \hat{b}^{\dagger}|n,m\rangle=\sqrt{m+1}\,|n,m+1\rangle$$

$$|n,m\rangle=\frac{\left(\hat{a}^{\dagger}\right)^{n}}{\sqrt{n!}}\frac{\left(\hat{b}^{\dagger}\right)^{m}}{\sqrt{m!}}|0,0\rangle$$

d) What are all the eigenvalues of the Hamiltonian (in terms of the integers n and m) ?

We define position eigenstates in 2D as $|x,y\rangle$. The 2D wavefunction of an energy eigenstate is therefore, $\phi_{n,m}(x,y)=\langle x,y|n,m\rangle$. We need to find the wavefunction of the lowest energy eigenstate, $\phi_{0,0}(x,y)=\langle x,y|0,0\rangle$. We have two facts at our disposal: $\hat{a}|0,0\rangle=0$ and $\hat{b}|0,0\rangle=0$.

e) Use the relations $\hat{a}|0,0\rangle=0$ and $\hat{b}|0,0\rangle=0$, and find the wavefunction $\phi_{0,0}(x,y)=\langle x,y|0,0\rangle$ of the lowest energy eigenstate. Hint: try a product wavefunction of the type: $\phi_{0,0}(x,y)=f(x)g(y)$.

The problem is pretty much solved. But we have a problem ………… you will notice that the energy eigenstates are very much degenerate. For example, the state $|n=0,m=1\rangle$ and $|n=1,m=0\rangle$ have the same energy. This means that unlike in the 1D SHO case, the Hamiltonian alone does not form a CSCO in the 2D SHO. We need another observable, which commutes with the Hamiltonian, to form a CSCO. The rotational symmetry of the potential suggests that we look at the z-component of the angular momentum observable,

$$\hat{L}_z=\hat{x}\hat{p}_y-\hat{y}\hat{p}_x$$

First some rough qualitative remarks:

For any state $|\psi\rangle$,

$$\langle x,y|\hat{L}_z|\psi\rangle=\langle x,y|\hat{x}\hat{p}_y-\hat{y}\hat{p}_x|\psi\rangle=\frac{\hbar}{i}\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)\psi(x,y)$$

If we switch to the polar coordinates in 2D for which,

$$r = \sqrt{x^2 + y^2} \qquad \phi = \tan^{-1}(y/x)$$

Then we get,

$$\langle x,y|\hat{L}_z|\psi\rangle = \langle x,y|\hat{x}\hat{p}_y - \hat{y}\hat{p}_x|\psi\rangle = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\psi(x,y) = \frac{\hbar}{i}\frac{\partial}{\partial \phi}\psi(x,y)$$

This means that the wavefunction of any eigenstate of the angular momentum operator must have the angular dependence going as $\psi(x,y) \sim e^{\pm i\lambda\phi}$ because then,

$$\frac{\hbar}{i}\frac{\partial}{\partial \phi}\psi(x,y) = \pm\hbar\lambda\psi(x,y)$$

Furthermore, if we want the wavefunction to be single-valued everywhere in space (meaning $\psi(\phi + 2\pi) = \psi(\phi)$) then λ must be an integer. So the eigenvalues of the angular momentum operator must be of the form $\pm\hbar\lambda$ where λ is an integer.

f) Show that the angular momentum $\hat{L}_z$ commutes with the Hamiltonian, i.e. $\left[\hat{L}_z, \hat{H}\right] = 0$.

g) Show that the energy eigenstates $|n,m\rangle$ are in general **NOT** the eigenstates of the angular momentum $\hat{L}_z$, i.e. show that $\hat{L}_z|n,m\rangle$ is not proportional to $|n,m\rangle$. Hint: write $\hat{L}_z$ in terms of the creation and destruction operators and use the commutation relations you found in earlier parts. You should find: $\hat{L}_z = i\hbar\left(\hat{b}^\dagger\hat{a} - \hat{a}^\dagger\hat{b}\right)$.

Even though the energy eigenstates $|n,m\rangle$ are not in general the eigenstates of $\hat{L}_z$, we know that since $\left[\hat{L}_z, \hat{H}\right] = 0$ the Hamiltonian $\hat{H}$ and $\hat{L}_z$ must have a common set of eigenstates.

h) Show that the ground state $|n=0, m=0\rangle$ of the Hamiltonian, found earlier, is an eigenstate of the angular momentum $\hat{L}_z$ and find the corresponding eigenvalue.

i) Consider the following two sets of new creation and destruction operators,

$$\hat{d}_+^\dagger = \frac{1}{\sqrt{2}}\left(\hat{a}^\dagger + i\hat{b}^\dagger\right) \qquad \hat{d}_-^\dagger = \frac{1}{\sqrt{2}}\left(\hat{a}^\dagger - i\hat{b}^\dagger\right)$$

$$\hat{d}_+ = \frac{1}{\sqrt{2}}\left(\hat{a} - i\hat{b}\right) \qquad \hat{d}_- = \frac{1}{\sqrt{2}}\left(\hat{a} + i\hat{b}\right)$$

and the only non-zero commutators are,

$$\left[\hat{d}_+, \hat{d}_+^\dagger\right] = 1 \qquad \left[\hat{d}_-, \hat{d}_-^\dagger\right] = 1$$

j) Show that the Hamiltonian and the angular momentum operators can be written as,

$$\hat{H} = \hbar\omega_o \left(\hat{d}_+^\dagger \hat{d}_+ + \hat{d}_-^\dagger \hat{d}_- + 1 \right)$$

$$\hat{L}_z = \hbar \left(\hat{d}_+^\dagger \hat{d}_+ - \hat{d}_-^\dagger \hat{d}_- \right)$$

k) Show that the state defined as,

$$|p,s\rangle_c = \frac{\left(\hat{d}_+^\dagger\right)^p}{\sqrt{p!}} \frac{\left(\hat{d}_-^\dagger\right)^s}{\sqrt{s!}} |0,0\rangle \qquad \begin{cases} p = 0,1,2,3...... \\ s = 0,1,2,3...... \end{cases}$$

is an eigenstate of both the Hamiltonian with eigenvalue $\hbar\omega_o (p + s + 1)$ and of the angular momentum $\hat{L}_z$ with eigenvalue $\hbar(p - s)$. **Note that the orthogonality relation is,** ${}_c\langle p',s'|p,s\rangle_c = \delta_{p,p'}\delta_{s,s'}$ **and we also have,**

$$\hat{d}_+ |p,s\rangle_c = \sqrt{p}\,|p-1,s\rangle_c \qquad \hat{d}_+^\dagger |p,s\rangle_c = \sqrt{p+1}\,|p+1,s\rangle_c$$

$$\hat{d}_- |p,s\rangle_c = \sqrt{s}\,|p,s-1\rangle_c \qquad \hat{d}_-^\dagger |p,s\rangle_c = \sqrt{s+1}\,|p,s+1\rangle_c$$

PS: Note that $|p=0,s=0\rangle_c = |n=0,m=0\rangle = |0,0\rangle$ **from part (h).**

l) Show that the 2D wavefunction of the $+\hbar$ angular momentum eigenstate $|p=1,s=0\rangle_c$ is,

$$\langle x,y|p=1,s=0\rangle_c = \langle x,y|\hat{d}_+^\dagger|0,0\rangle = \frac{1}{\sqrt{\pi}}\left(\frac{m\omega_o}{\hbar}\right) r\, e^{-\frac{m\omega_o}{2\hbar}r^2} e^{i\phi} \qquad \left\{ r = \sqrt{x^2+y^2} \quad \phi = \tan^{-1}(y/x) \right.$$

PS: The above state is rotating in the anti-clockwise direction when viewed from the positive z-axis.

k) Show that the wavefunction of the $-\hbar$ angular momentum eigenstate $|p=0,s=1\rangle_c$ is,

$$\langle x,y|p=0,s=1\rangle_c = \langle x,y|\hat{d}_-^\dagger|0,0\rangle = \frac{1}{\sqrt{\pi}}\left(\frac{m\omega_o}{\hbar}\right) r\, e^{-\frac{m\omega_o}{2\hbar}r^2} e^{-i\phi} \qquad \left\{ r = \sqrt{x^2+y^2} \quad \phi = \tan^{-1}(y/x) \right.$$

PS: The above state is rotating in the clockwise direction when viewed from the positive z-axis.