ECE 4060: Quantum Physics and Engineering

Fall 2020

Homework 6 **Due on Oct. 28, 2020 by 5:00 PM (via email)**

Problem 6.1: (A Driven Two Level System (TLS): A Spin Qubit)

The mean values of time-dependent physical observables can be computed directly without first finding the time-dependent quantum state. In this problem, we will consider a spin 1/2 qubit in a magnetic field.

Suppose the magnetic field is representative by a vector $\vec{B}$. The Hamiltonian of the spin in the magnetic field is,

$$\hat{H} = -\hat{\vec{m}}.\vec{B} = -\gamma\hat{\vec{S}}.\vec{B} = -\gamma\left[\hat{S}_x B_x + \hat{S}_y B_y + \hat{S}_z B_z\right]$$

where the gyromagnetic ratio for the electron is, $\gamma = -e/m$.

The mean value of the spin vector is,

$$\left\langle\hat{\vec{S}}\right\rangle(t) = \langle\psi(t)|\hat{\vec{S}}|\psi(t)\rangle = \left\langle\hat{S}_x\right\rangle(t)\,\hat{x} + \left\langle\hat{S}_y\right\rangle(t)\,\hat{y} + \left\langle\hat{S}_x\right\rangle(t)\,\hat{z}$$

a) Show that,

$$\frac{d\left\langle\hat{\vec{S}}(t)\right\rangle}{dt} = \gamma\left\langle\hat{\vec{S}}\right\rangle(t)\times\vec{B}$$

Which can also be written as,

$$\frac{d\left\langle\hat{\vec{m}}(t)\right\rangle}{dt} = \gamma\left\langle\hat{\vec{m}}\right\rangle(t)\times\vec{B}$$

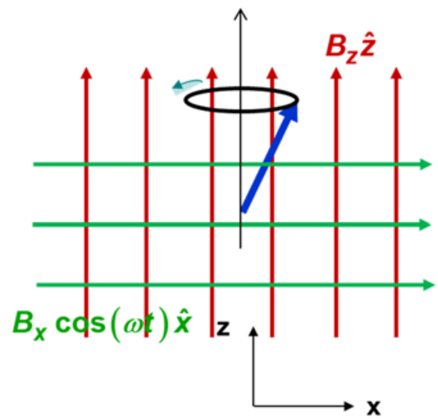
Which in turn matches the classical equation for a magnetic moment in a magnetic field.

Hint: Recall that for any observable O, irrespective of the quantum state $|\psi(t)\rangle$,

$$i\hbar\frac{d\left\langle\hat{O}\right\rangle(t)}{dt} = \left\langle\left[\hat{O},\hat{H}\right]\right\rangle(t)$$

$$\Rightarrow i\hbar\frac{d}{dt}\langle\psi(t)|\hat{O}|\psi(t)\rangle = \langle\psi(t)|\left[\hat{O},\hat{H}\right]|\psi(t)\rangle$$

We now specialize to the problem discussed in the text, where a strong DC magnetic field is applied along the z-axis and a weak AC magnetic field is applied in the x-direction, as shown below. We have already solved this problem in the lecture handouts, where we found the time-dependent quantum state. In this problem, we will obtain the time-dependent mean values of spin components without solving for the quantum state.



b) Using your results from part (a), show that,

$$\frac{d}{dt}\begin{bmatrix}\langle\hat{S}_x\rangle(t)\\ \langle\hat{S}_y\rangle(t)\\ \langle\hat{S}_z\rangle(t)\end{bmatrix}=\begin{bmatrix}0 & -\frac{\Delta}{\hbar} & 0\\ \frac{\Delta}{\hbar} & 0 & -\frac{2\kappa}{\hbar}\cos(\omega t)\\ 0 & \frac{2\kappa}{\hbar}\cos(\omega t) & 0\end{bmatrix}\begin{bmatrix}\langle\hat{S}_x\rangle(t)\\ \langle\hat{S}_y\rangle(t)\\ \langle\hat{S}_z\rangle(t)\end{bmatrix}=\bar{\bar{P}}\begin{bmatrix}\langle\hat{S}_x\rangle(t)\\ \langle\hat{S}_y\rangle(t)\\ \langle\hat{S}_z\rangle(t)\end{bmatrix}\qquad\begin{cases}\Delta=2\mu_B B_z\\ \kappa=\mu_B B_x\end{cases}$$

Now since the spin is precessing really fast in the x-y plane at an angular frequency $\Delta/\hbar$, it is useful to go into a rotating coordinate system that is rotating at an angular frequency ω with respect to the normal coordinate system and one hopes that in that rotating coordinate system one will at least not see the fast precession at the angular frequency $\Delta/\hbar$ when the detuning is zero, but one might see more clearly everything else interesting that is happening. Convince yourself that in a coordinate system that is rotating in the x-y plane (with z-axis as the rotation axis) the mean values of the spin components $\langle\hat{\bar{U}}\rangle(t)$ are related to the mean values of the spin components $\langle\hat{\bar{S}}\rangle(t)$ in the stationary coordinate system by the matrix:

$$\begin{bmatrix}\langle\hat{U}_x\rangle(t)\\ \langle\hat{U}_y\rangle(t)\\ \langle\hat{U}_z\rangle(t)\end{bmatrix}=\begin{bmatrix}\cos(\omega t) & \sin(\omega t) & 0\\ -\sin(\omega t) & \cos(\omega t) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\langle\hat{S}_x\rangle(t)\\ \langle\hat{S}_y\rangle(t)\\ \langle\hat{S}_z\rangle(t)\end{bmatrix}=\bar{\bar{M}}\begin{bmatrix}\langle\hat{S}_x\rangle(t)\\ \langle\hat{S}_y\rangle(t)\\ \langle\hat{S}_z\rangle(t)\end{bmatrix}$$

c) Using the two matrix equations in part (b), show that:

$$\frac{d}{dt}\begin{bmatrix}\langle\hat{U}_x\rangle(t)\\ \langle\hat{U}_y\rangle(t)\\ \langle\hat{U}_z\rangle(t)\end{bmatrix}=\begin{bmatrix}0 & -\frac{\Delta}{\hbar}+\omega & 0\\ \frac{\Delta}{\hbar}-\omega & 0 & -\frac{\kappa}{\hbar}\\ 0 & \frac{\kappa}{\hbar} & 0\end{bmatrix}\begin{bmatrix}\langle\hat{U}_x\rangle(t)\\ \langle\hat{U}_y\rangle(t)\\ \langle\hat{U}_z\rangle(t)\end{bmatrix}=\bar{\bar{R}}\begin{bmatrix}\langle\hat{U}_x\rangle(t)\\ \langle\hat{U}_y\rangle(t)\\ \langle\hat{U}_z\rangle(t)\end{bmatrix}\qquad\begin{cases}\Delta=2\mu_B B_z\\ \kappa=\mu_B B_x\end{cases}$$

Hint: You can start by showing that the matrix $\bar{\bar{R}}$ that appears above is really:

$$\bar{\bar{R}} = \frac{d\bar{\bar{M}}}{dt}.\bar{\bar{M}}^{-1} + \bar{\bar{M}}.\bar{\bar{P}}.\bar{\bar{M}}^{-1}$$

And where in the final expression for $\bar{\bar{R}}$ we have ignored sinusoids that carry a 2ω frequency dependence and made the following approximations:

$$2\cos(\omega t)\sin(\omega t) = \sin(2\omega t) \rightarrow 0$$

$$2\cos^2(\omega t) = 1 + \cos(2\omega t) \rightarrow 1$$

$$2\sin^2(\omega t) = 1 - \cos(2\omega t) \rightarrow 1$$

NOTE: There is a very interesting interpretation of the equation for the vector spin components $\left\langle \hat{\vec{U}} \right\rangle(t)$. In the rotating frame, the spin vector equation for $\left\langle \hat{\vec{U}} \right\rangle(t)$ can be written as,

$$\frac{d}{dt}\left\langle \hat{\vec{U}} \right\rangle(t) = \gamma \vec{B}' \times \left\langle \hat{\vec{U}} \right\rangle(t)$$

Where the fictitious magnetic field $\vec{B}'$ is entirely DC and its components are given by,

$$\vec{B}' = B_x \hat{x} + B_z \left(1 - \frac{\hbar\omega}{\Delta}\right)\hat{z}$$

d) Assuming zero detuning, i.e. $\hbar\omega = \Delta$ (the RF frequency is tuned to match the Zeeman splitting), show that the solution of the equations for $\left\langle \hat{\vec{U}} \right\rangle(t)$ obtained in part (c) is,

$$\begin{bmatrix} \left\langle \hat{U}_x \right\rangle(t) \\ \left\langle \hat{U}_y \right\rangle(t) \\ \left\langle \hat{U}_z \right\rangle(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\left(\frac{\kappa}{\hbar}t\right) & -\sin\left(\frac{\kappa}{\hbar}t\right) \\ 0 & \sin\left(\frac{\kappa}{\hbar}t\right) & \cos\left(\frac{\kappa}{\hbar}t\right) \end{bmatrix} \begin{bmatrix} \left\langle \hat{U}_x \right\rangle(t=0) \\ \left\langle \hat{U}_y \right\rangle(t=0) \\ \left\langle \hat{U}_z \right\rangle(t=0) \end{bmatrix}$$

e) Using your solutions in part (d) find the matrix below that will give you the spin components in the stationary coordinate system,

$$\begin{bmatrix} \left\langle \hat{S}_x \right\rangle(t) \\ \left\langle \hat{S}_y \right\rangle(t) \\ \left\langle \hat{S}_z \right\rangle(t) \end{bmatrix} = \begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix} \begin{bmatrix} \left\langle \hat{S}_x \right\rangle(t=0) \\ \left\langle \hat{S}_y \right\rangle(t=0) \\ \left\langle \hat{S}_z \right\rangle(t=0) \end{bmatrix}$$

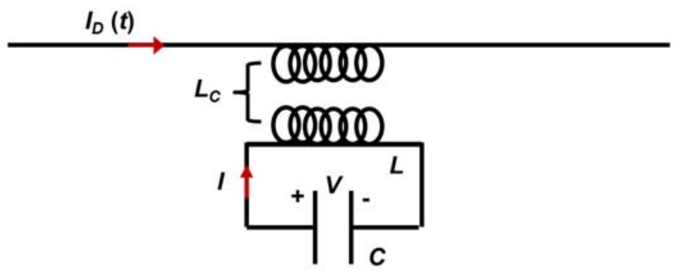
f) Suppose the initial condition is that the electron spin was pointing in the +z-direction, i.e.,

$$\begin{bmatrix} \left\langle \hat{S}_x \right\rangle(t=0) \\ \left\langle \hat{S}_y \right\rangle(t=0) \\ \left\langle \hat{S}_z \right\rangle(t=0) \end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Find the vector $\begin{bmatrix} \langle \hat{S}_x \rangle(t) \\ \langle \hat{S}_y \rangle(t) \\ \langle \hat{S}_z \rangle(t) \end{bmatrix}$ and show that your result matches the one in the lecture handouts obtained by finding the quantum state $|\psi(t)\rangle$ explicitly.

Problem 6.2: (A Driven Two Level System (TLS): A Superconducting Qubit)

Consider the following current driven superconducting qubit. The state of the qubit can be controlled by applying a short current pulse $I_D(t) = I_{DO}\cos(\omega t)$ at center frequency ω in the external wire.



The weak coupling between the wire current and the LC circuit via the transformer-styled coupling results in the following classical equations:

$$\frac{dV}{dt} = -\frac{I}{C}$$

$$L\frac{dI}{dt} = V - L_c\frac{dI_D}{dt}$$

Or in terms of the scaled variables (small letters),

$$\frac{dv}{dt} = -L\omega_o^2 i$$

$$L\frac{di}{dt} = v - L_c\frac{di_D}{dt}$$

The quantum Hamiltonian can be written as,

$$\hat{H} = \frac{\hat{v}^2}{2L} + \frac{1}{2}L\omega_o^2\hat{i}^2 - \hat{v}\frac{L_c}{L}\frac{di_D(t)}{dt} = \hat{H}_o - \hat{v}\frac{L_c}{L}\frac{di_D(t)}{dt}$$

Note that the wire current $i_D(t) = I_D(t)/\omega_o$ remains a classical time-dependent variable in the above Hamiltonian. We will assume that,

$$i_D(t) = i_{DO}\cos(\omega t)$$

$$\frac{di_D(t)}{dt} = -\omega i_{DO}\sin(\omega t)$$

Writing the above Hamiltonian in terms of creation and destructon operator gives,

$$\hat{H} = \hbar\omega_o\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + i\frac{L_c}{L}\sqrt{\frac{\hbar L\omega_o}{2}}\left(\hat{a} - \hat{a}^\dagger\right)\frac{di_D(t)}{dt}$$

$$= \hat{H}_o + i\frac{L_c}{L}\sqrt{\frac{\hbar L\omega_o}{2}}\left(\hat{a} - \hat{a}^\dagger\right)\frac{di_D(t)}{dt}$$

NOTE: The " i " appearing in the above equation is iota.

We will now try to write this Hamiltonian in matrix form. We assume that the LC circuit qubit (in the absence of the coupling to the circuit wire) has only two relevant eigenstates $|0\rangle$ and $|1\rangle$ spaced apart in energy by $\hbar\omega_o$. And that is the entire Hilbert space. Why are we ignoring the other higher energy eigenstates? In practice, the potential term in the Hamiltonian is slightly non-quadratic in the current $\hat{i}$ (which is another way of saying that the inductor L of the qubit is slightly nonlinear) such that the energy difference between all the successive eigenstates is not the same $\hbar\omega_o$ and the higher energy eigenstates are further apart in energy than the first two and so we can ignore the higher energy eigenstates in what follows.

We then make the following mapping (a change of basis, if you wish),

$$|1\rangle \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |0\rangle \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

a) Show that under this mapping the operators $\hat{a}$ and $\hat{a}^\dagger$ can be written as,

$$\hat{a} \rightarrow \frac{1}{2}\left(\hat{\sigma}_x - i\hat{\sigma}_y\right) \qquad \hat{a}^\dagger \rightarrow \frac{1}{2}\left(\hat{\sigma}_x + i\hat{\sigma}_y\right)$$

NOTE: Operators in quantum physics are defined by their actions on the states in the Hilbert space.

b) Show that under this mapping, the original Hamiltonian $\hat{H}_o$ can be written as,

$$\hat{H}_o = \hbar\omega_o\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \rightarrow \hbar\omega_o\begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \hbar\omega_o + \frac{\hbar\omega_o}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \hbar\omega_o + \frac{\hbar\omega_o}{2}\hat{\sigma}_z$$

Finally, the driving term in the Hamiltonian can be written as,

$$i\frac{L_c}{L}\sqrt{\frac{\hbar L\omega_o}{2}}\left(\hat{a} - \hat{a}^\dagger\right)\frac{di_D(t)}{dt} \rightarrow -\omega\frac{L_c}{L}\sqrt{\frac{\hbar L\omega_o}{2}}i_{DO}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\sin(\omega t) = -\omega\frac{L_c}{L}\sqrt{\frac{\hbar L\omega_o}{2}}i_{DO}\hat{\sigma}_y\sin(\omega t)$$

So the final form of the Hamiltonian becomes,

$$\hat{H} = \hbar\omega_o + \frac{\hbar\omega_o}{2}\hat{\sigma}_z + \kappa\,\hat{\sigma}_y\sin(\omega t)$$

Where,

$$\kappa = -\omega\frac{L_c}{L}\sqrt{\frac{\hbar L\omega_o}{2}}\,i_{DO}$$

Other than an energy offset of $\hbar\omega_o$, the above Hamiltonian looks like that of a spin 1/2 qubit in a strong z-directed DC magnetic field and a weak y-directed AC magnetic field.

Since an overall energy offset has no effect on dynamics and can be safely ignored, we will write the Hamiltonian of a current driven superconducting qubit as,

$$\hat{H} = \frac{\hbar\omega_o}{2}\hat{\sigma}_z + \kappa\,\hat{\sigma}_y \sin(\omega t)$$

c) Suppose at time t=0, the state of the qubit was,

$$|\psi(t=0)\rangle = |0\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Find the quantum state of the qubit $|\psi(t)\rangle$ at time t assuming that the driving frequency ω equals ω_o .

Hint: Follow the steps in the lecture handouts for the case where the time dependent magnetic field was in the x-direction. You will need to make suitable changes to take into account the fact that the time-dependent magnetic field in the present case is in the y-direction.

d) Suppose one desires to realize a quantum operation (a quantum gate) in which an initial quantum state $|0\rangle$ of the superconducting qubit is turned into an equal weight superposition of $|0\rangle$ and $|1\rangle$. One accomplishes this by turning on the drive current in the wire for a duration T and then turning it off. What ought to be the duration T of the RF drive current in the wire to realize this quantum operation?