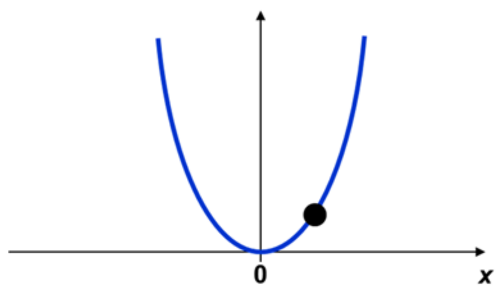
Department of Electrical and Computer Engineering, Cornell University

ECE 4060: Quantum Physics and Engineering

Fall 2020

Homework 5 **Due on Oct. 14, 2020 by 5:00 PM (via email)**

Problem 5.1: (Quantum SHO)



$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega_o^2 \hat{x}^2$$

Consider a particle in the ground state $|0\rangle$ of the quantum simple harmonic oscillator,

$$\langle x|0\rangle = \phi_0(x) = \left(\frac{m\omega_o}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega_o}{2\hbar}x^2}$$

Suppose one wants to find the average value of the position with respect to the ground state. One can certainly evaluate the following integral,

$$\langle 0|\hat{x}|0\rangle = \int_{-\infty}^{\infty} dx\, x\, |\phi_0(x)|^2 = 0$$

But the above would be a poor way to do the problem. A better way would be to realize that,

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega_o}}\left(\hat{a} + \hat{a}^\dagger\right)$$

and then evaluate,

$$\langle 0|\hat{x}|0\rangle = \sqrt{\frac{\hbar}{2m\omega_o}}\langle 0|\left(\hat{a} + \hat{a}^\dagger\right)|0\rangle = \sqrt{\frac{\hbar}{2m\omega_o}}\langle 0|\left(\hat{a}|0\rangle + \hat{a}^\dagger|0\rangle\right) = \sqrt{\frac{\hbar}{2m\omega_o}}\langle 0|\left(0 + |1\rangle\right) = \sqrt{\frac{\hbar}{2m\omega_o}}\langle 0|1\rangle = 0$$

The operator techniques are far more useful and powerful. In the parts below you are NOT supposed to evaluate integrals and are supposed to use operator techniques.

Consider a particle in the state $|n\rangle$ of the quantum simple harmonic oscillator.

a) Find the mean values of the operators $\hat{x}$ and $\hat{p}$ in the state $|n\rangle$.

b) Assuming $\Delta\hat{x} = \hat{x} - \langle\hat{x}\rangle$, find the standard deviation $\sigma_x^2 = \left\langle \Delta\hat{x}^2 \right\rangle$ in the state $|n\rangle$.

c) Assuming, $\Delta\hat{p} = \hat{p} - \langle\hat{p}\rangle$, find the standard deviation $\sigma_p^2 = \langle\Delta\hat{p}^2\rangle$ in the state $|n\rangle$.

d) Find the mean value of the kinetic energy of the particle, i.e. $\left\langle \frac{\hat{p}^2}{2m} \right\rangle$, in the state $|n\rangle$.

e) Find the mean value of the potential energy of the particle, i.e. $\left\langle \frac{1}{2} m\omega_0^2 \hat{x}^2 \right\rangle$, in the state $|n\rangle$.

Now consider the quantum state,

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}\left[|n\rangle + |n+1\rangle\right]$$

f) Find the state $|\psi(t)\rangle$ at time *t*.

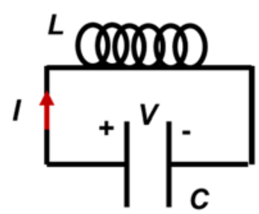
g) Find the mean values of position and momentum, i.e. $\langle\hat{x}\rangle(t)$ and $\langle\hat{p}\rangle(t)$, in the state $|\psi(t)\rangle$.

h) Do the mean values calculated in part (g) above satisfy the classical simple harmonic oscillator equations,

$$\frac{d}{dt}\langle\hat{p}\rangle(t) = -m\omega_o^2\langle\hat{x}\rangle(t)$$

$$m\frac{d}{dt}\langle\hat{x}\rangle(t) = \langle\hat{p}\rangle(t)$$

Problem 5.2: (Quantum Superconducting LC Circuit)



The Hamiltonian is,

$$\hat{H} = \frac{1}{2}C\hat{V}^2 + \frac{1}{2}L\hat{I}^2 = \frac{\hat{v}^2}{2L} + \frac{1}{2}L\omega_o^2\hat{i}^2 = \hbar\omega_o\left(\hat{n} + \frac{1}{2}\right)$$

a) Write the operators $\hat{v}$ and $\hat{i}$ in terms of the creation and destruction operators $\hat{a}$ and $\hat{a}^\dagger$.

In the following parts, use operator techniques.

Suppose the circuit is in its quantum ground state $|0\rangle$.

b) Find the mean values of the voltage and current in the ground state, i.e. find:

$$\langle\hat{V}\rangle = \omega_o\langle\hat{v}\rangle = ? \qquad \langle\hat{I}\rangle = \omega_o\langle\hat{i}\rangle = ?$$

c) Find the mean value of the energy stored in the capacitor, i.e. find

$$\left\langle \frac{1}{2} C \hat{V}^2 \right\rangle = \left\langle \frac{\hat{v}^2}{2L} \right\rangle = ?$$

in the ground state.

d) Find the mean value of the energy stored in the inductor, i.e. find

$$\left\langle \frac{1}{2} L \hat{I}^2 \right\rangle = \left\langle \frac{1}{2} L \omega_o^2 \hat{i}^2 \right\rangle = ?$$

in the ground state.

You will note in parts (c) and (d) that the mean square values of the voltage and current are not zero in the ground state. This does not mean that the voltage values are fluctuating in time (which happens when you have, for example, thermal noise in a classical circuit). What it means here is that the ground state is in a superposition of different voltage values and one doesn't know the actual voltage value unless one makes a measurement and the mean square voltage value calculated above represents the result obtained by making repeated voltage measurements on a circuit sitting in its quantum ground state.

Suppose the superconducting circuit is operated at a non-zero temperature. Forgetting all quantum mechanical effects for a second, one has to take into account the thermal noise in voltage and current at non-zero temperatures. This thermal noise results in the following mean square values for the voltage and current (now taken as classical variables),

$$\left\langle \frac{1}{2} C V^2 + \frac{1}{2} L I^2 \right\rangle = K_B T$$

$$\left\langle \frac{1}{2} C V^2 \right\rangle = \frac{1}{2} K_B T$$

$$\left\langle \frac{1}{2} L I^2 \right\rangle = \frac{1}{2} K_B T$$

Here, T is the temperature and $K_B \approx 1.38 \times 10^{-23}$ Jules/Kelvin is the Boltzmann constant. As the circuit is cooled down, thermal noise becomes smaller and smaller until the only "noise" seen upon making measurements is due to quantum effects. As stated earlier, this quantum "noise" is not noise in the usual sense of the word which implies fluctuations in time, but it represents uncertainty due to quantum superposition

e) Consider a superconducting circuit with the following parameters,

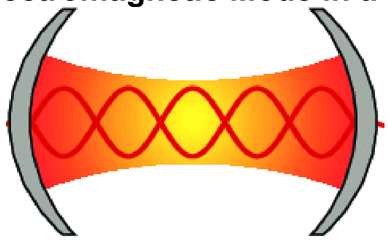
$L = 0.25$ nH

C=1 pF

At what temperature would you need to operate this circuit such that the thermal noise variances in voltage and current become less than the corresponding quantum variances in the ground state?

f) Find out (by searching on Google) the temperatures at which IBM and/or Google and/or other folks operate their superconducting qubit quantum computers and see how it compares with the number you calculated in part (e).

Problem 5.3: (A Quantized Electromagnetic Mode in a Photonic Cavity)



Consider an electromagnetic mode in an optical cavity. The Hamiltonian is,

$$\hat{H} = \frac{\hat{q}_E^2}{2\mu_o} + \frac{1}{2}\mu_o\omega_o^2\hat{q}_H^2 = \hbar\omega_o\left(\hat{n} + \frac{1}{2}\right)$$

The electric and magnetic fields, being observables, are also operators now:

$$\hat{\vec{E}}(\vec{r}) = \frac{\hat{q}_E}{\sqrt{\mu_o\varepsilon_o}}\vec{U}(\vec{r}) \qquad \hat{\vec{H}}(\vec{r}) = -\frac{\hat{q}_H}{\sqrt{\mu_o\varepsilon_o}}\nabla\times\vec{U}(\vec{r})$$

a) Write the operators $\hat{q}_E$ and $\hat{q}_H$ in terms of the creation and destruction operators $\hat{a}$ and $\hat{a}^\dagger$.

In the following parts, use operator techniques.

Consider an eigenstate of the Hamiltonian $|n\rangle$ with a very large number of photons (i.e. $n \sim 10^9$).

b) Find the average value of the electric field in the state $|n\rangle$, i.e. find,

$$\left\langle\hat{\vec{E}}(\vec{r})\right\rangle = \frac{\langle\hat{q}_E\rangle}{\sqrt{\mu_o\varepsilon_o}}\vec{U}(\vec{r})$$

c) Find the average value of the magnetic field in the state $|n\rangle$, i.e. find,

$$\left\langle\hat{\vec{H}}(\vec{r})\right\rangle = -\frac{\langle\hat{q}_H\rangle}{\sqrt{\mu_o\varepsilon_o}}\nabla\times\vec{U}(\vec{r})$$

The result in (b) and (c) might not be what you would have been expecting – that an eigenstate of the Hamiltonian with lots of photons and lots of energy, has zero mean values for the electric and magnetic fields.

Now consider the following state of the field which is a superposition of energy eigenstates,

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}\left[|n\rangle + |n+1\rangle\right]$$

d) Find the state $|\psi(t)\rangle$ at time *t*.

e) Find the average value of the electric field in the state $|\psi(t)\rangle$ at time *t*, i.e. find,

$$\left\langle \hat{\vec{E}}(\vec{r}) \right\rangle(t) = \frac{\langle \hat{q}_E \rangle(t)}{\sqrt{\mu_o \varepsilon_o}} \vec{U}(\vec{r})$$

f) Find the average value of the magnetic field in the state $|\psi(t)\rangle$ at time *t*, i.e. find,

$$\left\langle \hat{\vec{H}}(\vec{r}) \right\rangle(t) = -\frac{\langle \hat{q}_H \rangle(t)}{\sqrt{\mu_o \varepsilon_o}} \nabla \times \vec{U}(\vec{r})$$

Parts (e) and (f) tell you something about the nature of electric and magnetic fields which you are used to in classical physics; that if the field average values are not zero then the corresponding quantum state must be a superposition of energy eigenstates (i.e. a superposition of states with different photon number).

g) Do the average time-dependent values of the field operators, i.e. $\left\langle \hat{\vec{E}}(\vec{r}) \right\rangle(t)$ and $\left\langle \hat{\vec{H}}(\vec{r}) \right\rangle(t)$, that you found in parts (e) and (f), satisfy the classical Maxwell's equations (i.e Ampere's Law and Faraday's Law)? Show this explicitly.

BONUS CHALLENGE PROBLEM (worth 20 points):

The average values,

$$\left\langle \hat{\vec{E}}(\vec{r}) \right\rangle(t) = \frac{\langle \hat{q}_E \rangle(t)}{\sqrt{\mu_o \varepsilon_o}} \vec{U}(\vec{r})$$

$$\left\langle \hat{\vec{H}}(\vec{r}) \right\rangle(t) = -\frac{\langle \hat{q}_H \rangle(t)}{\sqrt{\mu_o \varepsilon_o}} \nabla \times \vec{U}(\vec{r})$$

of the field operators with respect to any arbitrary time-dependent quantum state, $|\psi(t)\rangle$, must satisfy Maxwell's equations (otherwise, humans would never have discovered Maxwell's equations in the 1800s). Prove this. Your proof will establish the classical classical-quantum correspondence for the electromagnetic fields.

Problem 5.4: (Commutation Relations and Inner Products)

There is an intimate connection between the commutation relation $[\hat{x}, \hat{p}] = i\hbar$ and the inner product, $\langle x | p \rangle = \frac{e^{i\frac{p}{\hbar}x}}{\sqrt{\hbar}}$. In this course, we had first established $\langle x | p \rangle = \frac{e^{i\frac{p}{\hbar}x}}{\sqrt{\hbar}}$ and then we had used it to establish $[\hat{x}, \hat{p}] = i\hbar$ (and it will be a good exercise for you to figure out the chain of reasoning, spread over many lectures, that established the latter from the former).

The arguments can go equally well in the reverse direction as well, and the commutator $[\hat{x}, \hat{p}] = i\hbar$ can be used to show that $\langle x | p \rangle = \frac{e^{i\frac{p}{\hbar}x}}{\sqrt{\hbar}}$, and thereby establish the Fourier transform relationship between position and momentum in quantum physics (even if one weren't aware of the De Broglie hypothesis or the Schrodinger's wave equation). This is what we are going to do in this problem and this is how Heisenberg developed his version of the quantum theory.

a) Starting from $\left[\hat{x},\hat{p}\right]=i\hbar$ and sandwiching it between the bra $\langle x'|$ and the ket $|x\rangle$ show that,

$$\int_{-\infty}^{\infty}\frac{dp}{2\pi}\ (x'-x)\,p\langle x'|p\rangle\langle p|x\rangle=i\hbar\delta(x'-x)$$

b) The integral equation obtained in part (a) can be solved to find out the value of the inner product $\langle x|p\rangle$. One easy way to solve integral equations (just like in the case of differential equations) is to guess a solution. Show that the solution $\langle x|p\rangle=\frac{e^{i\frac{p}{\hbar}x}}{\sqrt{\hbar}}$ satisfies the integral equation found in part (a).