ECE 4060: Quantum Physics and Engineering

Fall 2020

Homework 4 **Due on Oct. 07, 2020 by 5:00 PM (via email)**

Problem 4.1: (Commutators)

We know that:

$$\left[\hat{x},\hat{p}\right]=i\hbar$$

Find:

a) $\left[\hat{x},\hat{p}^2\right]=?$

b) $\left[\hat{x}^2,\hat{p}\right]=?$

c) $\left[\hat{x},\hat{p}^n\right]=?$ (n = 1,2,3,…….)

d) $\left[\hat{x}^n,\hat{p}\right]=?$ (n = 1,2,3,…….)

e) Suppose $F(\hat{x})$ is a polynomial function of the operator $\hat{x}$. Find $\left[F(\hat{x}),\hat{p}\right]=?$

f) Suppose the Hamiltonian is:

$$\hat{H}=\frac{\hat{p}^2}{2m}+V(\hat{x})$$

Find $\left[\hat{H},\hat{p}\right]$ and then figure out whether momentum and Hamiltonian operators can have a common set of eigenvectors or not.

Problem 4.2: (Eigenfunction Properties)

a) Consider the time-independent Schrodinger equation:

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x)\right]\phi(x)=E\phi(x)$$

Argue that if $\phi(x)$ is an eigenfunction of $\hat{H}$ with energy E, then $\phi^*(x)$ (the complex conjugate function) is also an eigenfunction of $\hat{H}$ with the same energy E.

Now, it is not necessary that $\phi^*(x)$ will always be a different eigenfunction than $\phi(x)$. In situations where it does turn out to be a different eigenfunction (for example, $e^{i\frac{p}{\hbar}x}$ and $e^{-i\frac{p}{\hbar}x}$ when $V(x)=0$), one can make linear superpositions and get two new (unnormalized) eigenfunctions, $\left[\phi(x)+\phi^*(x)\right]/2$

and $\left[\phi(x)-\phi^{*}(x)\right]/2i$, that are both purely real. The conclusion here is that energy eigenfunctions of the time-independent Schrodinger equation can always be chosen to be purely real.

b) Suppose,

$$\psi(x,t)=\phi(x)e^{-i\frac{E}{\hbar}t}$$

and suppose $\phi(x)$ is real everywhere. If $\phi(x)$ is real everywhere, then show that the mean value of the momentum $\langle\hat{p}\rangle$ is exactly zero, and conclude that any eigenfunction of the momentum operator cannot be purely real.

Problem 4.3: (Parity and Inversion Symmetry)

In quantum physics one often encounters potentials that are even functions in space, i.e.

$$V(-x)=V(x)$$

In such cases, we say that the potential has even parity. We also say that the potential has "inversion symmetry". The infinite potential well is such an example (if the x=0 point is chosen to be the center of the well).

Consider the time-independent Schrodinger equation with an **even** potential:

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x)\right]\phi(x)=E\phi(x)$$

We will assume that the energy eigenfunctions $\phi_n(x)$ are all real (as discussed in problem 4.2).

a) Show that if $\phi_n(x)$ is a eigenfunction of the above equation with eigenvalue E_n, then it must be that the function $\phi_n(-x)$ is also an eigenfunction of the above equation with the same eigenvalue E_n.

Parity Operator:
Symmetries in quantum physics are generally expressed in terms of symmetry operators. Here we will explore the consequences of the symmetry of the Hamiltonian with respect to spatial inversion which occurs when the potential is even, i.e. $V(-x)=V(x)$.

Define a parity operator $\hat{\Pi}$ with the following property,

$$\hat{\Pi}|x\rangle=|-x\rangle$$

This implies,

$$\hat{\Pi}^2|x\rangle=\hat{\Pi}\hat{\Pi}|x\rangle=\hat{\Pi}|-x\rangle=|x\rangle$$

$$\Rightarrow\hat{\Pi}^2=\hat{1}$$

$$\Rightarrow\hat{\Pi}=\hat{\Pi}^{-1}$$

Now,

$$\langle x|\hat{\Pi}^{\dagger} = \langle -x|$$

$$\Rightarrow \langle x|\hat{\Pi}^{\dagger}\hat{\Pi}|x'\rangle = \langle -x|-x'\rangle = \delta(x-x')$$

$$\Rightarrow \hat{\Pi}^{\dagger}\hat{\Pi} = \hat{1}$$

$$\Rightarrow \hat{\Pi}^{-1} = \hat{\Pi}^{\dagger}$$

Thus,

$$\hat{\Pi} = \hat{\Pi}^{-1} = \hat{\Pi}^{\dagger}$$

The above relations imply,

$$\langle x|\hat{\Pi}|\phi\rangle = \langle -x|\phi\rangle = \phi(-x)$$

Also,

$$\langle x|\hat{\Pi}|p\rangle = \langle -x|p\rangle = \langle x|-p\rangle$$

$$\Rightarrow \hat{\Pi}|p\rangle = |-p\rangle$$

Finally, start from:

$$\hat{x}|x\rangle = x|x\rangle$$

$$\Rightarrow \hat{x}\hat{\Pi}^{\dagger}\hat{\Pi}|x\rangle = x|x\rangle$$

$$\Rightarrow \hat{\Pi}\hat{x}\hat{\Pi}^{\dagger}\hat{\Pi}|x\rangle = x\hat{\Pi}|x\rangle$$

$$\Rightarrow \hat{\Pi}\hat{x}\hat{\Pi}^{\dagger}|-x\rangle = x|-x\rangle$$

$$\Rightarrow \hat{\Pi}\hat{x}\hat{\Pi}^{\dagger} = -\hat{x}$$

b) Show that,

$$\hat{\Pi}\hat{p}\hat{\Pi}^{\dagger} = -\hat{p}$$

c) Show that,

$$\hat{\Pi}\hat{p}^2\hat{\Pi}^{\dagger} = \hat{p}^2$$

d) Show that when the potential is even,

$$\hat{\Pi}\hat{H}\hat{\Pi}^{\dagger} = \hat{H}$$

The above statement is the mathematical expression of inversion symmetry of the Hamiltonian.

e) Show that when the potential is even,

$$\left[\hat{\Pi},\hat{H}\right] = 0$$

Part (e) result shows that the parity operator commutes with the Hamiltonian when the potential is even. If the parity operator commutes with the Hamiltonian then both these operators can have the same set of eigenvectors. So we look for the eigenvectors of the parity operator that satisfy,

$$\hat{\Pi}|\phi\rangle = \lambda|\phi\rangle$$

The above implies,

$$\hat{\Pi}^2|\phi\rangle = \lambda\hat{\Pi}|\phi\rangle = \lambda^2|\phi\rangle$$

But since $\hat{\Pi}^2 = \hat{1}$, the eigenvalue λ of the parity operator must satisfy, $\lambda^2 = 1$, which implies that all the eigenvectors of the parity operator (and there can be infinite number of them) can be divided into two sets. One set has the eigenvalue $\lambda = +1$ and the other set has the eigenvalue $\lambda = -1$.

f) Since all the eigenvectors $|\phi_n\rangle$ of the Hamiltonian (with an even potential) that satisfy,

$$\hat{H}|\phi_n\rangle = E_n|\phi_n\rangle$$

can also be chosen to be the eigenvectors of the parity operator and therefore must satisfy,

$$\hat{\Pi}|\phi_n\rangle = \lambda|\phi_n\rangle$$

where λ is either +1 and -1, argue that all the energy eigenfunctions must either satisfy ,

$$\phi_n(-x) = \phi_n(x)$$

or,

$$\phi_n(-x) = -\phi_n(x)$$

In other words, when the Hamiltonian as inversion symmetry, the energy eigenfunctions can be chosen such that they have a definite even or odd parity.

g) Consider the infinite potential well problem. Suppose the x=0 point is chosen to be in the center of the well such that the potential then has inversion symmetry and $\left[\hat{\Pi}, \hat{H}\right] = 0$. Do the eigenfunctions have a definite even or odd parity as the result of part (f) suggests?

Problem 4.4: (Angular Momentum Operator)

In classical physics, the angular momentum $\vec{L}$ of a particle equals,

$$\vec{L} = \vec{r} \times \vec{p}$$

In quantum physics, the angular momentum operator becomes,

$$\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}}$$

In component notation,

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y$$

$$\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z$$

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$$

a) Show that:

$$\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}} = -\hat{\vec{p}} \times \hat{\vec{r}}$$

b) Show that,

$$\left[\hat{L}_x, \hat{L}_y\right] = i\hbar\hat{L}_z$$

$$\left[\hat{L}_y, \hat{L}_z\right] = i\hbar\hat{L}_x$$

$$\left[\hat{L}_z, \hat{L}_x\right] = i\hbar\hat{L}_y$$

Angular momentum is a vector operator but its components don't commute!!

c) Consider a quantum state which is an eigenstate of the z-component of the angular momentum operator, $\hat{L}_z|\phi\rangle = \hbar|\phi\rangle$. This means in this quantum state, the z-component of the particle's angular momentum has a definite value. With respect to this state, what can you say about the uncertainty product,

$$\left\langle \Delta\tilde{L}_x^2 \right\rangle \left\langle \Delta\tilde{L}_y^2 \right\rangle$$