ECE 4060: Quantum Physics and Engineering

Fall 2020

Homework 3 **Due on Sep. 30, 2020 by 5:00 PM (via email)**

Problem 3.1: (Quantum Operators and Basis)

a) In lectures it was shown that: $\langle x|\hat{p}|\psi(t)\rangle = \frac{\hbar}{i}\frac{\partial \psi(x,t)}{\partial x}$

Show that: $\langle p|\hat{x}|\psi(t)\rangle = i\hbar\frac{\partial \psi(p,t)}{\partial p}$

b) The basis-independent form of the Schrodinger equation is:

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

In the position basis this equation becomes:

$$i\hbar\frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x)\psi(x,t)$$

Find the Schrodinger equation in the momentum basis. You will need to multiple from the left with the bra $\langle p|$ on both sides of the basis-independent Schrodinger equation to get:

$$i\hbar\frac{\partial}{\partial t}\langle p|\psi(p,t)\rangle = \langle p|\hat{H}|\psi(t)\rangle$$

Write your final answer in terms of the Fourier transform of the potential:

$$V(k) = \int dx\, V(x)e^{-ikx}$$

You will learn from this why we always like to solve the Schrodinger equation in the position basis and not in the momentum basis.

c) Show that if the Hamiltonian involves a zero potential everywhere (free-particle Hamiltonian):

$$\hat{H} = \frac{\hat{p}^2}{2m}$$

then momentum operator eigenstates are also energy operator eigenstates and find the corresponding eigenvalues.

Consider the position and momentum operator eigenstates defined by:

$$\hat{p}|p\rangle = p|p\rangle \qquad \hat{x}|x\rangle = x|x\rangle \qquad \langle x|p\rangle = \frac{e^{i\frac{p}{\hbar}x}}{\sqrt{\hbar}}$$

d) Find:

$\langle x|\hat{p}|p\rangle = ?$

$\langle p|\hat{x}|x\rangle = ?$

e) Find:

$\langle p|\hat{p}|x\rangle = ?$

$\langle x|\hat{x}|p\rangle = ?$

Problem 3.2: (Time Development and Measurement in Quantum Physics)

Consider a quantum particle described by the Hamiltonian and its eigenstates (e.g. a particle in an infinite well):

$$\hat{H}|\phi_n\rangle = E_n|\phi_n\rangle$$

Assume that all the eigenvalues of the Hamiltonian are different.

Suppose that at time t=0 the quantum state is:

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{1}{2}|\phi_2\rangle + \frac{i}{2}|\phi_3\rangle$$

a) Show that the state at time t=0, as written above, has norm equal to unity (i.e. it is properly normalized).

b) Suppose at time t=0, a measurement is made to determine the energy of the particle. What are the possible values of the result and what is the a-priori probability of getting each result?

c) Suppose at time t=0, a measurement is made to determine the position of the particle. What are the possible values of the result and what is the a-priori probability of getting each result?

d) Find the state for time t > 0: $|\psi(t)\rangle = ?$

e) Suppose at time t, a measurement is made to determine the energy of the particle. What are the possible values of the result and what is the a-priori probability of getting each result?

f) Are the a-priori probabilities you calculated in part (e) changing with time?

g) Suppose at time t, a measurement is made to determine the position of the particle. What are the possible values of the result and what is the a-priori probability of getting each result?

h) Are the a-priori probabilities you calculated in part (g) changing with time?

i) You will notice a big difference in the results obtained in parts (f) and (h). Can you explain why the difference is present?

j) Find the expectation value or mean value of the energy at time t. Is it time-dependent?

k) Find the expectation value or mean value of the position at time t. Is it time-dependent?

l) Suppose someone is trying to determine if the particle is sitting stationary or oscillating. If he plots the mean position of the particle as a function of time what different frequencies will be observe in his plot?

Suppose now that all eigenvalues of the Hamiltonian are not different. More specifically, $E_2 = E_3$.

m) A measurement is made to determine the energy of the particle at time t. The result E_1 is obtained. What is the quantum state right after the measurement?

n) A measurement is made to determine the energy of the particle at time t. The result E_2 is obtained. What is the quantum state right after the measurement?

Problem 3.3: (Commutation Relations)

Quantum operators should never be confused with ordinary numbers or variables. An important property of operators is that they do not commute:

$$\hat{A}\hat{B} \neq \hat{B}\hat{A}$$

And this property is usually expressed in terms of the operator commutation relations:

$$\left[\hat{A},\hat{B}\right] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

Suppose someone tells you that the position and momentum operators do not commute and that:

$$\left[\hat{x},\hat{p}\right] = \alpha\,\hat{1}$$

Where α is some complex number. Your task is to find this complex number.

You can start by sandwiching the equation, $\left[\hat{x},\hat{p}\right] = \alpha\,\hat{1}$, on both sides by some arbitrary state $|\psi\rangle$ from the right and the bra $\langle x|$ from the left. So on the right hand side you get:

$$\langle x|\alpha\,\hat{1}|\psi\rangle = \alpha\psi(x)$$

Find out what happens on the left hand side and thereby find the value of the complex number α.