ECE 4060: Quantum Physics and Engineering

Fall 2020

Homework 10 **Due on Dec. 09, 2020 by 5:00 PM (via email)**

Problem 10.1: (Orbital Angular Momentum via Matrices)

In this course we considered particles of internal angular momentum (or spin) s equal to $\frac{1}{2}$ and their states were given by:

$$\hat{S}_z \left| s, m = \pm\frac{1}{2} \right\rangle = \pm\frac{\hbar}{2} \left| s, m = \pm\frac{1}{2} \right\rangle$$

$$\hat{S}^2 \left| s, m \right\rangle = s(s+1)\hbar^2 \left| s, m \right\rangle = \frac{3}{4}\hbar^2 \left| s, m \right\rangle$$

We mapped the Hilbert space of the two spin states to the Hilbert space of 2-component column vectors such that under this mapping:

$$\left| s, m = -\frac{1}{2} \right\rangle \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \left| s, m = +\frac{1}{2} \right\rangle \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

And then the spin operators, under this mapping, became 2x2 Hermitian matrices,

$$\hat{S}_x = \frac{\hbar}{2}\hat{\sigma}_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \hat{S}_y = \frac{\hbar}{2}\hat{\sigma}_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \hat{S}_z = \frac{\hbar}{2}\hat{\sigma}_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Here we will do the same for the orbital angular momentum.

Consider a particle with a fixed **orbital angular momentum** ℓ equal to 1. This particle, for example, could be a particle in one of the three p-orbitals of a Hydrogen atom. Its Hilbert space is then effectively limited to three states:

$$\left| \ell = 1, m = -1 \right\rangle, \left| \ell = 1, m = 0 \right\rangle, \left| \ell = 1, m = +1 \right\rangle$$

And,

$$\hat{L}_z \left| 1, m \right\rangle = m\hbar \left| 1, m \right\rangle$$

$$\hat{L}^2 \left| 1, m \right\rangle = \ell(\ell+1)\hbar^2 \left| 1, m \right\rangle = 2\hbar^2 \left| 1, m \right\rangle$$

If one maps these three states to the 3-component column vectors as follows:

$$\left| \ell = 1, m = -1 \right\rangle \rightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \left| \ell = 1, m = 0 \right\rangle \rightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \left| \ell = 1, m = +1 \right\rangle \rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

then we need to find the 3x3 matrices that represent the operators $\hat{L}_x, \hat{L}_y, \hat{L}_z,$ and $\hat{L}^2$ under this mapping. The parts below will help you get there. I will do the simplest one for you; I will find the 3x3 matrix that represents $\hat{L}_z$. First note that the matrix elements of $\hat{L}_z$ are,

$$\langle \ell, m' | \hat{L}_z | \ell, m \rangle = \delta_{m,m'} m\hbar$$

This means that $\hat{L}_z$ must be a diagonal matrix, and the diagonal elements must be the eigenvalues of $\hat{L}_z$. This gives us the matrix,

$$\hat{L}_z \to \hbar \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

a) Find the matrix element:

$$\langle \ell, m' | \hat{L}_x | \ell, m \rangle$$

b) Find the 3x3 matrix that represents the operator $\hat{L}_x$

c) Find the matrix element:

$$\langle \ell, m' | \hat{L}_y | \ell, m \rangle$$

d) Find the 3x3 matrix that represents the operator $\hat{L}_y$

e) Find the matrix element:

$$\langle \ell, m' | \hat{L^2} | \ell, m \rangle$$

f) Find the 3x3 matrix that represents the operator $\hat{L^2}$

g) What are the eigenvalues of the matrices that represent $\hat{L}_x$ and $\hat{L}_y$?

g) Verify that your matrices satisfy the commutation relation,

$$\left[\hat{L}_x, \hat{L}_y\right] = i\hbar \hat{L}_z$$

h) Suppose the quantum state of the particle is $|\psi\rangle = |1, m = 0\rangle$. If a measurement of the angular momentum along x-axis is made, i.e. $\hat{L}_x$ is measured, what are the possible values and what are the a-priori probabilities for measuring each of these values?

Problem 10.2: (Spinors: test your concepts)

Consider the quantum state of an electron described by the two-component spinor (in the eigenbasis of $\hat{S}_z$) as:

$$\langle \vec{r} | \psi \rangle = \psi(\vec{r}) = \begin{bmatrix} \chi(\vec{r}) \\ \phi(\vec{r}) \end{bmatrix}$$

a) What is the probability density of finding the electron at the location $\vec{r}$ with spin up ?

b) What is the probability density of finding the electron at the location $\vec{r}$ with spin down ?

c) What is the total probability density of finding the electron at the location $\vec{r}$?

d) What condition should the functions $\chi(\vec{r})$ and $\phi(\vec{r})$ satisfy such that the wavefunction is properly normalized?

e) If the spin angular momentum along the x-axis is measured, what is the probability density of finding the electron with spin pointing along the +ve x-axis at location $\vec{r}$?

f) If the spin angular momentum along the x-axis is measured, what is the probability of finding the electron with spin pointing along the +ve x-axis anywhere in space?