ECE 4060: Quantum Physics and Engineering

Fall 2020

Homework 1 **Due on Sep. 16, 2020 by 5:00 PM (via email)**

Problem 1.1: (Fourier Transforms and Quantum Wavefunctions)

The Fourier transforms in 1D are defined as:

$$\psi(k) = \int_{-\infty}^{\infty} dx\, \psi(x)\, e^{-ikx}$$

$$\psi(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \psi(k)\, e^{ikx}$$

We call $\psi(k)$ the "wavefunction" in Fourier space. $\psi(k)$ and $\psi(x)$ are Fourier transform pairs.

a) Prove the Parseval's identity for Fourier Transform pairs:

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} |\psi(k)|^2 = \int_{-\infty}^{\infty} dx\, |\psi(x)|^2$$

Now for the remaining parts, suppose:

$$\psi(x) = \frac{1}{\left(2\pi\sigma_x^2\right)^{1/4}} e^{-\frac{(x-x_o)^2}{4\sigma_x^2}} e^{ik_o x}$$

b) Find the average value of the particle position $\langle x \rangle$ with respect to the probability distribution $|\psi(x)|^2$.

c) Find the standard deviation $\langle \Delta x^2 \rangle = \left\langle \left(x - \langle x \rangle\right)^2 \right\rangle$ with respect to $|\psi(x)|^2$. Here, $\Delta x = x - \langle x \rangle$.

d) Find the Fourier Transform $\psi(k)$ of $\psi(x)$.

e) Find the average value of momentum $\langle \hbar k \rangle$ with respect to the probability distribution $|\psi(k)|^2$.

f) Find the standard deviation $\langle \hbar^2 \Delta k^2 \rangle = \left\langle \left(\hbar k - \langle \hbar k \rangle\right)^2 \right\rangle$. Here, $\Delta k = k - \langle k \rangle$.

g) If you make the probability distribution $\left|\psi(x)\right|^2$ really narrow in space (i.e. make the position of the particle more certain) by decreasing σ_x, you will reduce $\left\langle \Delta x^2 \right\rangle$. But what happens to the distribution $\left|\psi(k)\right|^2$ in Fourier space? Does it become narrow too or does it spread out? Does $\left\langle \hbar^2 \Delta k^2 \right\rangle$ increase or decrease?

h) Find the value of the product:

$$\sqrt{\left\langle \Delta x^2 \right\rangle}\sqrt{\left\langle \hbar^2 \Delta k^2 \right\rangle} = ?$$

Problem 1.2: (Newton's Second Law in Quantum Physics)

Newton's second law forms the basis of classical physics:

$$ma(t) = \frac{dp(t)}{dt} = F$$

If the particle potential energy is $V(x)$ then the force acting on the particle is:

$$F = -\frac{\partial V(x)}{\partial x}$$

And,

$$\frac{dp(t)}{dt} = -\frac{\partial V(x)}{\partial x}$$

In this problem, we wish to see how what happens to Newton's second law in quantum physics. You will work with the 2D Schrodinger equation:

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi(\vec{r},t)}{\partial x^2} + V(x)\psi(x,t)$$

And its complex conjugate:

$$-i\hbar \frac{\partial \psi^*(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi^*(x,t)}{\partial x^2} + V(x)\psi^*(x,t)$$

In lecture 5 handout, we showed that the average momentum of a particle is:

$$\langle p \rangle(t) = \int_{-\infty}^{\infty} dx \;\; \psi^*(x,t)\left[\frac{\hbar}{i}\frac{\partial}{\partial x}\right]\psi(x,t) \qquad \text{........ (1)}$$

In this problem you need to find:

$$\frac{d\langle p \rangle(t)}{dt} = ??$$

The best way to start is to differentiate both sides of equation (1) with respect to time and see where it gets you.

Hint: the end result should look *somewhat* like Newton's second law.

Problem 1.3: (Linear Algebra Review)

Consider the matrix:

$$\hat{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

a) Find all the eigenvalues of the above matrix.

b) Find the eigenvectors corresponding to the eigenvalues found in part (a), Make sure your eigenvectors are orthonormal.

Consider another matrix:

$$\hat{B} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

c) Is the above matrix self-adjoint?

d) Find the matrix commutator $\left[\hat{A}, \hat{B}\right] = ?$

e) Consider an arbitrary 2x2 Hermitian matrix $\hat{C}$ with eigenvalues λ_1 and λ_2. The corresponding orthonormal eigenvectors are $|e_1\rangle$ and $|e_2\rangle$. Show that the matrix can be written as the following sum of exterior products:

$$\hat{C} = \lambda_1 |e_1\rangle\langle e_1| + \lambda_2 |e_2\rangle\langle e_2|$$