Lecture 9

Magnetoquasistatics

In this lecture you will learn:

• Basic Equations of Magnetoquasistatics
• The Vector Potential
• The Vector Poisson’s Equation
• The Biot-Savart Law
• Magnetic Field of Some Simple Current Carrying Elements
• The Magnetic Current Dipole

Equations of Magnetoquasistatics

Equations of Electroquasistatics

\[ \nabla \cdot \varepsilon_0 \vec{E} = \rho(\vec{r}, t) \]
\[ \nabla \times \vec{E} = 0 \]
\[ \nabla \times \vec{H} = \vec{J}(\vec{r}, t) + \frac{\partial \varepsilon_0 \vec{E}}{\partial t} \]

• Electric fields are produced by only electric charges
• Once the electric field is determined, the magnetic field can be found by the last equation above

In magnetoquasistatics the source of the magnetic field is electrical current

Equations of Magnetoquasistatics

\[ \nabla \cdot \mu_0 \vec{H} = 0 \]
\[ \nabla \times \vec{H} = \vec{J}(\vec{r}, t) \]
\[ \nabla \times \vec{E} = -\frac{\partial \mu_0 \vec{H}}{\partial t} \]

• Magnetic fields are produced by only electric currents
• Once the magnetic field is determined, the electric field can be found by the last equation above
• Currents in magnetoquasistatics are solenoidal (i.e. with zero divergence)

\[ \nabla \cdot \vec{J}(\vec{r}, t) = \nabla \cdot \left( \nabla \times \vec{H} \right) = 0 \]
Ampere’s Law for Magnetoquasistatics

\[ \oint \vec{H} \cdot d\vec{s} = \int \vec{J} \cdot d\vec{a} \]

\[ \nabla \times \vec{H} = \vec{J} \]

**Ampere’s Law:** The line integral of magnetic field over a closed contour is equal to the total current flowing through that contour.

**Right Hand Rule:** The positive directions for the surface normal vector and of the contour are related by the right hand rule.

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**Magnetic Field of an Infinite Line-Current**

Consider an infinitely long line-current carrying a total current \( I \) in the +z-direction, as shown below.

Use Ampere’s law on the closed contour shown by the dashed line:

\[ \oint \vec{H} \cdot d\vec{s} = \int \vec{J} \cdot d\vec{a} \]

\[ (2\pi r)H_\phi(r) = I \]

\[ H_\phi(r) = \frac{I}{2\pi r} \]

Magnetic field is entirely in the \( \hat{\phi} \) direction and falls off as \( \sim 1/r \) from the line-current.
Magnetic Field of a Solenoid

Consider a solenoid with \( N \) turns per unit length and carrying a current \( I \)

Assumptions:

• The magnetic field inside the solenoid is uniform and strong

• There is a fringing field outside the solenoid which is very weak and may be neglected

\[
\oint \mathbf{H} \cdot d\mathbf{s} = \int \mathbf{J} \cdot d\mathbf{a}
\]

\[ L \mathbf{H}_y = (LN) I \]

\[ \mathbf{H}_y = NI \]

The Vector Potential - I

A Vector Identity:

For any vector \( \mathbf{F} \) the divergence of the curl is always zero:

\[ \nabla \cdot (\nabla \times \mathbf{F}) = 0 \]

The Vector Potential:

In magnetoquasistatics the divergence of the B-field is always zero:

\[ \nabla \cdot (\mathbf{B}) = \nabla \cdot (\mu_0 \mathbf{H}) = 0 \]

So one may represent the B-field as the curl of another vector:

\[ \mathbf{B} = \mu_0 \mathbf{H} = \nabla \times \mathbf{A} \]

\( \mathbf{A} \) is called the vector potential

\[ \nabla \cdot \mathbf{B} = \nabla \cdot \mu_0 \mathbf{H} = \nabla \cdot (\nabla \times \mathbf{A}) = 0 \]
The Vector Potential - II

In **electroquasistatics** we had: \( \nabla \times \mathbf{E} = 0 \)

Therefore we could represent the E-field by the scalar potential: \( \mathbf{E} = -\nabla \phi \)

In **magnetoequasistatics** we have: \( \nabla \cdot (\mathbf{B}) = \nabla \cdot (\mu_0 \mathbf{H}) = 0 \)

Therefore we can represent the B-field by the vector potential:
\[
\mathbf{B} = \mu_0 \mathbf{H} = \nabla \times \mathbf{A}
\]

A vector field can be specified (up to a constant) by specifying its curl and its divergence.

Our definition of the vector potential \( \mathbf{A} \) is not yet unique – we have only specified its curl.

For simplicity we fix the divergence of the vector potential \( \mathbf{A} \) to be zero:
\[
\nabla \cdot \mathbf{A} = 0
\]

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Differential Equation for The Vector Potential

Start from Ampere’s law in differential form: \( \nabla \times \mathbf{H} = \mathbf{J} \)

Use: \( \mu_0 \mathbf{H} = \nabla \times \mathbf{A} \)

To get: \( \nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J} \)

**A Vector Identity:** \( \nabla \times (\nabla \times \mathbf{F}) = \nabla (\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F} \)

\[
\nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}
\]

\[\Rightarrow \nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \quad \text{Use: } \nabla \cdot \mathbf{A} = 0 \quad \text{The differential equation for the vector potential (also called the Vector Poisson’s Equation)}\]

This is in fact 3 different equations (one for each component of \( \mathbf{A} \))

\[
\begin{align*}
\nabla^2 A_x &= -\mu_0 J_x \\
\nabla^2 A_y &= -\mu_0 J_y \\
\nabla^2 A_z &= -\mu_0 J_z
\end{align*}
\]
The Superposition Principle for The Vector Potential

- Suppose for the current distribution $\mathbf{J}_1(\mathbf{r})$ we have found the vector potential $\mathbf{A}_1(\mathbf{r})$.
- Suppose for some other current distribution $\mathbf{J}_2(\mathbf{r})$ we have also found the vector potential $\mathbf{A}_2(\mathbf{r})$.
- Then the vector potential $(\mathbf{A}_1(\mathbf{r}) + \mathbf{A}_2(\mathbf{r}))$ is the solution for the current distribution $(\mathbf{J}_1(\mathbf{r}) + \mathbf{J}_2(\mathbf{r}))$.

Proof:

\[
\nabla^2 \mathbf{A}_1(\mathbf{r}) = -\mu_0 \mathbf{J}_1(\mathbf{r}) + \nabla^2 \mathbf{A}_2(\mathbf{r}) = -\mu_0 \mathbf{J}_2(\mathbf{r}) = \nabla^2 (\mathbf{A}_1(\mathbf{r}) + \mathbf{A}_2(\mathbf{r})) = -\mu_0 (\mathbf{J}_1(\mathbf{r}) + \mathbf{J}_2(\mathbf{r}))
\]

Recall the Superposition Integral for the Potential

In the most general scenario, one has to solve the Poisson equation:

\[
\nabla^2 \phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0}
\]

We know that the solution for a point charge sitting at the origin:

\[
\phi(\mathbf{r}) = \frac{q}{4\pi \varepsilon_0 r}
\]

To find the potential at any point one can sum up the contributions from different portions of a charge distribution treating each as a point charge:

\[
\phi(\mathbf{r}) = \int \int \int -\frac{\rho(\mathbf{r}')}{4\pi \varepsilon_0 |\mathbf{r} - \mathbf{r}'|} dV'
\]

A formal solution of the vector differential equation is the vector superposition integral:

\[
\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}
\]

\[
\mathbf{A}(\mathbf{r}) = \int \int \int \frac{\mu_0 \mathbf{J}(\mathbf{r}')}{4\pi |\mathbf{r} - \mathbf{r}'|} dV'
\]
The Biot-Savart Law - I

Start from the superposition integral for the vector potential:

\[ \mathbf{A}(\mathbf{r}) = \frac{1}{4\pi} \oint \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, dV' \]

Now find the magnetic field:

\[ \mathbf{H} = \nabla \times \mathbf{A} \]

\[ \mathbf{H}(\mathbf{r}) = \nabla \times \left[ \oint \frac{\mathbf{j}(\mathbf{r}')}{{4\pi |\mathbf{r} - \mathbf{r}'|}} \, dV' \right] \]

Biot-Savart Law:

\[ \mathbf{H}(\mathbf{r}) = \oint \frac{1}{4\pi |\mathbf{r} - \mathbf{r}'|} \nabla \times \mathbf{J}(\mathbf{r}') \, dV' \]

The unit vector \( \hat{n}_{r' \to r} \) is directed from the point \( \mathbf{r}' \) on the current source to the observation point \( \mathbf{r} \).

A vector identity:

\[ \nabla \times (\mathbf{\varphi} \mathbf{F}) = \mathbf{\varphi} \nabla \times \mathbf{F} + (\nabla \mathbf{\varphi}) \times \mathbf{F} \]

Recall that:

\[ \nabla \times \left( \frac{1}{r} \right) = \frac{\hat{r}}{r^2} \]

\[ \nabla \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} = \frac{n_{r' \to r}}{|\mathbf{r} - \mathbf{r}'|^2} \]

The Biot-Savart Law - II

Biot-Savart Law:

\[ \mathbf{H}(\mathbf{r}) = \oint \frac{\mathbf{j}(\mathbf{r}')}{{4\pi |\mathbf{r} - \mathbf{r}'|}} \nabla \times \hat{n}_{r' \to r} \, dV' \]

The unit vector \( \hat{n}_{r' \to r} \) is directed from the point \( \mathbf{r}' \) on the current source to the observation point \( \mathbf{r} \).
Need to find a formula that gives the total magnetic field at a point due to a current carrying wire as a superposition of magnetic field contributions from all the small pieces of the wire.

Start from the Biot-Savart law:

\[
H(\vec{r}) = \frac{\int (\vec{r}') \times \hat{n}_{\vec{r}' \rightarrow \vec{r}} \, dV'}{4\pi |\vec{r}' - \vec{r}|} = \int \frac{I(\vec{r}') \times \hat{n}_{\vec{r}' \rightarrow \vec{r}} \, ds'}{4\pi |\vec{r}' - \vec{r}|} 
\]

Integrate over the cross-section area of the wire to get the total current carried by the wire.

Consider a line-current of length \( L \) with current \( I \) in the +x-direction. Find the magnetic field at the point \( P \) on the y-axis (as shown).

Use the Biot-Savart law:

\[
H(0, y, 0) = \frac{I}{4\pi} \left\{ \frac{ds'}{|\vec{r} - \vec{s}'|^2} \times \hat{n}_{\vec{s}' \rightarrow \vec{r}} \cdot \hat{\vec{y}} \, dx' \right\} = \frac{I}{4\pi} \left\{ \frac{y \, dx'}{x'^2 + y^2} \right\} 
\]

As \( L \rightarrow \infty \), \( H(0, y, 0) \rightarrow \frac{I}{2\pi y} \) ...recover the previous result.
### Magnetic Field of a Current Loop – Near Field

Consider a line-current in the form of a circular loop of radius $a$ and carrying a current $I$, as shown in the figure.

Find the magnetic field at the point $P$ in the center of the loop.

Use the Biot-Savart law:

$$ H(0,0,0) = \frac{I}{4\pi} \int \frac{\hat{r}' \times \hat{n}_{s',r'}}{|\vec{r} - \vec{s}'|^2} \, ds' $$

$$ ds' = \hat{r}' \, d\phi' $$

$$ |\vec{r} - \vec{s}'|^2 = a^2 $$

$$ = \hat{z} \frac{I}{4\pi} \int_0^{2\pi} \frac{a \, d\phi'}{a^2} = \hat{z} \frac{I}{2a} $$

### Magnetic Field of a Current Loop – Far Field (Magnetic Dipole)

Consider a line-current in the form of a circular loop of radius $a$ and carrying a current $I$, as shown in the figure ($r \gg a$).

Find the magnetic field at the point $P$ far away from the loop.

A small current loop such as this is a magnetic dipole.

Use the superposition integral for the A-field:

$$ \vec{A}(r, \theta, \phi = 0) = \left\{ \frac{\mu_0}{4\pi} \int \frac{J(r') \, dV'}{|\vec{r} - \vec{s}'|^2} \right\} \rightarrow \int - \int = \hat{z} \frac{I}{4\pi} |\vec{r} - \vec{s}'| $$

$$ ds' = \hat{r}' \, d\phi' $$

$$ |\vec{r} - \vec{s}'| \approx r - a \sin(\theta) \cos(\phi') $$

Integrate over the cross-section of the wire.

Be careful – tricky integral – the unit vector is changing directions within the integral.
Magnetic Field of a Current Loop – Far Field (Magnetic Dipole)

\[ \vec{A}(r, \theta, \phi = 0) = \vec{P}(r, \theta, \phi = 0) \]

\[ \approx \frac{\mu_0 l 2\pi}{4\pi} \left[ \cos(\phi') \hat{y} - \sin(\phi') \hat{x} \right] a d\phi' \]

\[ \approx \frac{\mu_0 l (\pi a^2)}{4\pi} \frac{\sin(\theta) \hat{y}}{r^2} \]

More generally:

\[ \vec{A}(r, \theta, \phi) = \frac{\mu_0 l (\pi a^2) \sin(\theta)}{4\pi r^2} \hat{\phi} \]

And:

\[ \vec{H}(r, \theta, \phi) = \frac{\nabla \times \vec{A}}{\mu_0} \]

\[ \approx \frac{\mu_0 l (\pi a^2)}{4\pi \mu_0 r^3} \left[ 2\cos(\theta) \hat{r} + \sin(\theta) \hat{\phi} \right] \]

Electric and Magnetic Dipoles and Dipole Moments

Electric dipole moment:

\[ \vec{p} = q \vec{d} \]

Magnetic dipole moment:

\[ \vec{m} = l (\pi a^2) \hat{\vec{n}} \]
Magnetic Flux and Vector Potential Line Integral

The magnetic flux $\lambda$ through a surface is the surface integral of the B-field through the surface

$$\lambda = \iint \mathbf{B} \cdot d\mathbf{a}$$

$$= \mu_0 \iint \mathbf{H} \cdot d\mathbf{a}$$

Since:

$$\mathbf{B} = \mu_0 \mathbf{H} = \nabla \times \mathbf{A}$$

We get:

$$\lambda = \iint \mathbf{B} \cdot d\mathbf{a}$$

$$= \iint (\nabla \times \mathbf{A}) \cdot d\mathbf{a}$$

$$= \oint \mathbf{A} \cdot d\mathbf{s}$$

The magnetic flux through a surface is equal to the line-integral of the vector potential along a closed contour bounding that surface.