Lecture 8
Boundary Value Problems

In this lecture you will learn:

• How to solve some interesting boundary value problems

Solutions of Laplace Equation in Spherical Coordinates

For all the following solutions $\nabla^2 \phi(\vec{r}) = 0$

Spherically Symmetric Solution

$\phi(\vec{r}) = \frac{A}{r} + B$

A Constant Uniform Electric Field Solution

$\phi(\vec{r}) = -A \, r \cos(\theta) = -A \, z$

$\vec{E}(\vec{r}) = -\nabla \phi(\vec{r}) = A \, \hat{z}$

A Dipole Oriented Along z-Axis Solution

$\phi(\vec{r}) = A \, \frac{\cos(\theta)}{r^2}$
A Non-Uniformly Charged Spherical Shell - I

Consider a charged spherical sheet where the surface charge density is fixed and is given by:

\[ \sigma = \sigma_0 \cos(\theta) \]

Looking from outside the potential sort of looks like that of a dipole. So try a dipole-like solution for \( r > a \)

For \( r > a \):

\[ \phi_{\text{out}}(\vec{r}) = A \frac{\cos(\theta)}{r^2} + C \]

Why??

For \( r < a \): Try something that does not go to infinity at \( r = 0 \)

\[ \phi_{\text{in}}(\vec{r}) = B r \cos(\theta) + D \]

Why??

A Non-Uniformly Charged Spherical Shell - II

How would you know if your solution is the right one??

Uniqueness Theorem:

There is only one right solution and if you have a solution that satisfies all the boundary conditions then you have the right solution.
**A Non-Uniformly Charged Spherical Shell - III**

\[ \phi_{\text{out}}(\vec{r}) = A \frac{\cos(\theta)}{r^2} \quad \phi_{\text{in}}(\vec{r}) = B r \cos(\theta) \]

**Boundary Conditions:**

1. Potential inside and outside must be continuous at the surface of the shell
   \[ \phi_{\text{in}}(\vec{r})|_{r=a} = \phi_{\text{out}}(\vec{r})|_{r=a} \]
   \[ \Rightarrow A \frac{\cos(\theta)}{a^2} = B a \cos(\theta) \]

2. Discontinuity in the radial electric field at the surface must be related to the local surface charge density
   \[ \varepsilon_0 \left( E_{\text{out}}|_{r=a} - E_{\text{in}}|_{r=a} \right) = \sigma \]
   \[ \Rightarrow -\varepsilon_0 \frac{\partial \phi_{\text{out}}(\vec{r})}{\partial r}|_{r=a} + \varepsilon_0 \frac{\partial \phi_{\text{out}}(\vec{r})}{\partial r}|_{r=a} = \sigma \cos(\theta) \]
   \[ \Rightarrow 2A \frac{\cos(\theta)}{a} + B \cos(\theta) = \frac{\sigma \cos(\theta)}{\varepsilon_0} \]

**A Non-Uniformly Charged Spherical Shell - IV**

**Solution:**

\[ \phi_{\text{out}}(\vec{r}) = \left( \frac{\sigma_o a^3}{3\varepsilon_0} \right) \frac{\cos(\theta)}{r^2} \]

\[ \phi_{\text{in}}(\vec{r}) = \left( \frac{\sigma_o}{3\varepsilon_0} \right) r \cos(\theta) = \left( \frac{\sigma_o}{3\varepsilon_0} \right) z \]

A surface charge density of the form:

\[ \sigma = \sigma_o \cos(\theta) \]

produces a uniform z-directed E-field inside the charged shell and a dipole-like E-field outside the charged shell.
A Perfect Metallic Sphere in a Uniform E-Field - I

Consider a perfect metal sphere placed inside a constant and uniform z-directed E-field.

The field lines shown on the right cannot be the right picture – there cannot be any E-field inside a perfect metal.

**Question:** So what happens when a metal sphere is placed inside a constant and uniform electric field??

**Answer:** Induced surface charges appear on the metal sphere that screen out the outside E-field.

A Perfect Metallic Sphere in a Uniform E-Field - II

**Question:** What kind of surface charge density gets induced on the metal sphere that produces a uniform E-field inside the sphere which completely cancels the applied E-field??

**Answer:** Induced surface charge density on the metal sphere must look like:

\[ \sigma \propto \cos(\theta) \]
A Perfect Metallic Sphere in a Uniform E-Field - III

Solution:

The outside potential must look like a combination of:

(i) The potential corresponding to a uniform z-directed E-field, and

(ii) The dipole-like potential from the induced surface charge density on the metal sphere

So try the following solution for \( r > a \):

For \( r > a \): \( \phi_{\text{out}}(r) = A \frac{\cos(\theta)}{r^2} - E_o r \cos(\theta) \)

For \( r < a \): Metal sphere is equipotential with zero potential \( \phi_{\text{in}}(r) = 0 \)

A Perfect Metallic Sphere in a Uniform E-Field - IV

For \( r > a \): \( \phi_{\text{out}}(r) = A \frac{\cos(\theta)}{r^2} - E_o r \cos(\theta) \)

For \( r < a \): Metal sphere is an equipotential with zero potential \( \phi_{\text{in}}(r) = 0 \)

Boundary Condition:

(i) The potential at the surface of the metal must be continuous (i.e. the same inside and outside) and, therefore, must be zero

\[ \phi_{\text{in}}(r = a) = \phi_{\text{out}}(r = a) \]

\[ A \frac{\cos(\theta)}{a^2} - E_o a \cos(\theta) = 0 \]

Solution:

\[ \phi_{\text{out}}(r) = E_o a^3 \frac{\cos(\theta)}{r^2} - E_o r \cos(\theta) \]
A Perfect Metallic Sphere in a Uniform E-Field - V

Surface Charge Density:

Now let's calculate the induced surface charge density on the metal sphere surface.

\[
\varepsilon_0 \left( E_{\text{out},r} \bigg|_{r=a} - E_{\text{in},r} \bigg|_{r=a} \right) = \sigma
\]

\[
\Rightarrow - \varepsilon_0 \frac{\partial \phi_{\text{out}}}{\partial r} \bigg|_{r=a} - 0 = \sigma
\]

\[
\Rightarrow \sigma = 3 \varepsilon_0 E_0 \cos(\theta)
\]

As expected, the induced surface charge density is just what is needed to completely cancel the applied E-field inside the metal sphere to give a net zero E-field.

Parallel Plate Capacitor with a Dielectric - I

Consider the parallel plate capacitor now with a dielectric between the plates.

\[
\phi = V
\]

\[
\phi = 0
\]

area = A

Need to find the potential \( \phi(x) \) between the plates.

Need to solve:

\[
\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} = 0
\]

With the two boundary conditions:

\[
\phi(x = -\frac{d}{2}) = V \quad \phi(x = +\frac{d}{2}) = 0
\]

Solution is:

\[
\phi(x) = V \left( 1 - \frac{x}{d} \right)
\]

\[
\Rightarrow E_x(x) = -\frac{\partial \phi(x)}{\partial x} = \frac{V}{d}
\]
Parallel Plate Capacitor with a Dielectric - II

Surface Charge Densities on Metal Plates

On the left plate ($\sigma_L$)
- The field inside the metal is zero
- The field just outside the metal is $V/d$

\[
\left( \varepsilon - \frac{V}{d} \right) = \sigma_L \quad \Rightarrow \quad \sigma_L = \varepsilon \frac{V}{d}
\]

On the right plate ($\sigma_R$)
- The field just outside the metal is $V/d$
- The field inside the metal is zero

\[
0 - \varepsilon \frac{V}{d} = \sigma_R \quad \Rightarrow \quad \sigma_R = -\varepsilon \frac{V}{d}
\]

Total charge on the positive plate: $Q = \frac{\varepsilon A}{d} V$

Capacitance: $C = \frac{Q}{V} = \frac{dQ}{dV} = \frac{\varepsilon A}{d}$
Parallel Plate Capacitor with a Non-Uniform Dielectric - I

In region 1 need to solve:
\[ \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} = 0 \]
assume a solution:
\[ \phi_1(x) = Ax + B \]

In region 2 need to solve:
\[ \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} = 0 \]
assume a solution:
\[ \phi_2(x) = Cx + D \]

Boundary Conditions: Need 4 boundary conditions to determine A, B, C, and D

1. \[ \phi_1 \left( x = -\frac{d}{2} \right) = V \]
2. \[ \phi_2 \left( x = +\frac{d}{2} \right) = 0 \]
3. Continuity of the potential at \( x = 0 \)
4. Discontinuity of the normal component of the E-field at \( x = 0 \)

Parallel Plate Capacitor with a Non-Uniform Dielectric - II

In region 1:
\[ \phi_1(x) = Ax + B \]

In region 2:
\[ \phi_2(x) = Cx + D \]
Remember Your Boundary Conditions

(1) The discontinuity of the normal component of the D-field at a material interface is related to the (unpaired) surface charge density at the interface

\[ D_2 - D_1 = \sigma_u \quad \text{or} \quad \varepsilon_2 E_2 - \varepsilon_1 E_1 = \sigma_u \]

or

\[ \varepsilon_0 (E_2 - E_1) = \sigma_u + \sigma_p \]

(2) The parallel component of the E-field at a material interface is always continuous at the interface (no change here)

\[ (E_2 - E_1) = 0 \]

Parallel Plate Capacitor with a Non-Uniform Dielectric - III

Surface Charge Densities on Metal Plates

On the left plate (\( \sigma_L \))

\[ \sigma_L = \varepsilon_1 E_x \left( x = -\frac{d}{2} \right) = -\frac{V 2 \varepsilon_1 \varepsilon_2}{d (\varepsilon_1 + \varepsilon_2)} \]

On the right plate (\( \sigma_R \))

\[ \sigma_R = -\varepsilon_2 E_x \left( x = \frac{d}{2} \right) = -\frac{V 2 \varepsilon_1 \varepsilon_2}{d (\varepsilon_1 + \varepsilon_2)} \]

Capacitance

\[ C = \frac{Q}{V} = \frac{\sigma_L A}{\varepsilon_1 + \varepsilon_2} = \frac{2 \varepsilon_1 \varepsilon_2 A}{\varepsilon_1 + \varepsilon_2 d} \]
The discontinuity of the E-field at the interface of the two dielectrics is related to the surface charge density due to the paired charges (plot for $\varepsilon_2 > \varepsilon_1$).

Paired Surface Charge Density at the Interface of the Two Dielectrics:

$$\varepsilon_0 (E_2 - E_1) = \sigma_u + \sigma_p$$

Use:

$$\varepsilon_0 (E_2(x=0) - E_1(x=0)) = \sigma_u + \sigma_p = \sigma_p$$

$$\sigma_p = -\varepsilon_0 \frac{V}{d} \frac{2(\varepsilon_2 - \varepsilon_1)}{\varepsilon_1 + \varepsilon_2}$$