Lecture 8

Boundary Value Problems

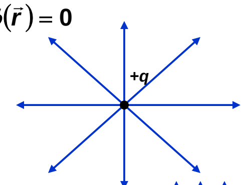
In this lecture you will learn:

- **How to solve some interesting boundary value problems**

Solutions of Laplace Equation in Spherical Coordinates

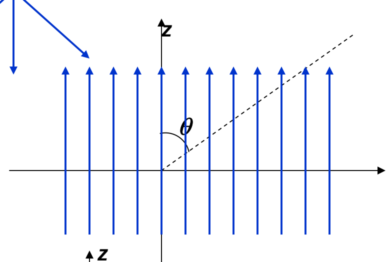
For all the following solutions $\nabla^2 \phi(\vec{r}) = 0$

Spherically Symmetric Solution



$$\phi(\vec{r}) = \frac{A}{r} + B$$

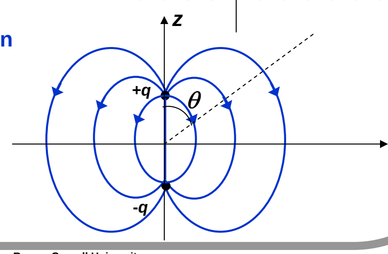
A Constant Uniform Electric Field Solution



$$\phi(\vec{r}) = -A\, r \cos(\theta) = -A\, z$$

$$\vec{E}(\vec{r}) = -\nabla \phi(\vec{r}) = A\,\hat{z}$$

A Dipole Oriented Along z-Axis Solution

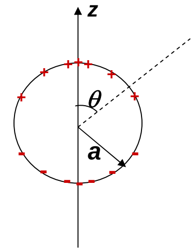


$$\phi(\vec{r}) = A \frac{\cos(\theta)}{r^2}$$

A Non-Uniformly Charged Spherical Shell - I

Consider a charged spherical sheet where the surface charge density is fixed and is given by:

$$\sigma = \sigma_o \cos(\theta)$$



Looking from outside the potential sort of looks like that of a dipole. So try a dipole-like solution for $r > a$

For $r > a$: $$\phi_{out}(\vec{r}) = A\frac{\cos(\theta)}{r^2} + \cancel{C}^{\,0}$$

Why??

For $r < a$: Try something that does not go to infinity at $r = 0$

$$\phi_{in}(\vec{r}) = B\, r\, \cos(\theta) + \cancel{D}^{\,0}$$

Why??

A Non-Uniformly Charged Spherical Shell - II

For $r < a$: $$\phi_{in}(\vec{r}) = B\, r\, \cos(\theta) + \cancel{D}^{\,0}$$

Why??

$$\phi(\vec{r}=0) - \phi(\vec{r}=\infty) = \int_{\vec{r}=0}^{\infty} \vec{E}.d\vec{s} = 0$$

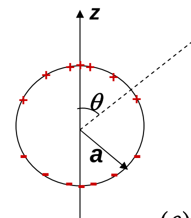
How would you know if your solution is the right one??

Uniqueness Theorem:

There is only one right solution and If you have a solution that satisfies all the boundary conditions then you have the right solution

A Non-Uniformly Charged Spherical Shell - III

$$\phi_{out}(\vec{r}) = A\,\frac{\cos(\theta)}{r^2} \qquad \phi_{in}(\vec{r}) = B\, r\, \cos(\theta)$$



$\sigma = \sigma_o \cos(\theta)$

Boundary Conditions:

(1) Potential inside and outside must be continuous at the surface of the shell

$$\phi_{in}(\vec{r})\big|_{r=a} = \phi_{out}(\vec{r})\big|_{r=a}$$

$$\Rightarrow \quad A\,\frac{\cos(\theta)}{a^2} = B\, a\, \cos(\theta)$$

(2) Discontinuity in the radial electric field at the surface must be related to the local surface charge density

$$\varepsilon_o \left(E_{out,r}\big|_{r=a} - E_{in,r}\big|_{r=a} \right) = \sigma$$

$$\Rightarrow \quad -\varepsilon_o \frac{\partial \phi_{out}(\vec{r})}{\partial r}\bigg|_{r=a} + \varepsilon_o \frac{\partial \phi_{out}(\vec{r})}{\partial r}\bigg|_{r=a} = \sigma_o \cos(\theta)$$

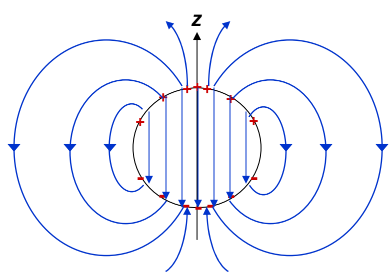
$$\Rightarrow \quad 2A\,\frac{\cos(\theta)}{a^3} + B\cos(\theta) = \frac{\sigma_o \cos(\theta)}{\varepsilon_o}$$

A Non-Uniformly Charged Spherical Shell - IV

Solution:

$$\phi_{out}(\vec{r}) = \left(\frac{\sigma_o\, a^3}{3\varepsilon_o} \right) \frac{\cos(\theta)}{r^2}$$

$$\phi_{in}(\vec{r}) = \left(\frac{\sigma_o}{3\varepsilon_o} \right) r\, \cos(\theta) = \left(\frac{\sigma_o}{3\varepsilon_o} \right) z$$



A surface charge density of the form:

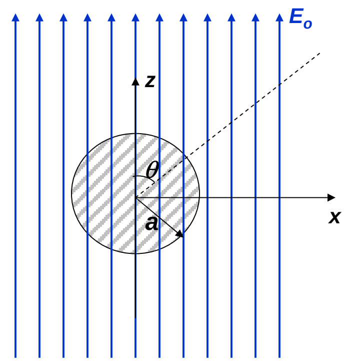
$\sigma = \sigma_o \cos(\theta)$

produces a uniform z-directed E-field inside the charged shell and a dipole-like E-field outside the charged shell

A Perfect Metallic Sphere in a Uniform E-Field - I

Consider a perfect metal sphere placed inside a constant and uniform z-directed E-field

The field lines shown on the right cannot be the right picture – there cannot be any E-field inside a perfect metal



Question: **So what happens when a metal sphere is placed inside a constant and uniform electric field ??**

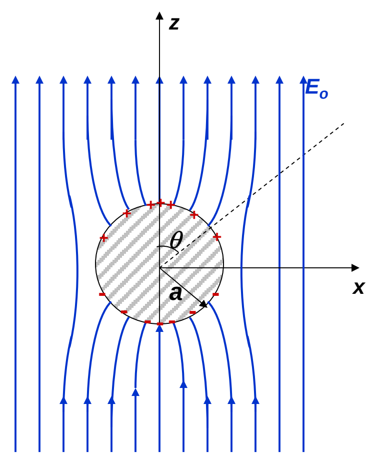
Answer: **Induced surface charges appear on the metal sphere that screen out the outside E-field**

A Perfect Metallic Sphere in a Uniform E-Field - II

Question: **What kind of surface charge density gets induced on the metal sphere that produces a uniform E-field inside the sphere which completely cancels the applied E-field??**

Answer: **Induced surface charge density on the metal sphere must look like:**

$$\sigma \propto \cos(\theta)$$



A Perfect Metallic Sphere in a Uniform E-Field - III

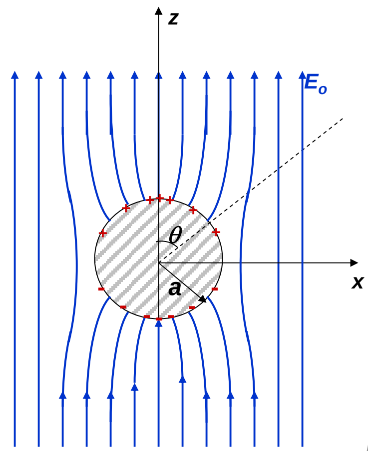
Solution:

The outside potential must look like a combination of:

(i) The potential corresponding to a uniform z-directed E-field, and

(ii) The dipole-like potential from the induced surface charge density on the metal sphere

So try the following solution for $r > a$:



For $r > a$: $\phi_{out}(\vec{r}) = A\dfrac{\cos(\theta)}{r^2} - E_o\, r\cos(\theta)$

For $r < a$: Metal sphere is equipotential with zero potential

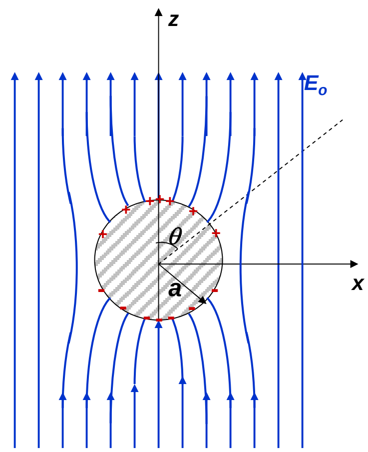
$$\phi_{in}(\vec{r}) = 0$$

A Perfect Metallic Sphere in a Uniform E-Field - IV

For $r > a$: $\phi_{out}(\vec{r}) = A\dfrac{\cos(\theta)}{r^2} - E_o\, r\cos(\theta)$

For $r < a$: Metal sphere is an equipotential with zero potential

$$\phi_{in}(\vec{r}) = 0$$



Boundary Condition:

(i) The potential at the surface of the metal must be continuous (i.e. the same inside and outside) and, therefore, must be zero

$$\phi_{in}(\vec{r})\big|_{r=a} = \phi_{out}(\vec{r})\big|_{r=a}$$

$$\Rightarrow \quad A\frac{\cos(\theta)}{a^2} - E_o\, a\cos(\theta) = 0$$

Solution:

$$\phi_{out}(\vec{r}) = E_o\, a^3\frac{\cos(\theta)}{r^2} - E_o\, r\cos(\theta)$$

A Perfect Metallic Sphere in a Uniform E-Field - V

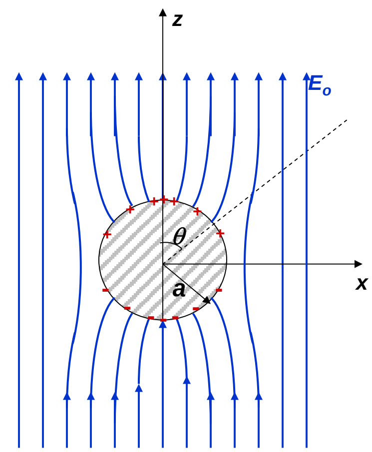
Surface Charge Density:

Now lets calculate the induced surface charge density on the metal sphere surface

$$\varepsilon_o \left(E_{out,r}\big|_{r=a} - E_{in,r}\big|_{r=a} \right) = \sigma$$

$$\Rightarrow \quad -\varepsilon_o \left.\frac{\partial \phi_{out}(\vec{r})}{\partial r}\right|_{r=a} - \quad 0 = \sigma$$

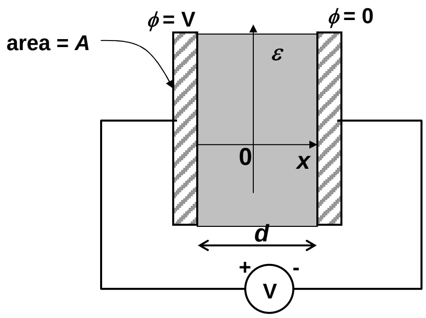
$$\Rightarrow \quad \sigma = 3\,\varepsilon_o\, E_o \cos(\theta)$$



As expected, the induced surface charge density is just what is needed to completely cancel the applied E-field inside the metal sphere to give a net zero E-field

Parallel Plate Capacitor with a Dielectric - I

Consider the parallel plate capacitor now with a dielectric between the plates



Need to find the potential $\phi(x)$ between the plates

Need to solve:

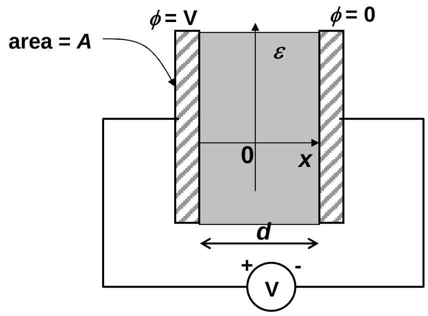
$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} = 0$$

With the two boundary conditions:

$$\phi\left(x = -\frac{d}{2}\right) = V \qquad \phi\left(x = +\frac{d}{2}\right) = 0$$

Solution is: $\phi(x) = V\left(\frac{1}{2} - \frac{x}{d}\right) \quad \Rightarrow \quad E_x(x) = -\frac{\partial \phi(x)}{\partial x} = \frac{V}{d}$

Parallel Plate Capacitor with a Dielectric - II



Solution is:

$$\phi(x) = V\left(\frac{1}{2} - \frac{x}{d}\right)$$

$$\Rightarrow \quad E_x(x) = -\frac{\partial \phi(x)}{\partial x} = \frac{V}{d}$$

Surface Charge Densities on Metal Plates

$$\varepsilon_2 E_2 - \varepsilon_1 E_1 = \sigma_u$$

On the left plate (σ_L)

- The field inside the metal is zero
- The field just outside the metal is V/d

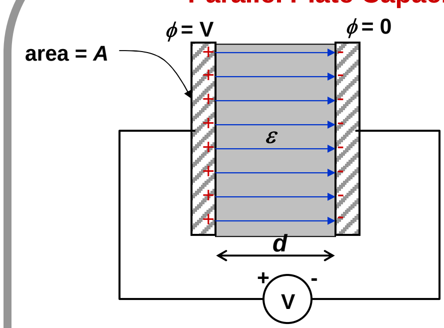
$$\left(\varepsilon \frac{V}{d} - 0\right) = \sigma_L \quad \Rightarrow \quad \sigma_L = \varepsilon \frac{V}{d}$$

On the right plate (σ_R)

- The field just outside the metal is V/d
- The field inside the metal is zero

$$\left(0 - \varepsilon \frac{V}{d}\right) = \sigma_R \quad \Rightarrow \quad \sigma_R = -\varepsilon \frac{V}{d}$$

Parallel Plate Capacitor with a Dielectric - III



Surface Charge Densities on Metal Plates

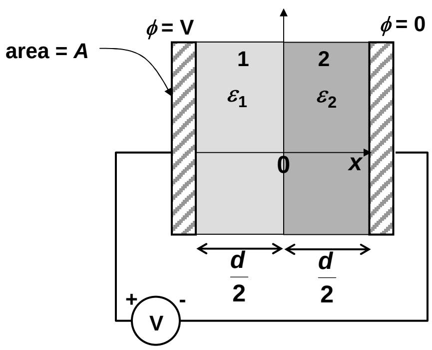
$$\sigma_L = \varepsilon \frac{V}{d}$$

$$\sigma_R = -\varepsilon \frac{V}{d}$$

Total charge on the positive plate: $Q = \frac{\varepsilon A}{d} V$

Capacitance: $C = \frac{Q}{V} = \frac{dQ}{dV} = \frac{\varepsilon A}{d}$

Parallel Plate Capacitor with a Non-Uniform Dielectric - I



In region 1 need to solve:

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} = 0$$

assume a solution:

$$\phi_1(x) = A\,x + B$$

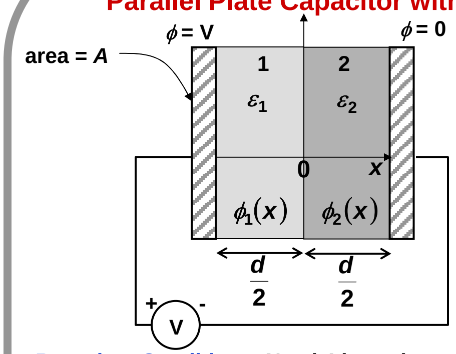
In region 2 need to solve:

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} = 0$$

assume a solution:

$$\phi_2(x) = Cx + D$$

Parallel Plate Capacitor with a Non-Uniform Dielectric - II



In region 1:

$$\phi_1(x) = A\,x + B$$

In region 2:

$$\phi_2(x) = Cx + D$$

Boundary Conditions: Need 4 boundary conditions to determine A, B, C, and D

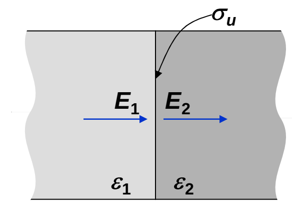
(1) $\phi_1\left(x = -\frac{d}{2}\right) = V$ (2) $\phi_2\left(x = +\frac{d}{2}\right) = 0$

(3) $\phi_1(x = 0) = \phi_2(x = 0)$ ⟶ Continuity of the potential at $x = 0$

(4) $\varepsilon_2 E_2(x = 0) - \varepsilon_1 E_1(x = 0) = -\varepsilon_2 \left.\frac{\partial \phi_2}{\partial x}\right|_{x=0} + \varepsilon_1 \left.\frac{\partial \phi_1}{\partial x}\right|_{x=0} = \sigma_u = 0$

Discontinuity of the normal component of the E-field at $x = 0$

Remember Your Boundary Conditions

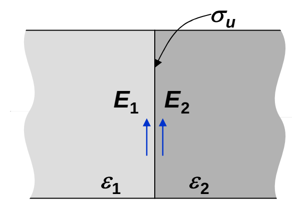


(1) The discontinuity of the normal component of the D-field at a material interface is related to the (unpaired) surface charge density at the interface

$$D_2 - D_1 = \sigma_u \quad \text{or} \quad \varepsilon_2 E_2 - \varepsilon_1 E_1 = \sigma_u$$

or

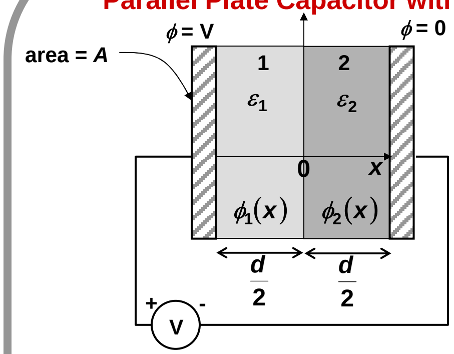
$$\varepsilon_o (E_2 - E_1) = \sigma_u + \sigma_p$$



(2) The parallel component of the E-field at a material interface is always continuous at the interface (no change here)

$$(E_2 - E_1) = 0$$

Parallel Plate Capacitor with a Non-Uniform Dielectric - III



In region 1:

$$\phi_1(x) = -V \frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2}\left(\frac{2x}{d} - \frac{\varepsilon_1}{\varepsilon_2}\right)$$

In region 2:

$$\phi_2(x) = -V \frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2}\left(\frac{2x}{d} - 1\right)$$

Surface Charge Densities on Metal Plates

On the left plate (σ_L)

$$\sigma_L = \varepsilon_1 E_x\left(x = -\frac{d}{2}\right) = \frac{V}{d}\frac{2\varepsilon_1 \varepsilon_2}{\varepsilon_1 + \varepsilon_2}$$

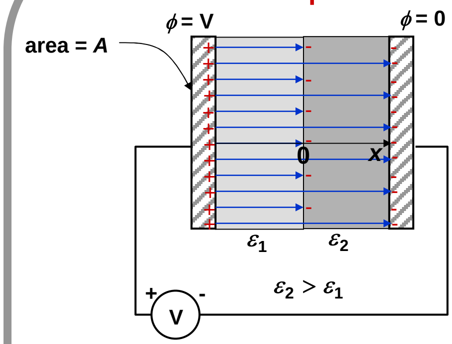
On the right plate (σ_R)

$$\sigma_R = -\varepsilon_2 E_x\left(x = \frac{d}{2}\right) = -\frac{V}{d}\frac{2\varepsilon_1 \varepsilon_2}{\varepsilon_1 + \varepsilon_2}$$

Capacitance

$$C = \frac{Q}{V} = \frac{\sigma_L A}{V} = \frac{2\varepsilon_1 \varepsilon_2}{\varepsilon_1 + \varepsilon_2}\frac{A}{d}$$

Parallel Plate Capacitor with a Non-Uniform Dielectric - IV



The discontinuity of the E-field at the interface of the two dielectrics is related to the surface charge density due to the paired charges (plot for $\varepsilon_2 > \varepsilon_1$)

Paired Surface Charge Density at the Interface of the Two Dielectrics:

$$\varepsilon_o \left(E_2 - E_1 \right) = \sigma_u + \sigma_p$$

Use: $\varepsilon_o \left(E_2(x=0) - E_1(x=0) \right) = \sigma_u + \sigma_p = \sigma_p$

$$\sigma_p = -\varepsilon_o \frac{V}{d} \frac{2(\varepsilon_2 - \varepsilon_1)}{\varepsilon_1 + \varepsilon_2}$$

ECE 303 – Fall 2007 – Farhan Rana – Cornell University