Lecture 7

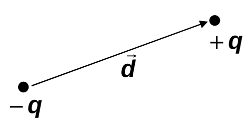
Polarization

In this lecture you will learn:

- Material Polarization
- Mathematics of Polarization
- Dielectric Permittivity
- Conductors Vs Dielectrics
- Appendix

Charge Dipoles and Dipole Moments

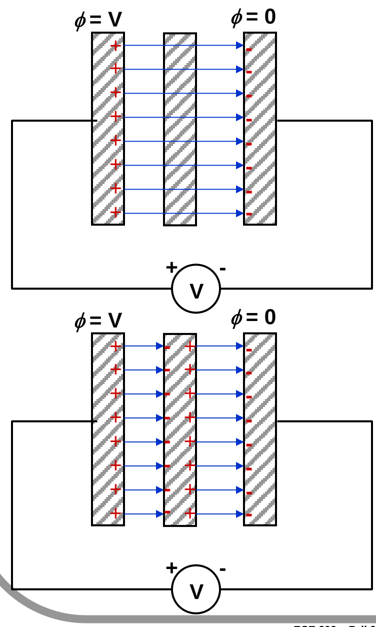
Consider a charge dipole:



Dipole moment of the charge dipole is a vector $\vec{p}$ such that:

$$\vec{p} = q\,\vec{d}$$

Conductors in Electric Fields – A Review



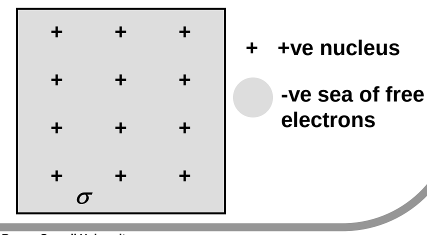
- Consider the problem when a conductive plate was placed inside an electric field

- Conductors have “free charges” that are able to move around (mostly these “free charges” are electrons that are not attached to any particular atom)

- Under the influence of external E-field these free charges move to completely screen out the E-field from within the conducting material

Conductors

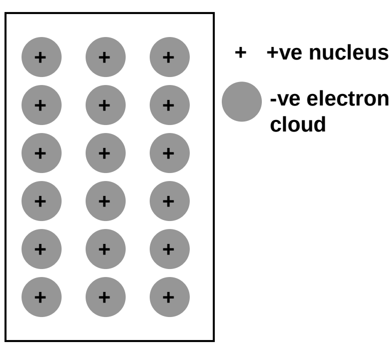
Sea of Free Electrons



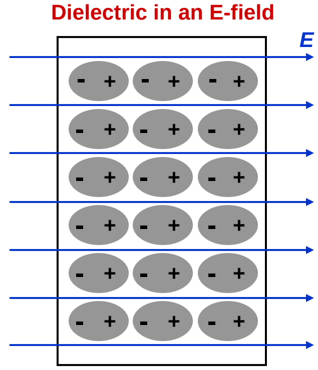
Dielectrics in Electric Fields - Polarization

Many materials do not have “free electrons” that can move around, but have electrons bound to atoms, as shown below

Dielectric

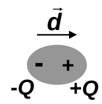


Dielectric in an E-field

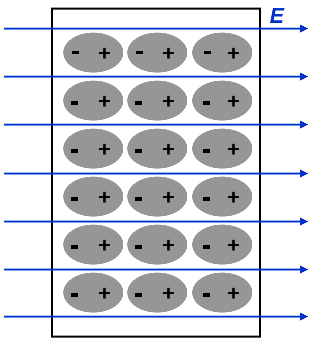


In the presence of an E-field, the electron cloud in each atom distorts ALMOST INSTANTANEOUSLY so that each atom looks like a charge dipole

Dielectrics in Electric Fields – Polarization Vector



$\vec{p} = Q\vec{d}$ = dipole moment of each dipole



The polarization vector $\vec{P}$ is a vector such that:

$$\vec{P} = N\,\vec{p}$$
$$= NQ\vec{d}$$

Where N is the number of charge dipoles per unit volume in the material

The units of $\vec{P}$ are: Coumlombs/m^2

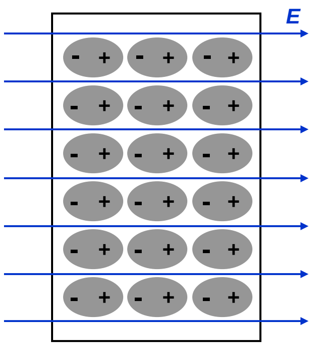
The polarization vector $\vec{P}$ characterizes the polarization density of the medium under the influence of the electric field

Dielectrics in Electric Fields – Electrical Susceptibility

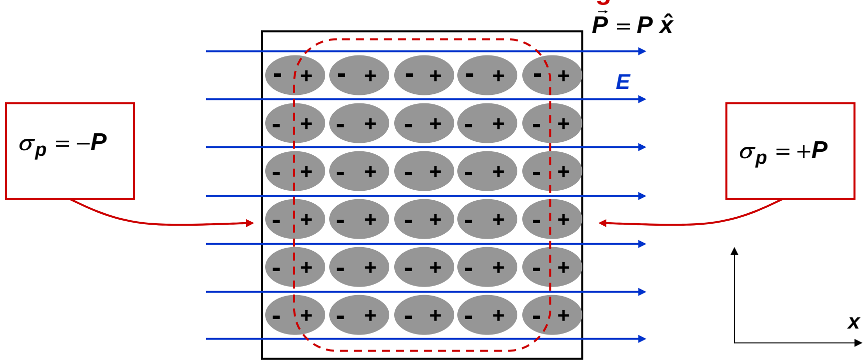
Naturally, one would expect the polarization of the material to be proportional to the strength of the electric field:

$$\vec{P} \propto \vec{E}$$
$$\Rightarrow \quad \vec{P} = \varepsilon_o\,\chi_e\,\vec{E}$$

The constant of proportionality χ_e is called the electrical susceptibility of the material



Material Polarization and Surface Charge Densities

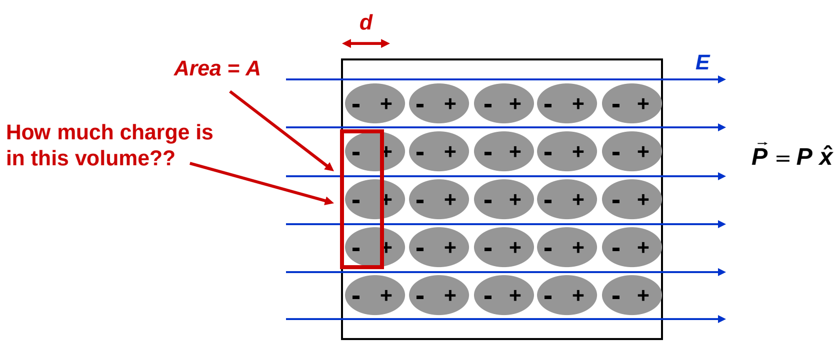


• The stuff inside the box in on the average charge neutral (same number of positive and negative charges)

• There is a net negative surface charge density on the left facet of the material as a result of material polarization

• There is a net positive surface charge density on the right facet of the material as a result of material polarization

Material Polarization and Surface Charge Densities



Total interface negative charge due to dipoles in the volume *Ad* = - *Q N A d*

If we divide the total interface charge by the area *A* we get the interface charge per unit area which would be the surface charge density σ_p

$$\Rightarrow \sigma_p = -\frac{QNAd}{A} = -NQd = -P$$

Material Polarization and Volume Charge Densities

More generally, one can write a volume polarization volume charge density due to material polarization as:

$$\rho_p = -\nabla . \vec{P}$$

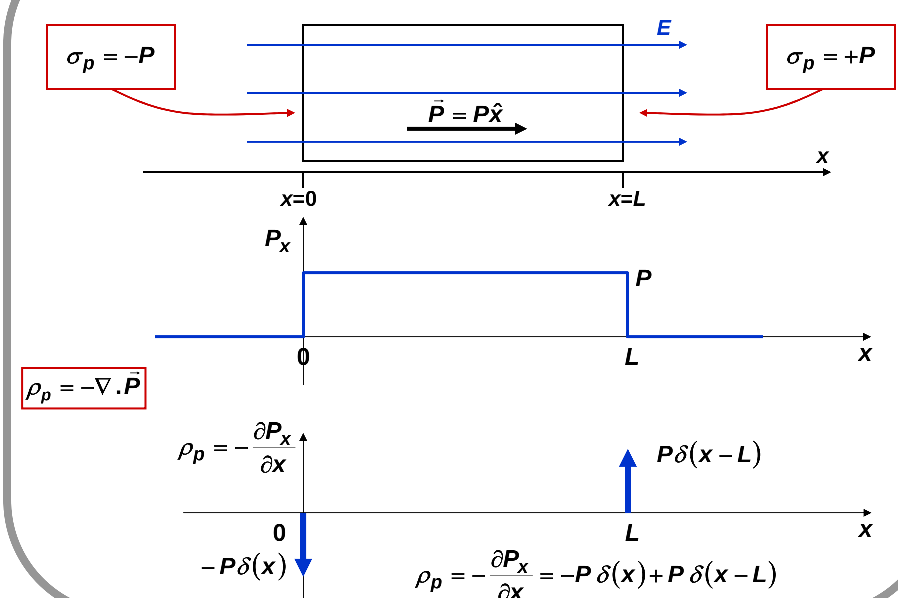
(A formal proof is given in the Appendix)

There will be a net non-zero volume charge density inside a material if the material polarization is varying in space

In 1D situations:

$$\rho_p(x) = -\frac{\partial P_x}{\partial x}$$

Material Polarization and Charge Densities



Mathematics of Polarization – The "D" Field

Gauss' Law states:

$$\nabla . \varepsilon_o \vec{E} = \rho$$

But charge densities could be of two types:

1) **Paired charge density ρ_p (due to material polarization)**
2) **Unpaired charge density ρ_u (due to everything else – the usual stuff)**

So:

$$\nabla . \varepsilon_o \vec{E} = \rho_u + \rho_p = \rho_u - \nabla . \vec{P}$$

$$\Rightarrow \quad \nabla . \left(\varepsilon_o \vec{E} + \vec{P} \right) = \rho_u$$

Using: $\rho_p = -\nabla . \vec{P}$

If one defines the D-field inside materials as:

$$\vec{D} = \varepsilon_o \vec{E} + \vec{P}$$

Then inside materials Gauss' Law becomes:

$$\nabla . \vec{D} = \rho_u$$

Mathematics of Polarization – Dielectric Permittivity

If one defines the D-field as:

$$\vec{D} = \varepsilon_o \vec{E} + \vec{P}$$

→ **Then:** $\nabla . \vec{D} = \rho_u$

Note that: $\vec{D} = \varepsilon_o \vec{E} + \vec{P} = \varepsilon_o \vec{E} + \varepsilon_o \chi_e \vec{E} = \varepsilon_o (1 + \chi_e) \vec{E}$

If one defines the dielectric permittivity of a material as:

$$\varepsilon = \varepsilon_o (1 + \chi_e)$$

then one can write the D-field inside materials as:

$$\vec{D} = \varepsilon \vec{E}$$

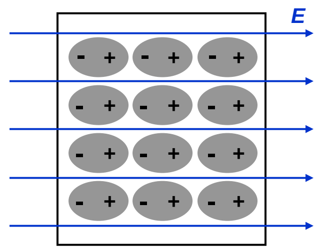
Inside materials the D-field obeys the Gauss' Law:

$$\boxed{\nabla . \vec{D} = \nabla . \varepsilon \vec{E} = \rho_u}$$

Mathematics of Polarization – Polarization Current Density

So far we have looked at the current density due to the motion of free unpaired charges:

$$\vec{J}_u = \sigma \vec{E}$$



The motion of paired charges also results in a current density:

$$\vec{J}_p = \frac{\partial \vec{P}}{\partial t} = \frac{\partial \left(\varepsilon_o \chi_e \vec{E}\right)}{\partial t}$$

$\vec{J}_p$ is called the polarization current density

Ampere's Law is correctly given by: $\nabla \times \vec{H} = \vec{J}_u + \vec{J}_p + \frac{\partial \left(\varepsilon_o \vec{E}\right)}{\partial t}$

Which can also be written as: $\nabla \times \vec{H} = \vec{J}_u + \frac{\partial \left(\varepsilon \vec{E}\right)}{\partial t}$

Mathematics of Polarization – Modified Maxwell's Equations

Putting it all together ….

One can forget all about material polarization, and polarization charge densities, and polarization current densities, as long as one uses the dielectric permittivity ε instead of the free space permittivity ε_o

Maxwell's equations in dielectric materials take the form:

$$\nabla . \left(\varepsilon_o \vec{E}\right) = \rho_u + \rho_p$$

$$\nabla . \mu_o \vec{H} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \mu_o \vec{H}}{\partial t}$$

$$\nabla \times \vec{H} = \vec{J}_u + \vec{J}_p + \frac{\partial \left(\varepsilon_o \vec{E}\right)}{\partial t}$$

OR

$$\nabla . \left(\varepsilon \vec{E}\right) = \rho_u$$

$$\nabla . \mu_o \vec{H} = 0$$

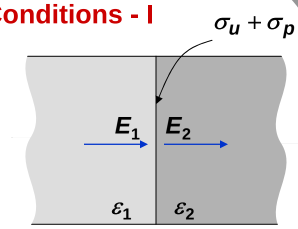
$$\nabla \times \vec{E} = -\frac{\partial \mu_o \vec{H}}{\partial t}$$

$$\nabla \times \vec{H} = \vec{J}_u + \frac{\partial \left(\varepsilon \vec{E}\right)}{\partial t}$$

Here the charge density ρ_u is the unpaired charge density

Here the current density is due to the unpaired charges

Dielectric Permittivity – Boundary Conditions - I



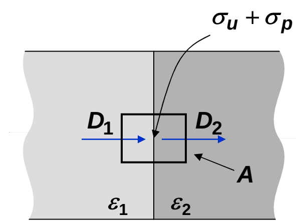
How to relate the electric fields on both sides of the dielectric interface ??

Gauss' Law in the presence of dielectric material is:

$$\nabla.\left(\vec{D}\right) = \nabla.\left(\varepsilon\,\vec{E}\right) = \rho_u$$

or $\quad \oiint \vec{D}\,.\,d\vec{a} = \iiint \rho_u\,dV$

Draw a Gaussian surface at the interface:

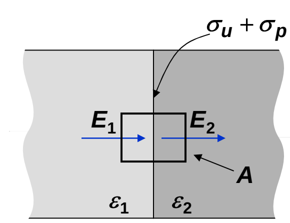


$$(D_2 - D_1)A = \sigma_u A$$

$$\Rightarrow \quad D_2 - D_1 = \sigma_u$$

or $\boxed{\varepsilon_2 E_2 - \varepsilon_1 E_1 = \sigma_u}$

Dielectric Permittivity – Boundary Conditions - II



How else can one relate the electric fields on both sides of the dielectric interface ??

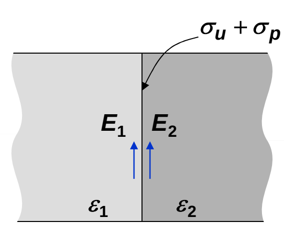
Gauss' Law in the presence of dielectric material is also:

$$\nabla.\left(\varepsilon_o\,\vec{E}\right) = \rho_u + \rho_p$$

Draw a Gaussian surface at the interface:

$$\varepsilon_o\left(E_2 - E_1\right)A = \left(\sigma_u + \sigma_p\right)A$$

or $\boxed{\varepsilon_o\left(E_2 - E_1\right) = \sigma_u + \sigma_p}$



The parallel component of the E-field at a material interface is always continuous at the interface (no change here)

$$\boxed{\left(E_2 - E_1\right) = 0}$$

Dielectrics Vs Conductors

Conductors:

Free unpaired charges move to completely screen the E-field on time scales longer than the dielectric relaxation time

Dielectrics:

Paired charges originating due to material polarization partially screen the E-field almost instantaneously

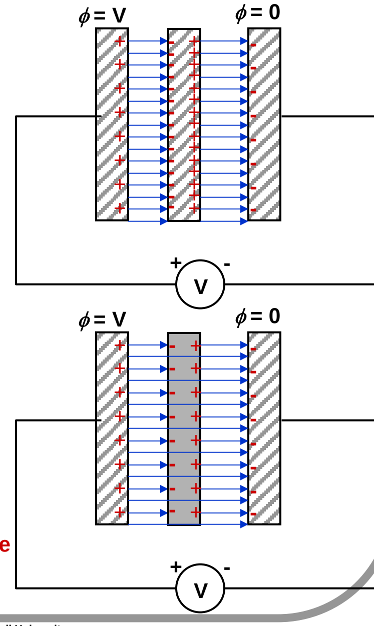
When $\sigma_u = 0$:

$$\varepsilon_2 E_2 - \varepsilon_1 E_1 = 0$$

also:

$$\varepsilon_o \left(E_2 - E_1\right) = \sigma_p$$

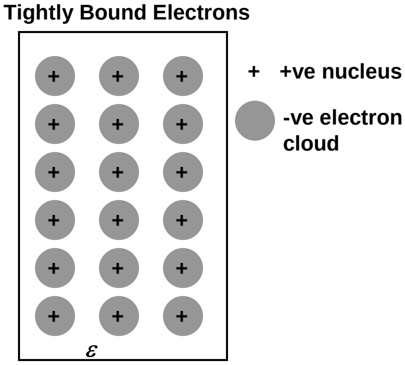
The discontinuity of the normal component of the E-field is due to the paired charges at the interface (even when $\sigma_u = 0$)



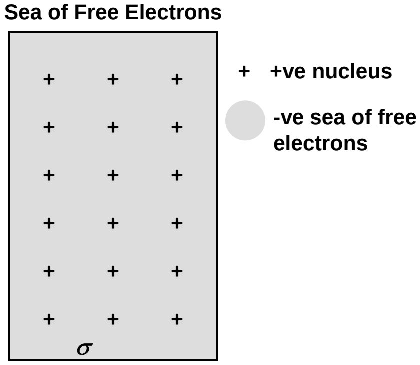
Conductors or Dielectrics

Some materials are conductors and some are dielectrics ………….

Dielectrics
Tightly Bound Electrons



Conductors
Sea of Free Electrons



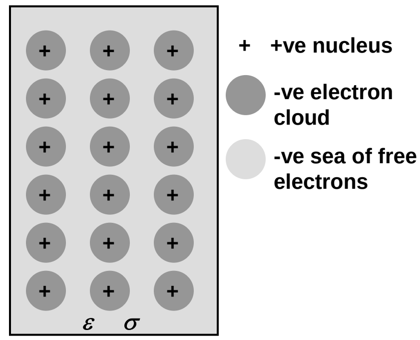
Conductors and Dielectrics

……. but many important materials are conductors as well as dielectrics

• Most conductors like gold, copper, and silver, and semiconductors like Silicon, are both conductors and dielectrics

• They have a sea of free electrons that results in a finite value of **conductivity** and they also have tightly bound core electrons that result in a value for the **dielectric permittivity**

Tightly bound core electrons and a sea of free electrons



Dielectric relaxation time for these materials $= \tau_d = \dfrac{\varepsilon}{\sigma}$

Appendix: Polarization Charge Density - I

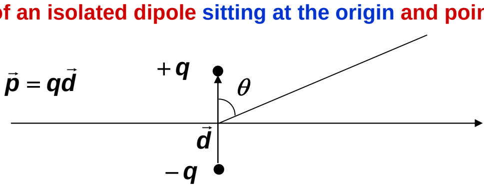
The expression relating the polarization charge density to the divergence of the polarization vector,

$$\rho_p = -\nabla . \vec{P}$$

can be proved more formally as shown below:

The potential of an isolated dipole sitting at the origin and pointing in the z-direction is:

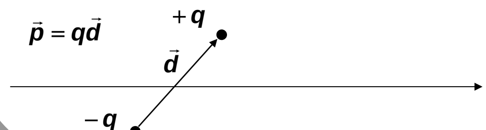
$$\vec{p} = q\vec{d}$$



$$\phi(\vec{r}) = \frac{|\vec{p}|}{4\pi\varepsilon_o} \frac{\cos(\theta)}{r^2}$$

More generally, the potential of a dipole sitting at position *r'* and pointing in an arbitrary direction is:

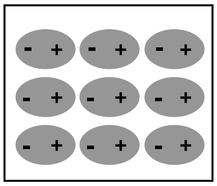
$$\vec{p} = q\vec{d}$$



$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_o} \frac{\vec{p} . (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

Appendix: Polarization Charge Density - II

If a material has a polarization density vector $\vec{P}(\vec{r}\,')$ then the potential due to all the dipoles can be found using superposition:



$$\phi(\vec{r}) = \iiint \frac{1}{4\pi\varepsilon_o} \frac{\vec{P}(\vec{r}\,') \cdot (\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} \, dV'$$

$$= \iiint \frac{1}{4\pi\varepsilon_o} \vec{P}(\vec{r}\,') \cdot \left[\vec{\nabla}' \left(\frac{1}{|\vec{r}-\vec{r}\,'|} \right) \right] dV'$$

Now integrate by parts in 3D……….

$$= \iiint \frac{1}{4\pi\varepsilon_o} \left[-\vec{\nabla}' \cdot \vec{P}(\vec{r}\,') \right] \left[\frac{1}{|\vec{r}-\vec{r}\,'|} \right] dV'$$

$$= \iiint \frac{-\vec{\nabla}' \cdot \vec{P}(\vec{r}\,')}{4\pi\varepsilon_o \, |\vec{r}-\vec{r}\,'|} \, dV'$$

This looks like the superposition integral for a volume charge density given by:

$$\rho_p(\vec{r}\,') = -\nabla \cdot \vec{P}(\vec{r}\,')$$