Lecture 6

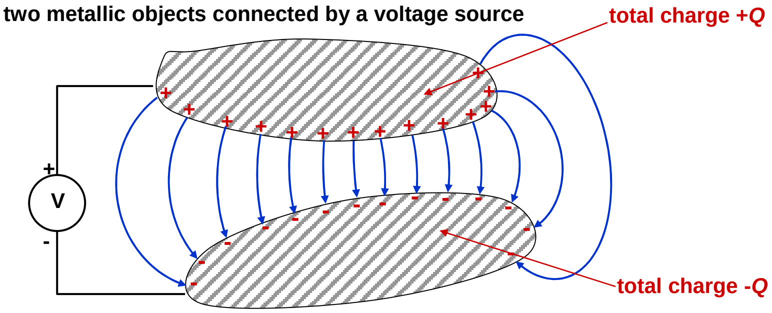
Capacitance

In this lecture you will learn:

- **Capacitance**
- **Capacitance of Some Simple Structures**
- **Some Simple Solutions of Laplace's Equation**

Capacitance

Consider two metallic objects connected by a voltage source

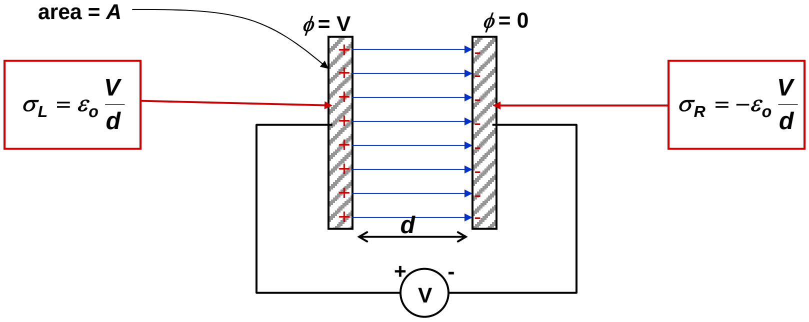


- **The total charge on both metal objects have the same magnitude but opposite sign**
- **The charge is proportional to the applied voltage:** $Q \propto V$
- **The constant of proportionality is called the capacitance *C* :** $Q = CV$
- **Capacitance has units Coulombs/Volts or Farads**
- **More generally, capacitance is also defined as:** $C = \dfrac{dQ}{dV}$

Capacitance of Parallel Plates

From previous lectures:

$$E_x = \frac{V}{d}$$



$$\sigma_L = \varepsilon_o \frac{V}{d}$$

$$\sigma_R = -\varepsilon_o \frac{V}{d}$$

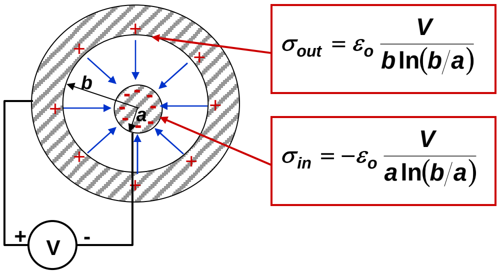
$$Q = \frac{\varepsilon_o A}{d} V$$

$$C = \frac{dQ}{dV} = \frac{Q}{V} = \frac{\varepsilon_o A}{d}$$

Capacitance of Concentric Cylinders

From previous lectures:

$$E_r(r) = -\frac{V}{r \ \ln\left(\frac{b}{a}\right)}$$



$$\sigma_{out} = \varepsilon_o \frac{V}{b \ln(b/a)}$$

$$\sigma_{in} = -\varepsilon_o \frac{V}{a \ln(b/a)}$$

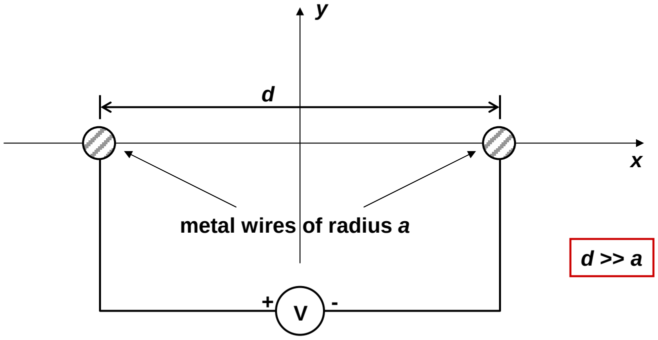
Since the structure is infinite in the z-direction (i.e. in the direction coming out of the slide) one can only talk about capacitance per unit length (units: Farads/m)

$$Q = \text{charge per unit length} = (2\pi b)\, \varepsilon_o \frac{V}{b \ln(b/a)}$$

$$C = \text{capacitance per unit length} = \frac{dQ}{dV} = \frac{Q}{V} = \frac{2\pi\, \varepsilon_o}{\ln(b/a)}$$

Capacitance of Parallel Metal Wires - I

Consider two infinitely long metal wires connected by a voltage source (the wires are infinitely long in the z-direction)



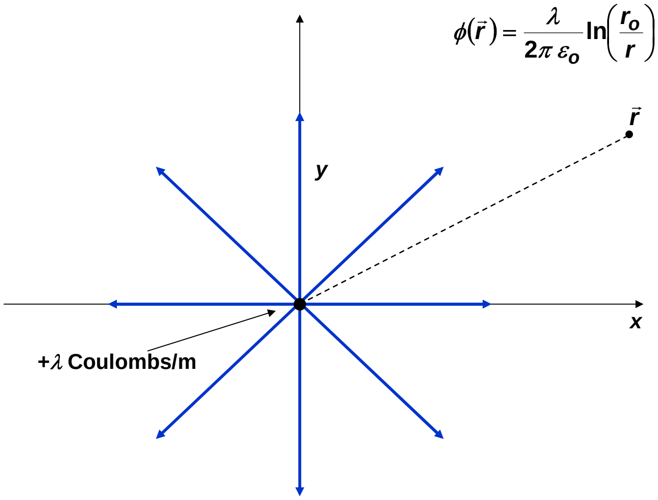
$d >> a$

Need to find the capacitance per unit length between them under the assumption that $d >> a$

Line Charge Revisited

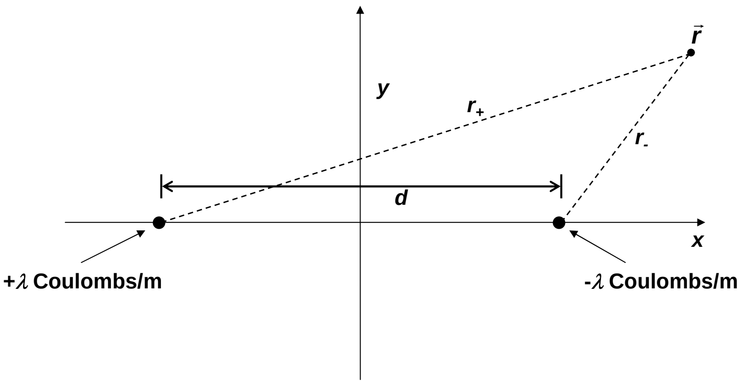
Start from the line charge ………….

$$\phi(\vec{r}) = \frac{\lambda}{2\pi\,\varepsilon_o}\ln\left(\frac{r_o}{r}\right)$$



Line Charge Dipole Revisited

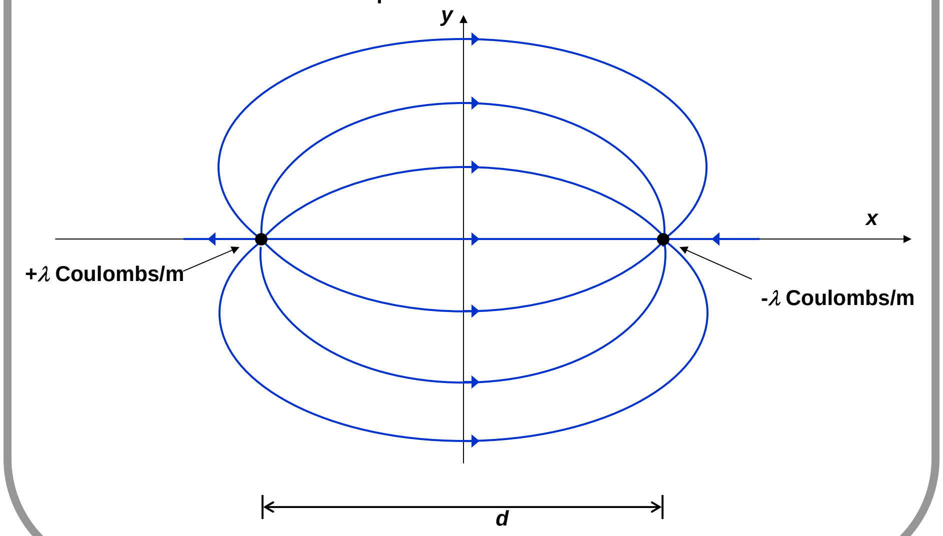
Then consider the line dipole……………



$$\phi(\vec{r}) = \frac{\lambda}{2\pi\,\varepsilon_o}\ln\left(\frac{r_-}{r_+}\right)$$

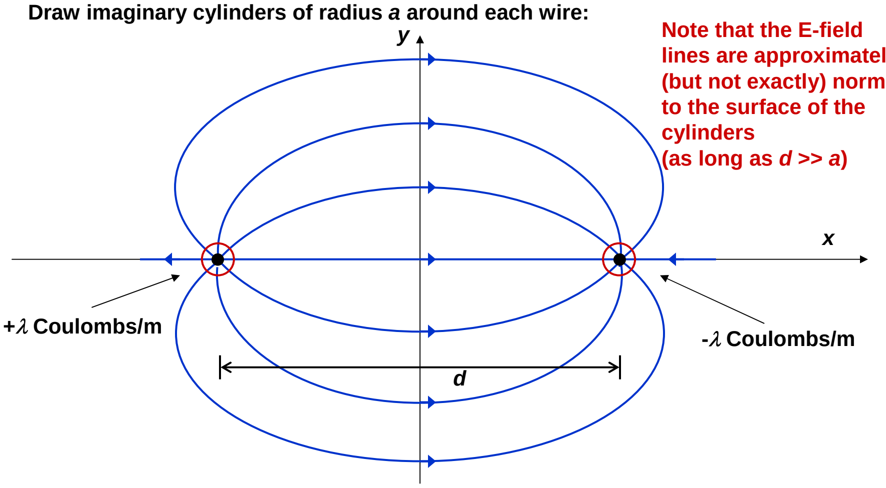
Capacitance of Parallel Metal Wires - II

Look at the E-fields of the line dipole:



Capacitance of Parallel Metal Wires - III

Draw imaginary cylinders of radius *a* around each wire:



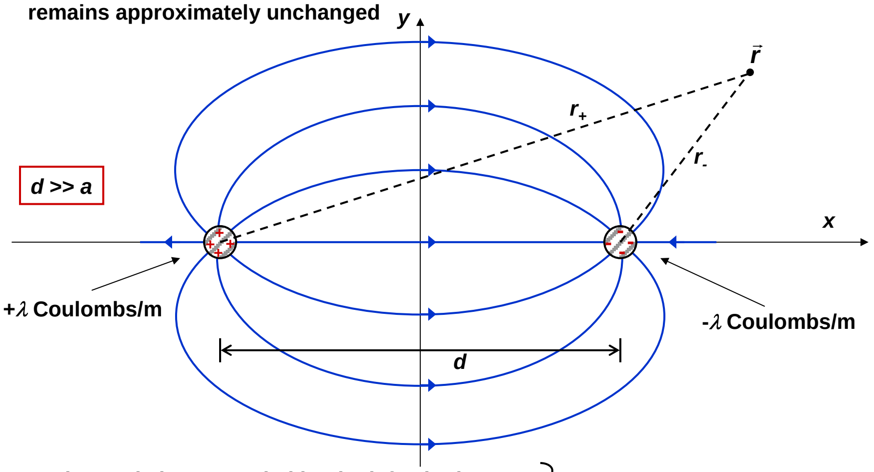
Note that the E-field lines are approximately (but not exactly) normal to the surface of the cylinders
(as long as *d* >> *a*)

The total electric flux per unit length coming out of the left cylinder is +λ

The total electric flux per unit length coming out of the right cylinder is -λ

Capacitance of Parallel Metal Wires - IV

Now if one replaces the cylinders and the line charges with metal wires of radius *a* that carry +λ and –λ coulombs of charge per unit length then the E-field outside remains approximately unchanged



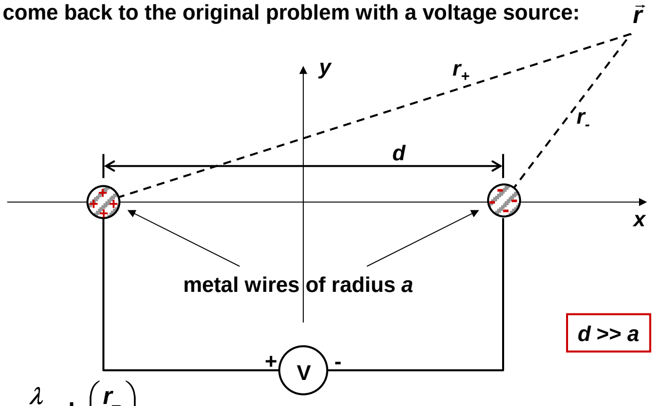
The total charge carried by the left wire is +λ

The total charge carried by the right wire is -λ

$$\phi(\vec{r}) = \frac{\lambda}{2\pi\,\varepsilon_o}\ln\left(\frac{r_-}{r_+}\right)$$

Capacitance of Parallel Metal Wires - V

Now come back to the original problem with a voltage source:



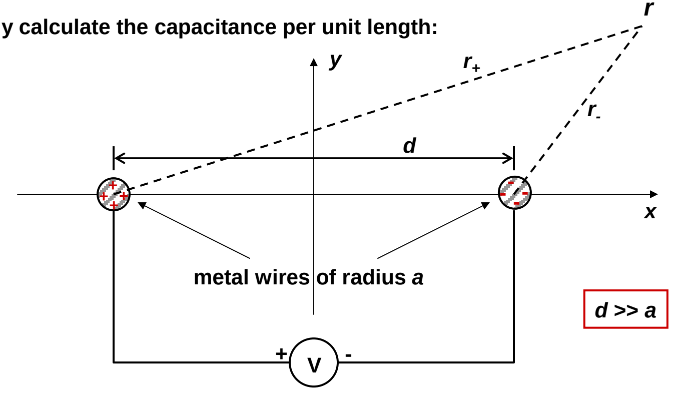
$$\phi(\vec r) = \frac{\lambda}{2\pi\,\varepsilon_o}\ln\left(\frac{r_-}{r_+}\right)$$

$$\Rightarrow \quad \text{Potential at the left cylinder} = \frac{\lambda}{2\pi\,\varepsilon_o}\ln\left(\frac{d}{a}\right) = \frac{V}{2}$$

$$\Rightarrow \quad \text{Potential at the right cylinder} = \frac{\lambda}{2\pi\,\varepsilon_o}\ln\left(\frac{a}{d}\right) = -\frac{V}{2}$$

Capacitance of Parallel Metal Wires - VI

Finally calculate the capacitance per unit length:

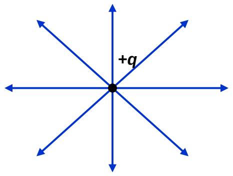


$$\text{Potential difference} = V = \frac{\lambda}{\pi\,\varepsilon_o}\ln\left(\frac{d}{a}\right)$$

$$\text{Capacitance per unit length} = C = \frac{dQ}{dV} = \frac{Q}{V} = \frac{\lambda}{\frac{\lambda}{\pi\,\varepsilon_o}\ln(d/a)} = \frac{\pi\,\varepsilon_o}{\ln\left(\frac{d}{a}\right)}$$

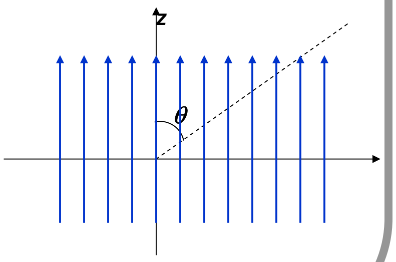
Solutions of Laplace’s Equation in Spherical Coordinates - I

Following are the solutions of $\nabla^2\phi(\vec{r}) = 0$



Spherically Symmetric Point Charge Solution:

$$\phi(\vec{r}) = \frac{A}{r}$$



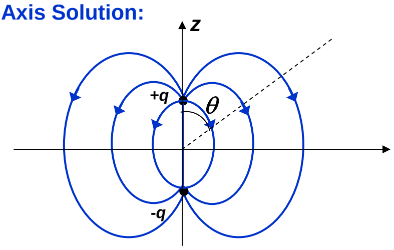
A Constant Uniform Electric Field Solution

$$\phi(\vec{r}) = -E_o\, r \cos(\theta) = -E_o\, z$$

$$\Rightarrow \quad \vec{E}(\vec{r}) = -\nabla\phi(\vec{r}) = E_o\,\hat{z}$$

Solutions of Laplace’s Equation in Spherical Coordinates - II

A Point Charge Dipole Oriented Along z-Axis Solution:



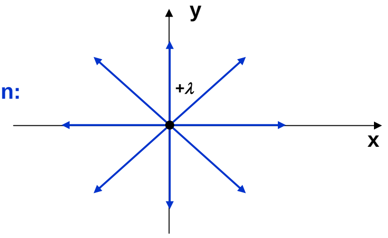
$$\phi(\vec{r}) = A\,\frac{\cos(\theta)}{r^2}$$

A Constant Potential Solution (Trivial Solution):

$$\phi(\vec{r}) = A$$

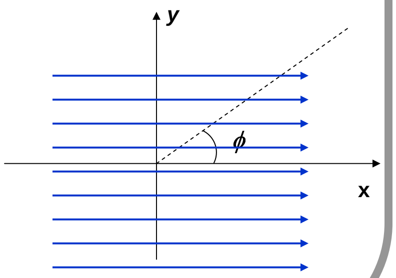
Solutions of Laplace’s Equation in Cylindrical Coordinates - I

Following are the solutions of $\nabla^2\phi(\vec{r})=0$



Cylindrically Symmetric Line Charge Solution:

$$\phi(\vec{r})=A\ \ln(r)$$

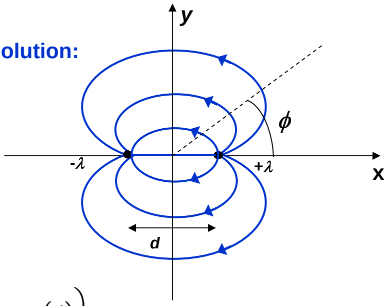


A Constant Uniform Electric Field Solution

$$\phi(\vec{r})=-E_o\ r\ \cos(\phi)=-E_o\ x$$

$$\Rightarrow\ \vec{E}(\vec{r})=-\nabla\phi(\vec{r})=E_o\ \hat{x}$$

Solutions of Laplace’s Equation in Cylindrical Coordinates - II



A Line Dipole Oriented Along x-Axis Solution:

$$\phi(\vec{r})=A\,\frac{\cos(\phi)}{r}$$

$$\phi(\vec{r})=\frac{\lambda}{2\pi\,\varepsilon_o}\ln\left(\frac{r_-}{r_+}\right)\approx\frac{\lambda}{2\pi\,\varepsilon_o}\ln\left(\frac{r+\frac{d}{2}\cos(\phi)}{r-\frac{d}{2}\cos(\phi)}\right)\approx\frac{\lambda\,d}{2\pi\,\varepsilon_o}\,\frac{\cos(\phi)}{r}$$

A Constant Potential Solution (Trivial Solution):

$$\phi(\vec{r})=A$$