## Lecture 5

## Electrical Conduction and Perfect Metals in Electroquasistatics

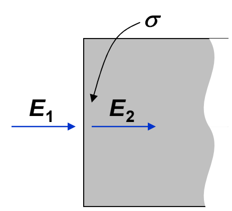
**In this lecture you will learn:**

- **Some More on Electric Field Boundary Conditions**
- **Electrical Conduction in Materials**
- **The Concept of Perfect Metals**
- **Electroquasistatics Problems with Perfect Metals**
- **Method of Images**



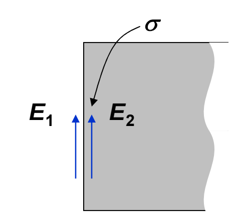
## Electric Field Boundary Conditions

**(1) The discontinuity of the normal component of the E-field at an interface is related to the surface charge density at the interface**




$$\boxed{\varepsilon_o\left(E_2 - E_1\right) = \sigma}$$

**(1) The parallel component of the E-field at an interface is always continuous at the interface**




$$\boxed{\left(E_2 - E_1\right) = 0}$$

****For formal proofs see the Appendix at the end of these lecture notes****

## Electrical Conductivity

### Electrical Conductivity

When E-field is present inside a material, it forces the charges inside the material to move causing an electric current

The current density $\vec{J}$ (units: Amps/m$^2$) is related to the E-field by the relation:

$$\vec{J}(\vec{r}) = \sigma\,\vec{E}(\vec{r})$$

where $\sigma$ is the material conductivity (units: 1/($\Omega$-m) or S/m). Don't confuse the conductivity $\sigma$ with sheet charge density $\sigma$ (both have the same symbol)

| Material | $\sigma$ (S/m) |
|---|---|
| Rubber | $10^{-15}$ |
| Water | $2\text{X}10^{-4}$ |
| Alcohol | $3\text{X}10^{-4}$ |
| Gold | $4\text{X}10^{7}$ |
| Aluminum | $3\text{X}10^{7}$ |
| Copper | $5\text{X}10^{7}$ |
| Silver | $6\text{X}10^{7}$ |



## Perfect Metals - I

**A perfect metal has infinite conductivity (i.e. $\sigma = \infty$)**

Of course, no real metal has infinite conductivity. However, some metals like Silver, Copper, and Gold have high enough conductivity that they may be considered "perfect metals" for simplicity in many calculations

**A perfect metal cannot have any E-field inside it**

The current density and E-field are related by:

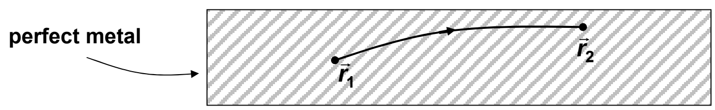
$$\vec{J}(\vec{r}) = \sigma\,\vec{E}(\vec{r})$$

An infinite conductivity implies that for any non-zero E-field one would get an infinite current density – and this is physically impossible. The only way such a catastrophe is avoided is if there is never an E-field inside a perfect metal.

(More on this later …)

## Perfect Metals - II

**Perfect metals are always "equipotential" (i.e. the electric potential inside a perfect metal has the same value everywhere)**




**The potential difference between any two points is given as:**

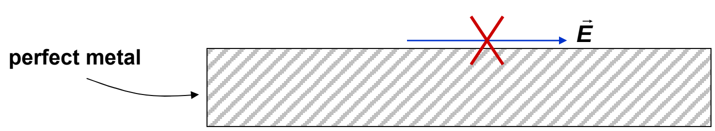
$$\phi(\vec{r}_1) - \phi(\vec{r}_2) = \int_{\vec{r}_1}^{\vec{r}_2} \vec{E} \cdot d\vec{s}$$

**$\Rightarrow$ If the E-field is zero inside a perfect metal then the potential difference between any two points inside a perfect metal must also be zero**



## Perfect Metals - III

**At the surface of a perfect metal the component of E-field parallel to the surface must be zero (in other words, there cannot be a component of E-field at the surface of a perfect metal that is parallel to the surface)**



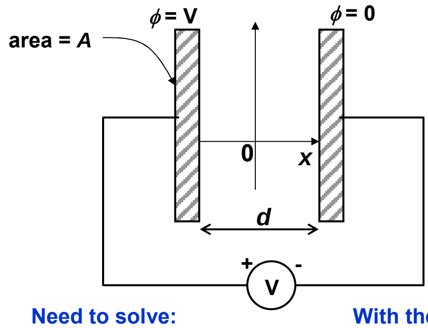

**The argument goes in two steps:**

**If there were a non-zero parallel component of E-field just outside the metal there must be an equal parallel component of E-field just inside the perfect metal (using the boundary condition that the components of E-field parallel to any interface are equal on both sides of the interface)**

**But since there cannot be E-field inside a perfect metal, there must not be a parallel component of E-field just outside a perfect metal**

## Parallel Plates – Potentials and Fields

Consider a problem with two perfect metal parallel plates as shown below




Need to find the potential $\phi(x)$ between the plates

Assume the separation $d$ between the plates is much smaller than the size of the plates so the plates could be assumed infinite in size

**Need to solve:**

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} = 0$$

**With the two boundary conditions:**

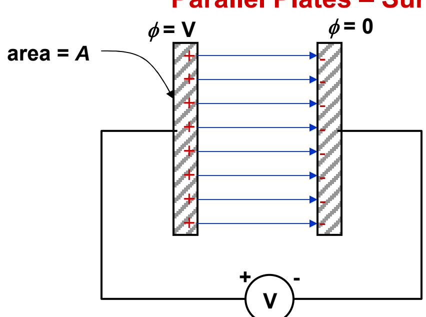
$$\phi\left(x = -\frac{d}{2}\right) = V \qquad \phi\left(x = +\frac{d}{2}\right) = 0$$

**Assume a solution:** $\phi(x) = A\,x + B$ **and use the boundary conditions to get:**

$$\phi(x) = V\left(\frac{1}{2} - \frac{x}{d}\right) \quad \Rightarrow \quad E_x(x) = -\frac{\partial \phi(x)}{\partial x} = \frac{V}{d}$$



## Parallel Plates – Surface Charge Densities




$$\phi(x) = V\left(\frac{1}{2} - \frac{x}{d}\right)$$

$$E_x(x) = -\frac{\partial \phi(x)}{\partial x} = \frac{V}{d}$$

The electric field must originate on positive charges on the surface of the left plate and must terminate on negative charges on the surface of the right plate

**Surface Charge Densities on Metal Plates**

Use the boundary condition: $\varepsilon_o\left(E_2 - E_1\right) = \sigma$

**On the left plate assume charge density $\sigma_L$:**

- The field inside the metal is zero
- The field just outside the metal is V/d

$$\varepsilon_o\left(\frac{V}{d} - 0\right) = \sigma_L \quad \Rightarrow \quad \sigma_L = \varepsilon_o \frac{V}{d}$$

**On the right plate assume charge density $\sigma_R$:**

- The field just outside the metal is V/d
- The field inside the metal is zero

$$\varepsilon_o\left(0 - \frac{V}{d}\right) = \sigma_R \quad \Rightarrow \quad \sigma_R = -\varepsilon_o \frac{V}{d}$$

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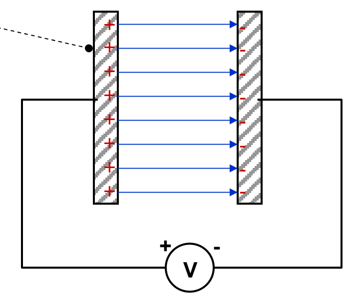
## Potential Differences and Voltage Sources

**A common mistake:**

- The potential on the left plate is V Volts
- Does it mean that the potential difference between infinity and the left plate is V Volts ?




**Answer:**

Be careful in the presence of voltage sources

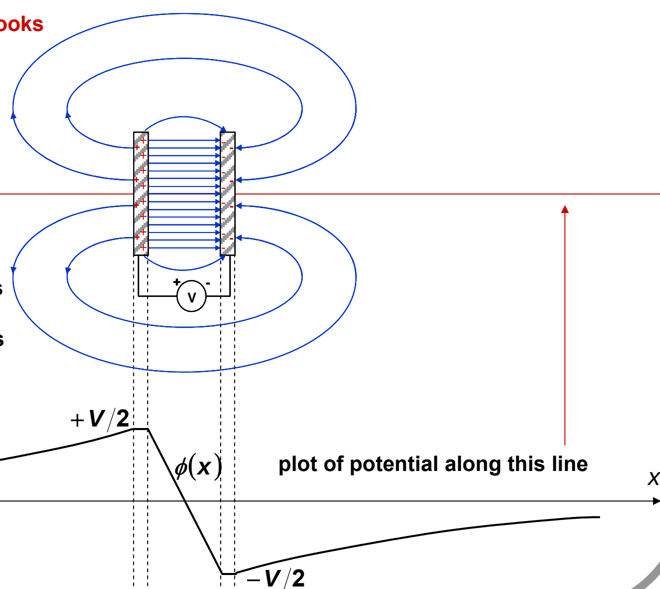
The voltage source only implies that the potential difference between the left and right plates is V Volts

The voltage source DOES NOT imply that the potential difference between the left plate and infinity is V Volts.



## Potential Differences and Voltage Sources

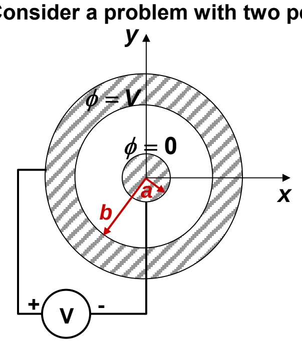
The actual situation looks like as shown here




Our previous analysis was accurate as long as absolute potentials w.r.t. infinity are not sought

## Concentric Cylinders - I

**Consider a problem with two perfect metal concentric cylinders, as shown below:**




- **Need to find the potential $\phi(r)$ between the cylinders (for $a \le r \le b$)**
- **The cylinders are infinite in the z-directions**
- **The outer cylinder is at a potential of $V$ Volts**
- **The inner cylinder is at 0 Volts**

**Need to solve:**

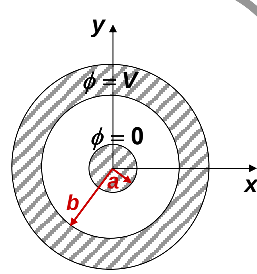
$$\nabla^2 \phi = 0$$
$$(a \le r \le b)$$

**With the two boundary conditions:**

$$\phi(r = b) = V \qquad \phi(r = a) = 0$$



## Concentric Cylinders - II




**Working in cylindrical co-ordinates**

**By symmetry, potential $\phi(r)$ cannot have any angular dependence**

$$\left.\begin{aligned} &\nabla^2 \phi = 0 \\ &\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) = 0 \end{aligned}\right\}$$

**For $a \le r \le b$, assume a solution:**

$$\phi(r) = B \ln(r) + D$$

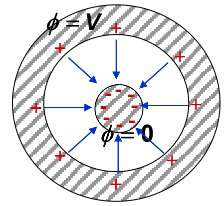
**The two unknowns, $B$ and $D$, are determined by the two boundary conditions:** $\phi(r = b) = V \qquad \phi(r = a) = 0$

**The final answer is:**

$$\phi(r) = V \frac{\ln\left(\frac{r}{a}\right)}{\ln\left(\frac{b}{a}\right)} \qquad \Rightarrow \qquad E_r(r) = -\frac{\partial \phi(r)}{\partial r} = -\frac{V}{r \ \ln\left(\frac{b}{a}\right)}$$

## Concentric Cylinders - III

$$\phi(r) = V\,\frac{\ln\left(\frac{r}{a}\right)}{\ln\left(\frac{b}{a}\right)} \qquad E_r(r) = -\frac{\partial \phi(r)}{\partial r} = -\frac{V}{r\,\ln\left(\frac{b}{a}\right)}$$




The electric field must originate on positive charges on the inner surface of the outer cylinder and must terminate on negative charges on the outer surface of the inner cylinder

Working in cylindrical co-ordinates

### Surface Charge Densities

**On the outer cylinder ($\sigma_{out}$)**

• The field inside the metal is zero

• The field just outside the metal is $\dfrac{V}{b\ln(b/a)}$

$$\left.\begin{aligned} &\varepsilon_o\left(\frac{V}{b\ln(b/a)} - 0\right) = \sigma_{out} \\ &\Rightarrow \quad \sigma_{out} = \varepsilon_o \frac{V}{b\ln(b/a)} \end{aligned}\right.$$

**On the inner cylinder ($\sigma_{in}$)**

• The field just outside the metal is $\dfrac{V}{a\ln(b/a)}$

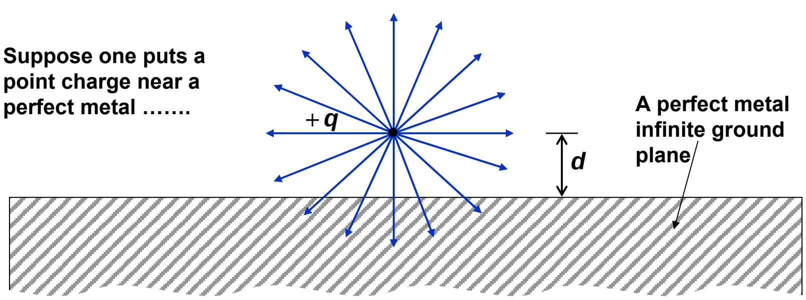
• The field inside the metal is zero

$$\left.\begin{aligned} &\varepsilon_o\left(0 - \frac{V}{a\ln(b/a)}\right) = \sigma_{in} \\ &\Rightarrow \quad \sigma_{in} = -\varepsilon_o \frac{V}{a\ln(b/a)} \end{aligned}\right.$$



## Charges Near Perfect Metals

Suppose one puts a point charge near a perfect metal …….



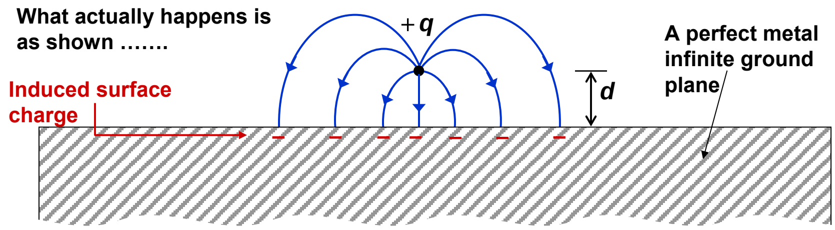

A perfect metal infinite ground plane

Clearly, there is something wrong in the picture above ……

The E-field from the charge is going into the metal – but perfect metals cannot have E-fields inside them!!

## Charges Near Perfect Metals

What actually happens is as shown …….




• Negative charges, under the influence of the E-field from the point charge, rush to the surface

• These negative charges (called the induced charges) terminate the E-field so that there is no E-field inside the metal (i.e. the induced charge screens the external field)

• The resulting field outside the metal has no component parallel to the surface of the metal

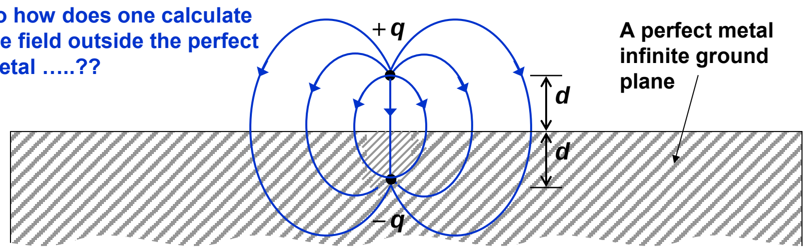
• There is negative surface charge density on the metal surface

• The resulting outside field is due to both the point charge and the induced surface charge density on the metal



## Image Charges in Perfect Metals

So how does one calculate the field outside the perfect metal …..??



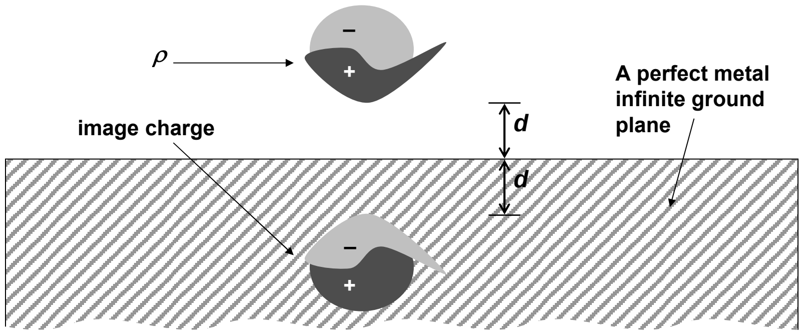

• The field outside the metal can be found by imagining a fictitious point charge of the same magnitude as the outside point charge but of opposite sign sitting inside the perfect metal at an equal distance below the interface

• This fictitious charge is called the image charge

• The field outside the metal can be determined as a superposition of the fields form the actual charge and the image charge

• Now you can see that the field outside resembles that of a charge dipole

## Method of Images

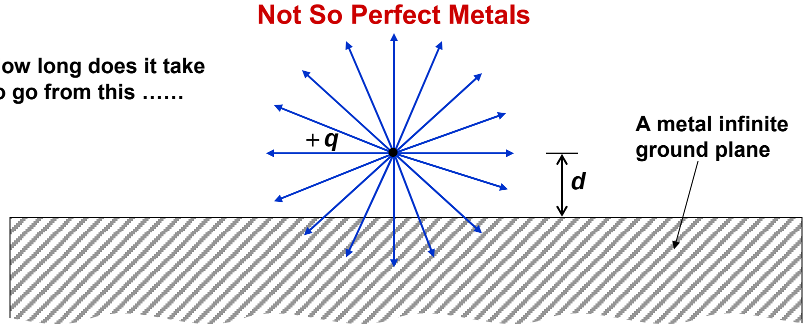



- The same principle works for any arbitrary charge distribution that is placed near a perfect metal
- The image charge is the mirror image of the charge distribution outside the metal but has the opposite polarity
- The field outside the metal can be determined as a superposition of the fields form the actual charge and the image charge

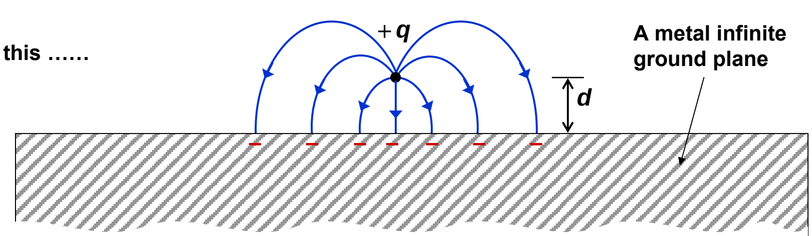


## Not So Perfect Metals

How long does it take to go from this ……




to this ……



## Not So Perfect Metals – Dielectric Relaxation Time

**How long does it take for the induced charges to screen out electric fields from within conducting materials ?**

**Equations of electroquasistatics are:**

$$\left.\begin{aligned} &\nabla . \varepsilon_o \vec{E} = \rho \\ &\nabla \times \vec{E} = 0 \\ &\overset{0}{\cancel{\nabla \times \vec{H}}} = \vec{J} + \frac{\partial \varepsilon_o \vec{E}}{\partial t} \end{aligned}\right\}$$

$$\vec{J} = \sigma \vec{E}$$

**So the last equation becomes:**

$$0 = \sigma \vec{E} + \frac{\partial \varepsilon_o \vec{E}}{\partial t}$$

**Or:** $$\frac{\partial \vec{E}(\vec{r},t)}{\partial t} + \frac{\vec{E}(\vec{r},t)}{\tau_d} = 0 \longrightarrow \tau_d = \text{dielectric relaxation time} = \frac{\varepsilon_o}{\sigma}$$

**Solution:** $$\vec{E}(\vec{r},t) = \vec{E}(\vec{r},0)\ \exp\left(-\frac{t}{\tau_d}\right) \longrightarrow$$

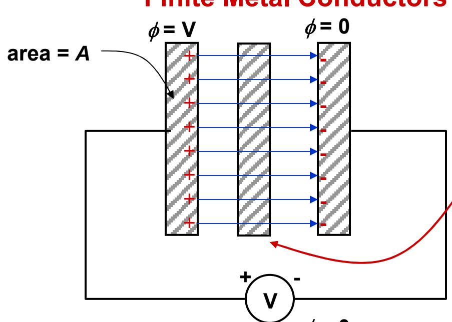
**Any initial E-field in a conductor decays exponentially on a time scale set by $\tau_d$**

**For a perfect metal $\tau_d$ = 0 s**
**For water $\tau_d$ = 0.05 μs**

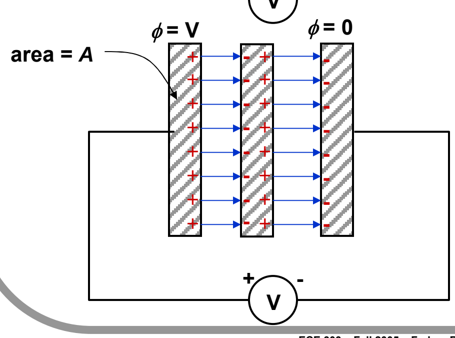
**Conductors behave like perfect metals on time scales longer than their dielectric relaxation times in the sense that they are able to screen out electric fields**



## Finite Metal Conductors and Charge Neutrality




- **Consider the problem with two perfect metal parallel plates and a voltage source**
- **Now insert an extra perfect metal plate in between the two metal plates**

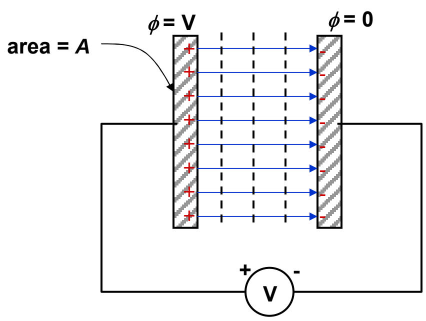



- **Charges will develop on the surface of the inserted metal plate that will screen out the electric field from the inserted plate**
- **The net charge on the inserted plate must be zero since the inserted plate was charge neutral to begin with.**

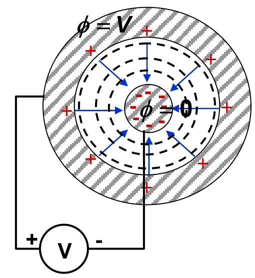
## Equipotential Surfaces

**Surfaces that have the same value of the potential are called equipotential surfaces**

**Example: Three equipotential surfaces In the form of planes shown by the dashed lines for the parallel plate problem**



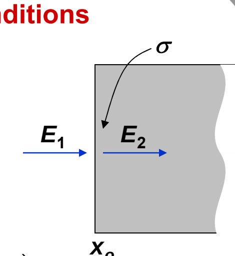

**Example: Three equipotential surfaces in the form of cylindrical shells shown by the dashed lines for the concentric cylinder problem**




**Equipotential surfaces are always normal to the direction of E-field at every point**



## Appendix: Electric Field Boundary Conditions




**Start from Gauss' Law:**

$$\nabla . \varepsilon_o \vec{E} = \rho$$

**For 1D problems:**

$$\frac{\partial \varepsilon_o E_x(x)}{\partial x} = \rho(x)$$

**For a surface charge density at $x = x_o$ :** $\rho(x) = \sigma\,\delta(x - x_o)$

$$\frac{\partial \varepsilon_o E_x(x)}{\partial x} = \sigma\,\delta(x - x_o)$$

**Integrate the above equation from a little behind $x_o$ to a little in front of $x_o$**

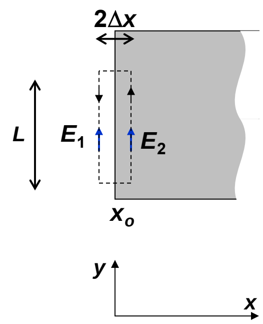
$$\int_{x_o-\Delta x}^{x_o+\Delta x} \frac{\partial \varepsilon_o E_x(x)}{\partial x} = \int_{x_o-\Delta x}^{x_o+\Delta x} \sigma\,\delta(x - x_o) \quad \Rightarrow \quad \varepsilon_o\left[E_x(x_o + \Delta x) - E_x(x_o - \Delta x)\right] = \sigma$$

**Take the limit that $\Delta x$ goes to zero**

$$\textit{Limit } \Delta x \to 0: \quad \varepsilon_o\left[E_x(x_o + \Delta x) - E_x(x_o - \Delta x)\right] = \sigma$$

$$\Rightarrow \quad \varepsilon_o(E_2 - E_1) = \sigma$$

## Appendix: Electric Field Boundary Conditions




Start from Faraday's Law:

$$\oint \vec{E} \, . \, d\vec{s} = -\frac{\partial}{\partial t} \oiint \mu_o \vec{H} \, . \, d\vec{a}$$

Integrate the above equation along the contour shown in the figure

$$\left(E_y(x_o + \Delta x) - E_y(x_o - \Delta x)\right) L = -2\Delta x \; L \; \frac{\partial \mu_o H_z(x_o)}{\partial t}$$

Take the limit that $\Delta x$ goes to zero – the right hand side will go to zero

$$Limit \;\; \Delta x \to 0: \quad E_y(x_o + \Delta x) = E_y(x_o - \Delta x)$$

$$\Rightarrow \quad E_2 = E_1$$