Lecture 4

Electric Potential

In this lecture you will learn:

• Electric Scalar Potential
• Laplace’s and Poisson’s Equation
• Potential of Some Simple Charge Distributions

Conservative or Irrotational Fields

Irrotational or Conservative Fields:

Vector fields \( \vec{F} \) for which \( \nabla \times \vec{F} = 0 \) are called “irrotational” or “conservative” fields

• This implies that the line integral of \( \vec{F} \) around any closed loop is zero

\[ \oint \vec{F} \cdot d\vec{s} = 0 \]

Equations of Electrostatics:

Recall the equations of electrostatics from a previous lecture:

\[ \nabla \cdot \epsilon_0 \vec{E} = \rho \]

\[ \nabla \times \vec{E} = 0 \]

⇒ In electrostatics or electroquasistatics, the E-field is conservative or irrotational

(But this is not true in electrodynamics)
Conservative or Irrotational Fields

More on Irrotational or Conservative Fields:

- If the line integral of $\mathbf{F}$ around any closed loop is zero …..

$$\int_{\text{path} A} \mathbf{F} \cdot d\mathbf{s} = 0$$

…. then the line integral of $\mathbf{F}$ between any two points is independent of any specific Path (i.e. the line integral is the same for all possible paths between the two points)

$$\int_{\text{path} A} \mathbf{F} \cdot d\mathbf{s} = 0$$

$$\Rightarrow \left( \int_{r_2}^{r_1} \mathbf{F} \cdot d\mathbf{s} \right)_{\text{path } A} + \left( \int_{r_1}^{r_2} \mathbf{F} \cdot d\mathbf{s} \right)_{\text{path } B} = 0$$

$$\Rightarrow \left( \int_{r_2}^{r_1} \mathbf{F} \cdot d\mathbf{s} \right)_{\text{path } A} - \left( \int_{r_1}^{r_2} \mathbf{F} \cdot d\mathbf{s} \right)_{\text{path } B} = 0$$

$$\Rightarrow \left( \int_{r_2}^{r_1} \mathbf{F} \cdot d\mathbf{s} \right)_{\text{path } A} = \left( \int_{r_1}^{r_2} \mathbf{F} \cdot d\mathbf{s} \right)_{\text{path } B}$$

The Electric Scalar Potential - I

The scalar potential:

Any conservative field can always be written (up to a constant) as the gradient of some scalar quantity. This holds because the curl of a gradient is always zero.

If $\mathbf{F} = \nabla \phi$

Then $\nabla \times (\mathbf{F}) = \nabla \times (\nabla \phi) = 0$

For the conservative E-field one writes: $\mathbf{E} = -\nabla \phi$

(The --ve sign is just a convention)

Where $\phi$ is the scalar electric potential

The scalar potential is defined only up to a constant

If the scalar potential $\phi(\mathbf{r})$ gives a certain electric field then the scalar potential $\phi(\mathbf{r}) + C$ will also give the same electric field (where $C$ is a constant)

The absolute value of potential in a problem is generally fixed by some physical reasoning that essentially fixes the value of the constant $C$
The Electric Scalar Potential - II

We know that:
\[ \vec{E} = -\nabla \phi \]

This immediately suggests that:

• The line integral of E-field between any two points is the difference of the potentials at those points

\[ \int_{r_1}^{r_2} \vec{E} \cdot d\vec{s} = \int_{r_1}^{r_2} (-\nabla \phi) \cdot d\vec{s} = \phi(r_1) - \phi(r_2) \]

• The line integral of E-field around a closed loop is zero

\[ \oint \vec{E} \cdot d\vec{s} = \oint (-\nabla \phi) \cdot d\vec{s} = 0 \]

The Electric Scalar Potential of a Point Charge

Assumption: The scalar potential is assumed to have a value equal to zero at infinity far away from any charges

Point Charge Potential

Do a line integral from infinity to the point \( r \) where the potential needs to be determined

\[ \int_{r}^{0} \vec{E} \cdot d\vec{s} = \int_{r}^{0} (-\nabla \phi) \cdot d\vec{s} = \phi(r) - \phi(\infty) = \phi(r) \]

\[ \phi(r) = \int_{r}^{0} \vec{E} \cdot d\vec{s} = \int_{r}^{0} \frac{q}{4\pi \epsilon_0 r^2} dr \]

\[ \phi(r) = \frac{q}{4\pi \epsilon_0 r} \]
Electric Scalar Potential and Electric Potential Energy

The electric scalar potential is the potential energy of a unit positive charge in an electric field.

- Electric force on a charge of $q$ Coulombs = $q \hat{E}$ (Lorentz Law)

Potential energy of a charge $q$ at any point in an electric field = Work done by the field in moving the charge $q$ from that point to infinity

\[
\text{Work done} = \int_{r}^{\infty} \mathbf{F} \cdot d\mathbf{s} = \int_{r}^{\infty} q\mathbf{E} \cdot d\mathbf{s} = q[\phi(\mathbf{r}) - \phi(\infty)] = q\phi(\mathbf{r})
\]

Work done on unit charge $= \frac{q\phi(r)}{q} = \phi(\mathbf{r})$

$\Rightarrow$ P.E. of unit charge $= \phi(r)$

$\Rightarrow$ Potential energy of a charge of $q$ Coulombs in electric field $= q\phi(\mathbf{r})$

Poisson’s and Laplace’s Equation

- It is not always easy to directly use Gauss’ Law and solve for the electric fields
- Need an equation for the electric potential

Start from: $\nabla \cdot \varepsilon_0 \mathbf{E} = \rho$

Use: $\mathbf{E} = -\nabla \phi$

To get: $\nabla \cdot (\varepsilon_0 (\nabla \phi)) = \rho$

$\Rightarrow \nabla^2 \phi = -\frac{\rho}{\varepsilon_0}$  

Poisson’s Equation

If the volume charge density is zero then Poisson’s equation becomes:

$\nabla^2 \phi = 0$  

Laplace’s Equation

Poisson’s or Laplace’s equation can be solved to give the electric scalar potential for charge distributions
Potential of a Uniformly Charged Spherical Shell - I

Use the spherical coordinate system \( \sigma \) Coulombs/m²

For \( a \leq r \leq \infty \):
\[
\nabla^2 \phi = 0 \\
\Rightarrow \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) = 0
\]
Assume a solution:
\[
\phi(r) = \frac{A}{r} + F
\]
\( F \) must be 0 so that the potential is 0 at \( r = \infty \)

For \( 0 \leq r \leq a \):
\[
\nabla^2 \phi = 0 \\
\Rightarrow \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) = 0
\]
Assume solution:
\[
\phi(r) = \frac{B}{r} + D
\]
Potential must not become infinite at \( r = 0 \) so \( B \) must be 0

Potential of a Uniformly Charged Spherical Shell - II

For \( 0 \leq r \leq a \)
\[
\phi(r) = D \\
E_r(r) = -\frac{\partial \phi}{\partial r} = 0
\]

For \( a \leq r \leq \infty \)
\[
\phi(r) = \frac{A}{r} \\
E_r(r) = -\frac{\partial \phi}{\partial r} = \frac{A}{r^2}
\]

Boundary conditions

We need two additional boundary conditions to determine the two unknown coefficients \( A \) and \( D \)

1. At \( r = a \) the potential is continuous (i.e. it is the same just inside and just outside the charged sphere)
\[
D = \frac{A}{a}
\]
2. At \( r = a \) the electric field is NOT continuous. The jump in the component of the field normal to the shell (i.e. the radial component) is related to the surface charge density
\[
\varepsilon_0 \left( E_r \right)_{\text{out}} - E_r \left( \right)_{\text{in}} = \sigma
\]
\[
\Rightarrow \varepsilon_0 \left( \frac{A}{a^2} - 0 \right) = \sigma
\]
Surface Charge Density Boundary Condition

Suppose we know the surface normal electric field on just one side of a charge plane with a surface charge density \( \sigma \).

**Question:** What is the surface normal field on the other side of the charge plane?

**Solution:**

- Draw a Gaussian surface in the form of a cylinder of area \( A \) piercing the charge plane.
- Total flux coming out of the surface = \( \varepsilon_0 (E_2 - E_1)A \).
- Total charge enclosed by the surface = \( \sigma A \).
- By Gauss’ Law: \( \varepsilon_0 (E_2 - E_1)A = \sigma A \).
  \[ \Rightarrow \varepsilon_0 (E_2 - E_1) = \sigma \]

\( \varepsilon_0 (E_2 - E_1) = \sigma \)

This an extremely important result that relates surface normal electric fields on the two sides of a charge plane with surface charge density \( \sigma \).

Potential of a Uniformly Charged Spherical Shell - III

For \( 0 \leq r \leq a \)

\[ \phi(r) = \frac{(4\pi \sigma a^2)}{4\pi \varepsilon_0 a} \]

For \( a \leq r \leq \infty \)

\[ \phi(r) = \frac{(4\pi \sigma a^2)}{4\pi \varepsilon_0 r} \]

Sketch of the Potential:
Potential of a Uniformly Charged Sphere a la Poisson and Laplace

In spherical co-ordinates potential can only be a function of $r$ (not of $\theta$ or $\phi$).

For $a \leq r \leq \infty$:

$$\nabla^2 \phi = 0$$

$$\Rightarrow \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) = 0$$

Assume a solution:

$$\phi(r) = \frac{A}{r} + F$$

$F$ must be 0 so that the potential is 0 at $r = \infty$.

For $0 \leq r \leq a$:

$$\nabla^2 \phi = -\frac{\rho}{\varepsilon_0}$$

$$\Rightarrow \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) = -\frac{\rho}{\varepsilon_0}$$

Assume solution:

$$\phi(r) = \frac{B}{r} + D + C r^2$$

By substituting the solution in the Poisson equation find $C$:

$$C = -\frac{\rho}{6 \varepsilon_0}$$

* Potential must not become infinite at $r = 0$ so $B$ must be 0.

Potential of a Uniformly Charged Sphere a la Poisson and Laplace

For $0 \leq r \leq a$:

$$\phi(r) = D - \frac{\rho}{6 \varepsilon_0} r^2$$

For $a \leq r \leq \infty$:

$$\phi(r) = \frac{A}{r}$$

Boundary conditions

We need two additional boundary conditions to determine the two unknown coefficients $A$ and $D$.

1. At $r = a$ the potential is continuous (i.e. it is the same just inside and just outside the charged sphere).
2. At $r = a$ the radial electric field is continuous (i.e. it is the same just inside and just outside the charged sphere).

$E_r = -\frac{\partial \phi}{\partial r}$

(1) gives:

$$D - \frac{\rho}{6 \varepsilon_0} a^2 = \frac{A}{a}$$

$$\Rightarrow A = \frac{\rho}{3 \varepsilon_0} a^3$$

(2) gives:

$$\frac{\rho}{3 \varepsilon_0} a = \frac{A}{a^2}$$

$$\Rightarrow D = \frac{\rho}{2 \varepsilon_0} a^2$$
Potential of a Uniformly Charged Sphere a la Poisson and Laplace

For $0 \leq r \leq a$

$$\phi(r) = \frac{\rho}{2\varepsilon_0} \left( a^2 - \frac{r^2}{3} \right)$$

For $a \leq r \leq \infty$

$$\phi(r) = \frac{\rho}{4\pi\varepsilon_0} \frac{4\pi a^3}{r}$$

Sketch of the Potential:

---

The Principle of Superposition for the Electric Potential

Poisson equation is LINEAR and allows for the superposition principle to hold

- Suppose for some charge density $\rho_1$ one has found the potential $\phi_1$
- Suppose for some other charge density $\rho_2$ one has found the potential $\phi_2$

The superposition principle says that the sum $(\phi_1 + \phi_2)$ is the solution for the charge density $(\rho_1 + \rho_2)$

A Simple Proof

$$\nabla^2 \phi_1 = -\frac{\rho_1}{\varepsilon_0} + \nabla^2 \phi_2 = -\frac{\rho_2}{\varepsilon_0} = \nabla^2 (\phi_1 + \phi_2) = -\frac{(\rho_1 + \rho_2)}{\varepsilon_0}$$
**Potential of a Charge Dipole**

Consider Two Equal and Opposite Charges

We are interested in the potential at a distance $r$ from the center of the pair in the plane of the charges, where $r \gg d$.

Work in spherical co-ordinates

Potential contributions from the two charges can be added algebraically

$$
\phi(\vec{r}) = q \left( \frac{1}{4\pi \varepsilon_0 \left( \frac{d}{2} \cos(\theta) \right)} - \frac{1}{4\pi \varepsilon_0 \left( r - \frac{d}{2} \cos(\theta) \right)} \right)
$$

$$
\approx \frac{qd}{4\pi \varepsilon_0 r^2} \cos(\theta)
$$

**Field of a Charge Dipole**

$$
\phi(\vec{r}) = \frac{qd}{4\pi \varepsilon_0 r^2} \cos(\theta)
$$

$$
\vec{E} = -\nabla \phi(\vec{r}) = -\frac{qd}{4\pi \varepsilon_0 r^3} \left( 2 \cos(\theta) \hat{r} + \sin(\theta) \hat{\theta} \right)
$$

Same result for the E-field was obtained in the previous lecture by superposing the individual E-fields (rather than the potentials) of the two charges.
Potential of a Line Charge

Consider an infinite line charge coming out of the plane of slide.

- The electric field, by symmetry, has only a radial component.
- Draw a Gaussian surface in the form of a cylinder of radius \( r \) and Length \( L \) perpendicular to the slide.

Using Gauss' Law:

\[
\varepsilon_0 E_r (2\pi r L) = \lambda L
\]

\[
\Rightarrow E_r = \frac{\lambda}{2\pi \varepsilon_0 r}
\]

But \( E_r = -\nabla \phi = -\frac{\partial \phi}{\partial r} \)

\[
\Rightarrow \frac{\partial \phi(r)}{\partial r} = -\frac{\lambda}{2\pi \varepsilon_0 r}
\]

Upon integrating from \( r_0 \) to \( r \) we get:

\[
\phi(r) - \phi(r_0) = \frac{\lambda}{2\pi \varepsilon_0} \ln\left(\frac{r}{r_0}\right)
\]

Where \( r_0 \) is a constant of integration and is some point where the potential is known.

The problem is that this solution becomes infinite at \( r = \infty \).

Potential of a Line Dipole

Consider two infinite equal and opposite line charges coming out of the plane of slide.

Using superposition, the potential can be written as:

\[
\phi(\vec{r}) = \frac{\lambda}{2\pi \varepsilon_0} \ln\left(\frac{r_0}{r_+}\right) - \frac{\lambda}{2\pi \varepsilon_0} \ln\left(\frac{r_0}{r_-}\right)
\]

\[
= \frac{\lambda}{2\pi \varepsilon_0} \ln\left(\frac{r_+}{r_-}\right)
\]

The final answer does not depend on the parameter \( r_0 \).

Question: where is the zero of potential?

Points for which \( r_+ \) equals \( r_- \) have zero potential. These points constitute the entire y-z plane.
The 3D Superposition Integral for the Potential

In the most general scenario, one has to solve the Poisson equation:

\[ \nabla^2 \phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0} \]

We know that the solution for a point charge sitting at the origin:

\[ \phi(\mathbf{r}) = \frac{q}{4\pi \varepsilon_0 r} \]

To find the potential at any point one can sum up the contributions from different portions of a charge distribution treating each as a point charge:

\[ \phi(\mathbf{r}) = \iiint \frac{\rho(\mathbf{r}')}{4\pi \varepsilon_0 |\mathbf{r} - \mathbf{r}'|} dV' \]

Check: For a point charge at the origin \( \rho(\mathbf{r}') = q \delta^3(\mathbf{r}') = q \delta(x') \delta(y') \delta(z') \)

\[ \phi(\mathbf{r}) = \iiint \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV' = \iiint \frac{q \delta^3(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV' = \frac{q}{4\pi \varepsilon_0 |\mathbf{r}|} = \frac{q}{4\pi \varepsilon_0 r} \]