In this lecture you will learn:

- Diffraction of electromagnetic radiation
- Gain and radiation pattern of aperture antennas

“Aperture antenna” usually refers to a (metallic) sheet with a hole (or an aperture) of some shape through which radiation comes out.

The natural spreading of electromagnetic waves in free space when emanating from a source is called “diffraction”.

Questions

- What happens on this side?
- How does the radiation coming out of the aperture looks like when the dimensions of the hole are of the order of the wavelength?
- What is the radiation pattern?
Aperture Antennas in Practice: Rectangular Waveguides

How does radiation coming out of a rectangular waveguide look like?

Metal rectangular waveguide

$\sigma = \infty

\text{Radiation coming out of a rectangular aperture}$

$\infty = \sigma$

Some fraction of the incident power is reflected from the open end and some is radiated out

Aperture Antennas in Practice: Dielectric Waveguides

Optical fiber

$\text{Radiation coming out of a circular aperture}$

Integrated Photonics (dielectric waveguides on a chip)

$\text{Radiation coming out of an integrated dielectric waveguide (e.g. a semiconductor laser)}$
Assumption and Goal

**Assumption:** Assume that we know the field for all time right at the aperture

\[ \mathbf{E}(x, y = 0, z, t) \]

This we could know for example from our knowledge of the incident (and reflected) fields behind the aperture.

**Goal:** To find the field for \( y > 0 \)

\[ \mathbf{E}(x, y, z, t) = ? \]

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H-field and Surface Current Density Boundary Condition

First recall the surface current boundary condition for the H-field (now in vector form):

\[ \mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K} \]

For a left-right symmetric problem:

\[ \mathbf{H}_1 = -\mathbf{H}_2 \]

\[ \mathbf{n} \times 2\mathbf{H}_2 = \mathbf{K} \]
Principle of Equivalence (Huygens Principle)

Principle of equivalence says that if one knows the radiation E- and H-fields at every point on an imaginary closed surface, then the radiation outside the closed surface can be described as the radiation generated from a surface current density that flows on the closed surface.

\[
\begin{align*}
|\mathbf{E}_s(\mathbf{r})| &= \eta_0 |\mathbf{H}_s(\mathbf{r})| \cos\theta
\end{align*}
\]

Principle of Equivalence is a mathematical statement of the old Huygens Principle that said that every point on a wave-front can be considered a source of radiation.

Aperture Antenna and the Equivalent Problem

Assumption: Knowing the E-field and H-field phasors at the aperture allows us to consider the equivalent problem of radiation by a current sheet density.

\[
\begin{align*}
\mathbf{E}(x, y = 0, z) &= -\hat{z} \mathbf{E}_s(x, z) \\
\mathbf{H}(x, y = 0, z) &= -\hat{x} \frac{\mathbf{E}_s(x, z)}{\eta_0}
\end{align*}
\]

\[
\begin{align*}
\mathbf{K}(x, z) &= \hat{y} \times 2\mathbf{H}_s(x, z) = \hat{y} \frac{2\mathbf{E}_s(x, z)}{\eta_0} \\
&= \hat{y} \frac{2\mathbf{E}_s(x, z)}{\eta_0} \\
\Rightarrow \mathbf{j}(x, y, z) &= \hat{z} \frac{2\mathbf{E}_s(x, z)}{\eta_0} \delta(y)
\end{align*}
\]
**Aperture Antennas: Analysis**

Knowing the current density, use the superposition integral for the vector potential to calculate the fields:

\[ \mathbf{\mathcal{A}}(\mathbf{r}) = \frac{j \eta_0}{4\pi} \int \mathbf{J}(\mathbf{r}') e^{-j k |\mathbf{r} - \mathbf{r}'|} d\mathbf{v}' \]

Make the far-field (or the Fraunhoffer) approximation:

\[ |\mathbf{r} - \mathbf{r}'| \approx |\mathbf{r} - \hat{\mathbf{r}} \cdot \mathbf{r}'| \]

\[ \Rightarrow \mathbf{\mathcal{A}}_f(\hat{\mathbf{r}}) = \frac{j \eta_0}{4\pi} \int \mathbf{J}(\mathbf{r}') e^{j k \hat{\mathbf{r}} \cdot \mathbf{r}'} d\mathbf{v}' \]

Compute the E-field in the far-field approximation:

\[ \mathbf{E}_f(\hat{\mathbf{r}}) = \frac{\epsilon_0 c^2}{j \omega} \nabla \times \nabla \times \mathbf{\mathcal{A}}_f(\hat{\mathbf{r}}) \approx \left( \text{in far field} \right) j \omega \int \hat{\mathbf{r}} \times \mathbf{\mathcal{A}}_f(\hat{\mathbf{r}}) d\mathbf{v}' 
= j \frac{\eta_0 k}{4\pi} e^{-j k r} \int \hat{\mathbf{r}} \times \mathbf{J}(\mathbf{r}') e^{j k \hat{\mathbf{r}} \cdot \mathbf{r}'} d\mathbf{v}' \]

Note that in the far-field: \( \nabla \times \mathbf{\mathcal{A}}_f(\hat{\mathbf{r}}) = -j k \times \mathbf{\mathcal{A}}_f(\hat{\mathbf{r}}) = -j k \hat{\mathbf{r}} \times \mathbf{\mathcal{A}}_f(\hat{\mathbf{r}}) \)

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**Rectangular Apertures: General Case**

\[ \mathbf{J}(x, y, z) = \begin{cases} \frac{2 r E_a(x, z)}{\eta_0} \delta(y) & \text{for } -\frac{L_x}{2} \leq x \leq \frac{L_x}{2} \text{ and } -\frac{L_z}{2} \leq z \leq \frac{L_z}{2} \\ 0 & \text{otherwise} \end{cases} \]

Use the formulas:

\[ \mathbf{E}_f(\hat{\mathbf{r}}) = j \frac{\eta_0 k}{4\pi} e^{-j k r} \int \hat{\mathbf{r}} \times \mathbf{J}(\mathbf{r}') e^{j k \hat{\mathbf{r}} \cdot \mathbf{r}'} d\mathbf{v}' \]

\[ \hat{\mathbf{r}} \times \mathbf{\mathcal{A}}_f(\hat{\mathbf{r}}) = -j k \hat{\mathbf{r}} \times \mathbf{\mathcal{A}}_f(\hat{\mathbf{r}}) = -j k \hat{\mathbf{r}} \times \mathbf{\mathcal{A}}_f(\hat{\mathbf{r}}) \]

To get:

\[ \mathbf{E}_f(\hat{\mathbf{r}}) = \frac{j k}{2\pi} \sin(\theta) e^{-j k r} \int_{-L_z/2}^{L_z/2} \int_{-L_x/2}^{L_x/2} E_a(x', z') e^{j k \hat{\mathbf{r}} \cdot \mathbf{r}'} dx' dz' \]

\[ k = k_x \hat{x} + k_y \hat{y} + k_z \hat{z} \]

\[ \hat{r}' = x' \hat{x} + z' \hat{z} \]

Or:

\[ \mathbf{E}_f(\hat{\mathbf{r}}) = \frac{j k}{2\pi} \sin(\theta) e^{-j k r} \int_{-L_z/2}^{L_z/2} \int_{-L_x/2}^{L_x/2} E_a(x', z') e^{j k x' \hat{x} - j k z' \hat{z}} dx' dz' \]

Far-field is proportional to the 2D Fourier transform of the field at the aperture
Rectangular Apertures with Uniform Field at the Aperture

\[ J(x, y, z) = \begin{cases} 2E_y/\eta_0 \delta(y) & \text{for } -L_z/2 \leq z \leq L_z/2 & \text{otherwise} \\ 0 & \end{cases} \]

\[ \mathcal{E}_n(r) = \hat{\theta} \frac{j}{2\pi r} E_x \sin(\theta)e^{-jkr} \int_{-L_z/2}^{L_z/2} \int_{-L_y/2}^{L_y/2} e^{jkx'x} e^{jkz'z} \, dx' \, dz' \]

Far-field is proportional to the 2D Fourier transform of the shape of the aperture

Or:

\[ \mathcal{E}_n(r) = \hat{\theta} \frac{j}{2\pi r} E_x \sin(\theta)e^{-jkr} \int_{-L_z/2}^{L_z/2} \int_{-L_y/2}^{L_y/2} e^{jkx'x} \, dx' \, dz' \]

Fourier Transforms and the Rectangular Aperture Far-Field

FT: \( F(k_x) = \int_{-\infty}^{\infty} f(x) e^{jkx} \, dx \)  
IFT: \( f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k_x) e^{-jkx} \, dk_x \)

Consider the 1D box function

\[ f(x) = \begin{cases} 1 & \text{for } \frac{-L_x}{2} \leq x \leq \frac{L_x}{2} \\ 0 & \text{otherwise} \end{cases} \]

FT of the 1D box function

\[ F(k_x) = L_x \frac{\sin(k_Lx/2)}{k_Lx/2} \]

Width of main lobe in k-space = \( \frac{4\pi}{L_x} \)

The far-field E-field is proportional to the 2D FT of the aperture shape

\[ \mathcal{E}_n(r) = \hat{\theta} \frac{j}{2\pi r} E_x \sin(\theta)e^{-jkr} \int_{-L_z/2}^{L_z/2} \int_{-L_y/2}^{L_y/2} e^{jkx'x} \, dx' \, dz' \]

\[ = \hat{\theta} \frac{j}{2\pi r} E_x \sin(\theta)e^{-jkr} \frac{\sin(k_Lx/2)}{k_Lx/2} \frac{\sin(k_Lz/2)}{k_Lz/2} \]
Rectangular Aperture: Far-Field

\[ E_{\|}(r) = \delta \frac{j k}{2\pi r} E_z \sin(\theta)e^{-jkr}(L_x L_z) \frac{\sin(k_x L_x/2)}{k_x L_x/2} \frac{\sin(k_z L_z/2)}{k_z L_z/2} \]

\[
\begin{align*}
  k_x &= k \sin(\theta) \cos(\phi) \\
  k_z &= k \cos(\theta)
\end{align*}
\]

Angular half-width is determined by when the term inside the sine function becomes \( \pm \pi \):

\[ \theta = \frac{\pi}{2} - \theta' \]

Angular width of main lobe in vertical direction is governed by the function:

\[ \sin(k_z L_z/2) = \frac{\sin(k L_z \sin(\theta')/2)}{k L_z \sin(\theta')/2} \]

The angular half-width is determined by when the term inside the sine function becomes \( \pm \pi \):

\[ \sin(\theta'_{null}) = \pm \frac{2\pi}{k L_z} = \pm \frac{\lambda}{L_z} \]
Rectangular Aperture: Angular Widths of the Main Lobe

Angular width of main lobe in horizontal direction is governed by the function:

\[ \frac{\sin(k_x L_z / 2)}{k_x L_z / 2} = \frac{\sin(k L_x \sin(\phi') / 2)}{k L_x \sin(\phi') / 2} \]

The angular half-width is determined by when the term inside the sine function becomes \( \pm \pi \)

\[ \sin(\phi_{null}) = \pm \frac{2\pi}{k L_x} = \pm \frac{\lambda}{L_x} \]

Rectangular Aperture: Radiation Pattern

\[ p(\theta = \pi/2, \phi) \]

\[ p(\theta, \phi = \pi/2) \]

\[ k L_x = 4\pi \]

\[ k L_z = 4\pi \]
Nulls in the Diffraction Pattern: Interference in Diffraction

For the first null in the $\delta$ direction one must have waves coming from one half of the slit interfere destructively with the waves coming from the other half of the slit:

$$k \left( \frac{L_x}{2} \right) \sin(\delta) = \pi$$

$$\Rightarrow \quad \sin(\delta) = \frac{2\pi}{kL_x} = \frac{\lambda}{L_x}$$

Can you guess what kind of interference is responsible for the second null? The third null? .......

$$\sin(\delta_{null}) = \frac{2\pi}{kL_x} = \frac{\lambda}{L_x}$$
Rectangular Aperture: Far-Field Intensity

Intensity on a plane perpendicular to the y-axis is plotted.

\[
S_H(\mathbf{r}, t) = \frac{1}{2\pi r} \left[ \frac{k E_a}{2\pi r} \right]^2 \sin^2(\theta) \left[ \frac{L_x L_y}{k_x L_x / 2} \frac{\sin(k_x L_x / 2)}{k_x L_x / 2} \right]^2
\]

Rectangular Aperture: Far-Field Intensity

Intensity on a plane perpendicular to the y-axis is plotted.

The lobes are wider in the direction in which the aperture dimension is smaller.

\[
\sin(\theta'_\text{null}) = \pm \frac{2\pi}{k L_x} = \pm \frac{\lambda}{L_x}
\]

\[
\sin(\phi'_\text{null}) = \pm \frac{2\pi}{k L_z} = \pm \frac{\lambda}{L_z}
\]
Rectangular Aperture: Total Radiated Power

\[
\langle S_h(\hat{r}, t) \rangle = \hat{r} \cdot \frac{1}{2\eta_0} \frac{kE_a^2}{2\pi r^2} \sin^2(\theta) \left[ \left( L_x L_z \right) \frac{\sin(k_x L_x/2)}{k_x L_x/2} \frac{\sin(k_z L_z/2)}{k_z L_z/2} \right]^2
\]

\[k_x = k \sin(\theta) \cos(\phi)\]

\[k_z = k \cos(\theta)\]

Total power radiated:

Calculate right at the aperture

\[
P_{\text{rad}} = \int_0^{\pi} \left( S_h(\hat{r}, t) \right) \cdot \hat{r}^2 \sin(\theta) d\theta d\phi
\]

\[= \frac{1}{2\eta_0} |E_a|^2 L_x L_z\]

Aperture Antennas: Gain and Effective Area

Gain: \(G(\theta, \phi) = \eta_{\text{rad}} \frac{\langle S_h(\hat{r}, t) \rangle \cdot \hat{r}}{P_{\text{rad}}/4\pi r^2}\)

\[= \frac{4\pi}{\lambda^2} \eta_{\text{rad}} L_x L_z \sin^2(\theta) \left[ \sin(k_x L_x/2) \sin(k_z L_z/2) \right]^2\]

\[= \frac{4\pi}{\lambda^2} A(\theta, \phi)\]

Effective Area: \(A(\theta, \phi)\)

\[= \eta_{\text{rad}} L_x L_z \sin^2(\theta) \left[ \sin(k_x L_x/2) \sin(k_z L_z/2) \right]^2\]

Maximum Effective Area: \(A(\theta, \phi)_{\text{max}} = A\left(\frac{\pi}{2}, \frac{\pi}{2}\right)\)

\[= \eta_{\text{rad}} L_x L_z = \eta_{\text{rad}} \text{ {aperture area}}\]

Maximum possible effective area of any aperture antenna (of any shape) is equal to its actual physical area