In this lecture you will learn:

• Scattering of electromagnetic waves from objects
• Rayleigh Scattering
• Why the sky is blue
• Radar range equation

Scattering of Electromagnetic Waves from a Plane Interface

Incident, transmitted, and reflected waves are all plane waves

The reflected and transmitted waves can also be called the scattered waves
Scattering of Electromagnetic Waves from Objects

*Incident plane wave:* $E_i(\vec{r})$

*Scattered wave:* $E_s(\vec{r})$

Questions:
- How does one find the scattered field?
- How much power from the incident field goes into the scattered field?
- In which direction(s) does the scattered power go?

Scattering of Electromagnetic Waves From Spherical Particles

**Assumption:**
- Assume that the particle radius is much smaller than the wavelength of the incident wave, i.e.: $ka << 1$
  - When the above condition holds the scattering is called “Rayleigh Scattering”
  - When the above condition does not hold the scattering is called “Mie Scattering”
- When $ka << 1$, the particle sees a uniform E-field that is slowly oscillating in time

*Incident plane wave:* $E_i(\vec{r})$
Rayleigh Scattering: Basic Mechanism

Incident plane wave: $\hat{E}_i(\hat{r})$

One way to understand scattering is as follows:

i) The incident E-field induces a time-varying dipole moment in the sphere (recall the electrostatics problem of a dielectric sphere in a uniform E-field from homework 3)

ii) The time-varying dipole radiates like a Hertzian dipole, and this radiation is the scattered radiation

Rayleigh Scattering: Induced Dipole

Incident plane wave: $\hat{E}_i(\hat{r})$

Suppose the z-directed E-field phasor for the incident plane wave at the location of the particle is: $\hat{E}(\hat{r} = 0) = \hat{z} E_i$

From homework (3), the z-directed dipole moment $p$ induced in a sphere in the presence of E-field $\hat{E}$ is:

$$ p = 4\pi \varepsilon_0 a^3 \left( \frac{\varepsilon_1 - \varepsilon_0}{\varepsilon_1 + 2\varepsilon_0} \right) \hat{z} E_k $$

In the present case, the dipole moment phasor $p$ induced in the sphere is therefore:

$$ p = 4\pi \varepsilon_0 a^3 \left( \frac{\varepsilon_1 - \varepsilon_0}{\varepsilon_1 + 2\varepsilon_0} \right) \hat{z} E_k $$
Rayleigh Scattering: Scattered Radiation

Hertzian Dipole

Induced Dipole

Dipole moment = \( c = j \omega q \)

\[ E_H (r) = \hat{\theta} j \eta_0 k j \omega q \sin(\theta) e^{-jkr} \]

\[ P_s = \frac{4\pi}{3\eta_0} k^4 a^6 \left( \frac{\epsilon_1 - \epsilon_0}{\epsilon_1 + 2\epsilon_0} \right)^2 |E_i|^2 \]

The far-field scattered radiation is:

\[ E_{s-H} (r) = \hat{\theta} j \eta_0 k j \omega p \sin(\theta) e^{-jkr} \]

\[ = -\hat{\phi} k^2 a^3 \left( \frac{\epsilon_1 - \epsilon_0}{\epsilon_1 + 2\epsilon_0} \right) E_i \sin(\theta) e^{-jkr} \]
Rayleigh Scattering: Scattering Cross-Section

Total scattered power $P_s$ from a dielectric sphere is:

$$P_s = \frac{2\pi}{3\eta_0^2} \left| \mathbf{E}_{\text{sca}}(\mathbf{r}) \right|^2 r^2 \sin(\theta) d\theta d\phi = \frac{4\pi}{3\eta_0} k^4 a^6 \left( \frac{\epsilon_1 - \epsilon_0}{\epsilon_1 + 2\epsilon_0} \right)^2 |\mathbf{E}_i|^2$$

The incident power per unit area was just the Poynting vector of the incident wave:

$$\frac{|\mathbf{E}_i(\mathbf{r})|^2}{2\eta_0}$$

The scattering cross-section $\sigma_s$ of a scatterer is defined as the area of a plane oriented perpendicular to the direction of incident wave that would intercept the same total incident power as the power $P_s$ that the scatterer radiates

$$\sigma_s = \frac{P_s}{|\mathbf{E}_i(\mathbf{r})|^2/2\eta_0}$$

$\sigma_s$ is also the ratio of the total scattered power to the power per unit area of the incident wave at the location of the scatterer.

For the dielectric sphere:

$$\sigma_s = \frac{P_s}{|E_i(\mathbf{r})|^2/2\eta_0} = \frac{8\pi}{3} k^4 a^6 \left( \frac{\epsilon_1 - \epsilon_0}{\epsilon_1 + 2\epsilon_0} \right)^2$$

Rayleigh Scattering: Why is the Sky Blue

The scattering cross-section of a dielectric sphere is:

$$\sigma_s = \frac{8\pi}{3} k^4 a^6 \left( \frac{\epsilon_1 - \epsilon_0}{\epsilon_1 + 2\epsilon_0} \right)^2$$

The scattered power is inversely proportional to the fourth power of the wavelength:

$$\sigma_s \propto k^4 \propto \frac{1}{\lambda^4}$$

Shorter wavelengths are scattered more than longer wavelengths in the Rayleigh limit.

Why is the sky blue?

Molecules/atoms in the atmosphere

Rayleigh scatter the sunlight

Sun

Sun is actually white – all wavelengths are present

Sun appears yellow/orange since shorter wavelengths have been scattered out in the direct line-of-sight

Sky appears blue since the shorter wavelengths are scattered more (and violet is absorbed in the upper atmosphere)
Example: Radar Range Equation

Power per unit area at the location of the target:

\[ S_{\text{target}} = \frac{P_{\text{in}}}{4\pi r^2} G(\theta, \phi) \]

If the scattering cross-section of the target is \( \sigma_s \) then the total scattered power \( P_s \) is:

\[ P_s = \sigma_s S_{\text{target}} = \sigma_s \frac{P_{\text{in}}}{4\pi r^2} G(\theta, \phi) \]

If the target scatters isotropically (equally in all directions) then the power \( P_{\text{out}} \) received by the matched antenna is:

\[ P_{\text{out}} = \eta_p \frac{P_s}{4\pi r^2} A(\theta, \phi) \]

\[ P_{\text{out}} = \eta_p \frac{P_{\text{in}}}{4\pi r^2} \sigma_s G(\theta, \phi) A(\theta, \phi) \]

Radar range equation

\[ \frac{P_{\text{out}}}{P_{\text{in}}} = \eta_p \frac{\sigma_s G^2(\theta, \phi)}{(4\pi r^2)^2} \frac{\lambda^2}{4\pi} = \eta_p \frac{\sigma_s G^2(\theta, \phi)\lambda^2}{(4\pi)^3 r^4} \]