Lecture 32

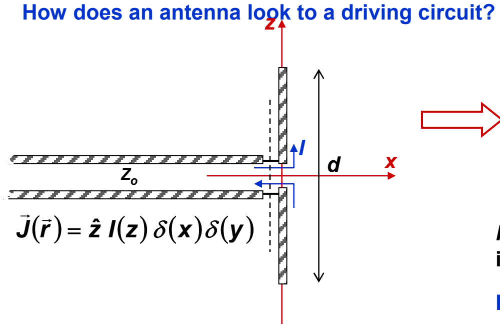
Circuit Properties of Antennas

In this lecture you will learn:

- Circuit properties of transmitting and receiving antennas
- Thevenin circuit models for receiving antennas
- Antenna theorem

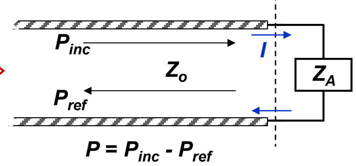
Circuit Properties of Transmitting Antennas - I

How does an antenna look to a driving circuit?



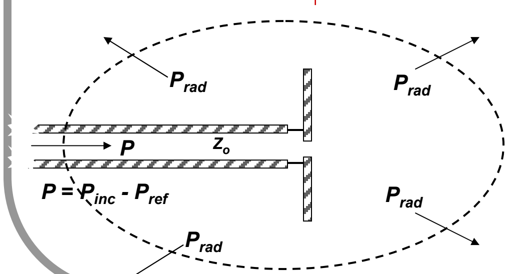
$$\vec{J}(\vec{r}) = \hat{z}\, I(z)\,\delta(x)\,\delta(y)$$

Antenna appears as a load impedance



$P = P_{inc} - P_{ref}$

P = Net time-average power going into the antenna



Poynting theorem:

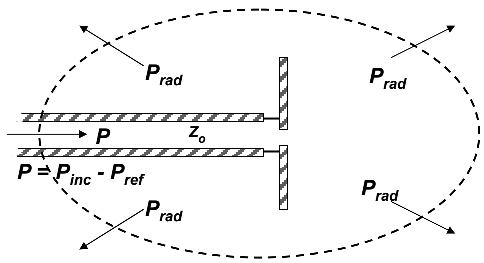
$$-\nabla.\vec{S}(\vec{r},t) = \frac{\partial W(\vec{r},t)}{\partial t} + \vec{J}(\vec{r},t).\vec{E}(\vec{r},t)$$

Time-average form for time-harmonic signals:

$$-\nabla.\left\langle \vec{S}(\vec{r},t)\right\rangle = \left\langle \vec{J}(\vec{r},t).\vec{E}(\vec{r},t)\right\rangle$$

Note the distinction between P and P_{inc} and P_{rad}

Circuit Properties of Transmitting Antennas - II

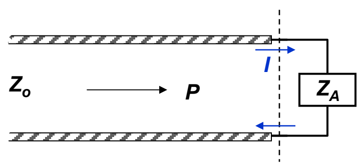


Time-average form for time-harmonic signals:

$$-\oint\!\!\oint \langle \vec{S}(\vec{r},t) \rangle \,.\, d\vec{a} = \iiint \langle \vec{J}(\vec{r},t).\vec{E}(\vec{r},t) \rangle \, dv$$

Net power going in — Net power dissipated

$$\Rightarrow \quad P - P_{rad} = \frac{1}{2}|I|^2 R_{diss}$$



But:

$$P_{rad} = \frac{1}{2}|I|^2 R_{rad}$$

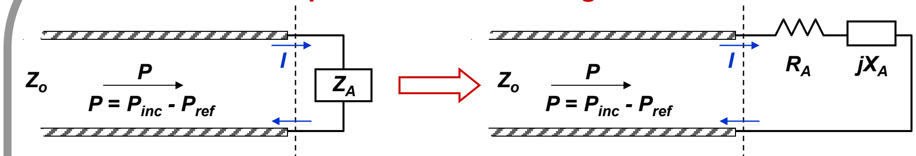
So we get:

$$P = \frac{1}{2}|I|^2 \left(R_{diss} + R_{rad}\right)$$

But from the circuit model:

$$P = \frac{1}{2}|I|^2 \,\mathrm{Re}[Z_A] \quad \Rightarrow \quad \mathrm{Re}[Z_A] = R_{rad} + R_{diss}$$

Circuit Properties of Transmitting Antennas - III



$$Z_A = R_A + jX_A = \left(R_{rad} + R_{diss}\right) + jX_A$$

The reactive part is due to the net inductance or capacitance of the antenna structure

Not all the power P going into an antenna is converted into radiated power P_{rad} – some is dissipated due to the less-than-infinite conductivity of real metals (i.e. non-zero R_{diss})

Definition of Antenna Gain for real antennas:

$$G(\theta,\phi) = \frac{\langle \vec{S}(\vec{r},t) \rangle.\hat{r}}{P/4\pi r^2}$$

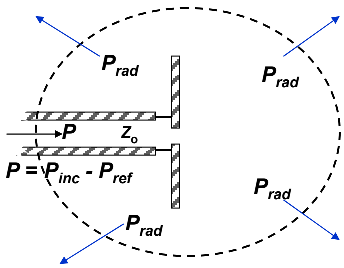
NOT: $G(\theta,\phi) = \dfrac{\langle \vec{S}(\vec{r},t) \rangle.\hat{r}}{P_{rad}/4\pi r^2}$

The Antenna Efficiency η_{rad} is defined as:

$$\eta_{rad} = \frac{P_{rad}}{P} = \left(\frac{R_{rad}}{R_{rad} + R_{diss}}\right)$$

$$\Rightarrow \quad \int_0^{2\pi}\int_0^{\pi} G(\theta,\phi)\ \sin(\theta)\,d\theta\, d\phi = 4\pi\,\eta_{rad}$$

Transmitting and Receiving Antenna Systems



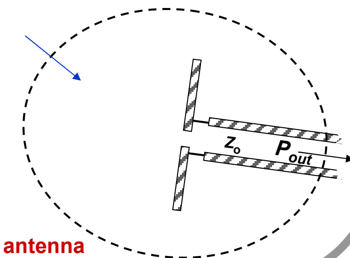
Transmitting antenna

Need to determine the following:

i) From the net power P going to the antenna how much is the total radiated power P_{rad}

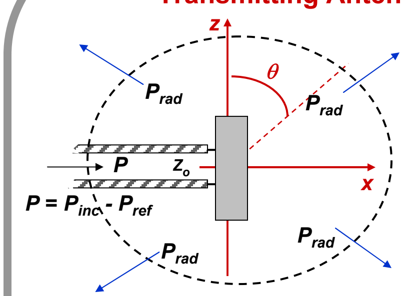
ii) From the total radiated power how much power is coupled into the receiving antenna

iii) From the total power coupled into the receiving antenna how much ends up as the useful power P_{out} in the receiving circuit



Receiving antenna

Transmitting Antennas: Gain and Radiated Power



Suppose we only know that the Gain of the antenna is: $G(\theta,\phi)$

$$G(\theta,\phi) = \frac{\langle \vec{S}(\vec{r},t)\rangle . \hat{r}}{P/4\pi r^2}$$

Question: If net power P is delivered to the antenna by the feeding transmission line then how much radiated power per unit area is going in the (θ, ϕ) direction at a distance r from the antenna?

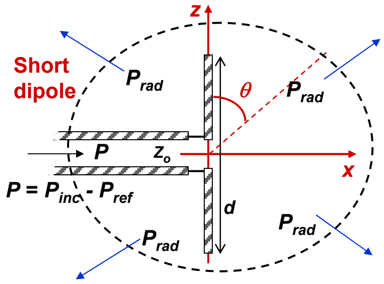
Answer: Radiated power per unit area going in the (θ, ϕ) direction at a distance r from the antenna is:

$$\langle \vec{S}(\vec{r},t)\rangle . \hat{r} = \frac{P}{4\pi r^2} G(\theta,\phi)$$

Remember that the unit vector in the radial (θ, ϕ) direction is:

$$\hat{r} = \hat{r}(\theta,\phi) = \hat{x}\ \sin(\theta)\cos(\phi) + \hat{y}\ \sin(\theta)\sin(\phi) + \hat{z}\ \cos(\theta)$$

A Short Dipole: Review of Some Formulas



Antenna Efficiency:

$$\eta_{rad} = \frac{P_{rad}}{P} = \left(\frac{R_{rad}}{R_{rad} + R_{diss}} \right)$$

Radiation Resistance:

$$R_{rad} = \frac{\eta_o}{6\pi} \left(k\, d_{eff} \right)^2$$

$$d_{eff} = \frac{d}{2}$$

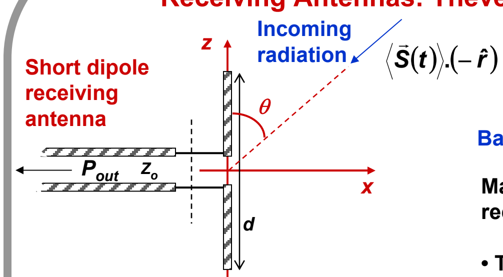
Gain:

$$G(\theta,\phi) = \frac{\langle \vec{S}(\vec{r},t) \rangle . \hat{r}}{P / 4\pi r^2} = \left(\frac{\langle \vec{S}(\vec{r},t) \rangle . \hat{r}}{P_{rad} / 4\pi r^2} \right) \left(\frac{P_{rad}}{P} \right)$$

$$= \left(\frac{3}{2} \sin^2(\theta) \right) \left(\eta_{rad} \right)$$

$$= \frac{3}{2} \eta_{rad} \sin^2(\theta)$$

Receiving Antennas: Thevenin Equivalent Circuit

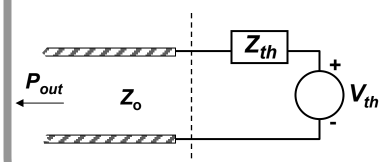


Basic Idea:

Make a Thevenin equivalent circuit for the receiving antenna

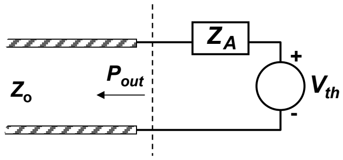
- **The Thevenin impedance Z_{th} is the impedance looking into the antenna and is therefore the same as the antenna impedance Z_A**

- **V_{th} is due to the the radiation power received by the antenna and is something that we need to find**

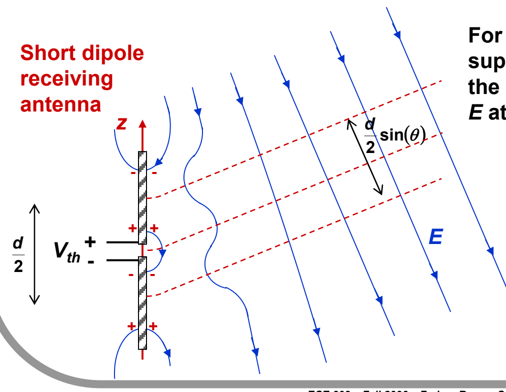


$$Z_{th} = Z_A$$
$$= R_{rad} + R_{diss} + jX_A$$

Receiving Antennas: Thevenin Equivalent Circuit



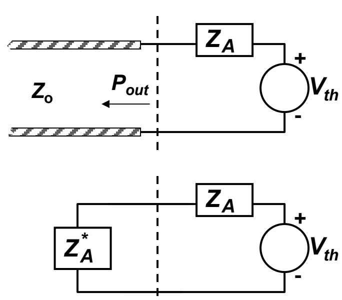
To find V_{th} assume the antenna inputs are open circuited and then determine how much voltage will be produced at the input terminals by the incoming radiation



For the short dipole under consideration, suppose the incoming radiation was from the θ-direction and had E-field of strength E at the location of the antenna, then:

$$V_{th} = E\frac{d}{2}\sin(\theta)$$
$$= E\,d_{eff}\sin(\theta)$$

Receiving Antennas: Maximum Power Delivered



$$V_{th} = E\,d_{eff}\sin(\theta)$$

But: $\left\langle \vec{S}(t)\right\rangle.(-\hat{r}) = \dfrac{|E|^2}{2\eta_o}$

$$\Rightarrow V_{th} = \sqrt{2\eta_o\left\langle \vec{S}(t)\right\rangle.(-\hat{r})}\;\; d_{eff}\sin(\theta)$$

Maximum power is delivered to the output circuit if its (transformed) impedance is matched to the antenna impedance Z_A

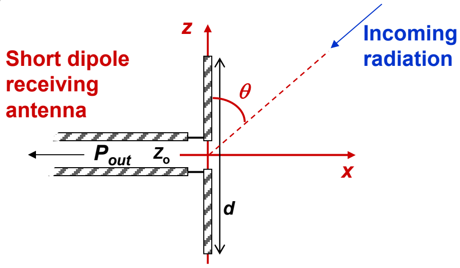
So the max. value of P_{out} is:

$$P_{out} = \frac{|V_{th}|^2}{8\,\mathrm{Re}[Z_A]}$$
$$= \frac{|V_{th}|^2}{8R_{rad}}\left(\frac{R_{rad}}{\mathrm{Re}[Z_A]}\right) = \frac{|V_{th}|^2}{8R_{rad}}\left(\frac{R_{rad}}{R_{rad}+R_{diss}}\right) = \frac{|V_{th}|^2}{8R_{rad}}\eta_{rad}$$
$$= \frac{\lambda^2}{4\pi}\left(\frac{3}{2}\eta_{rad}\sin^2(\theta)\right)\left\langle \vec{S}(t)\right\rangle.(-\hat{r})$$

Remember that for a short dipole:

$$R_{rad} = \frac{\eta_o}{6\pi}(k\,d_{eff})^2$$

Receiving Antennas: Effective Antenna Area



Assuming a matched case:

$$P_{out} = \frac{\lambda^2}{4\pi}\left(\frac{3}{2}\eta_{rad}\,\sin^2(\theta)\right)\left\langle \vec{S}(t)\right\rangle.(-\hat{r})$$
$$= A(\theta,\phi)\left\langle \vec{S}(t)\right\rangle.(-\hat{r})$$

Effective Antenna Area:

The effective antenna area $A(\theta,\phi)$ is defined as the ratio of the power delivered by the antenna to a matched load to the incident radiation power per unit area from the (θ,ϕ)-direction

For the short dipole:

$$A(\theta,\phi) = \frac{\lambda^2}{4\pi}\left(\frac{3}{2}\eta_{rad}\,\sin^2(\theta)\right)$$
$$= \frac{\lambda^2}{4\pi}G(\theta,\phi)$$

For the short dipole:

$$G(\theta,\phi) = \frac{3}{2}\eta_{rad}\,\sin^2(\theta)$$

Antenna Theorem

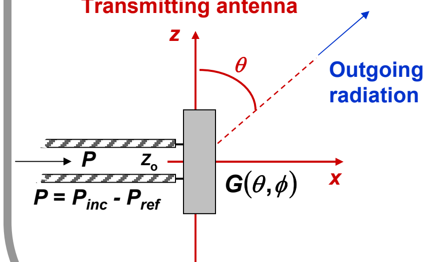
Antenna Theorem (see proof at the end):

The antenna theorem states that for any antenna the effective antenna area and antenna gain are related by:

$$A(\theta,\phi) = \frac{\lambda^2}{4\pi}G(\theta,\phi)$$

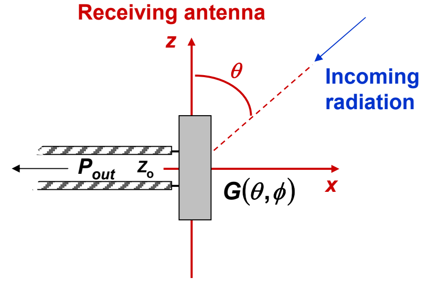
In the previous slides we proved it for the short dipole – but it holds for all types of antennas

Transmitting antenna



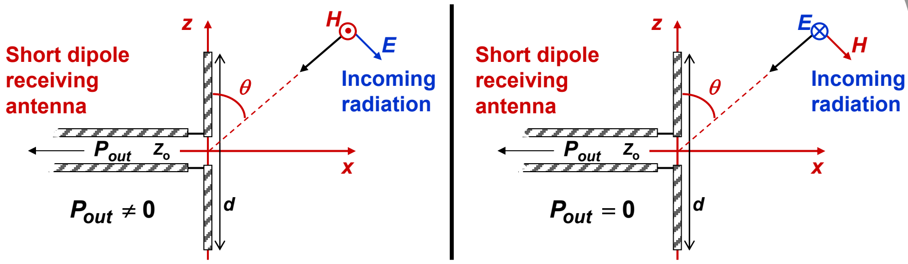
$$\left\langle \vec{S}(\vec{r},t)\right\rangle.\hat{r} = \frac{P}{4\pi r^2}G(\theta,\phi)$$

Receiving antenna



$$P_{out} = A(\theta,\phi)\left\langle \vec{S}(t)\right\rangle.(-\hat{r})$$

Receiving Antennas: Polarization Mismatch



The polarization of incident radiation is said to be not matched to the "polarization of the antenna" if it is not the same as that of the radiation emitted by the antenna if it were to radiate in the direction of the incident radiation

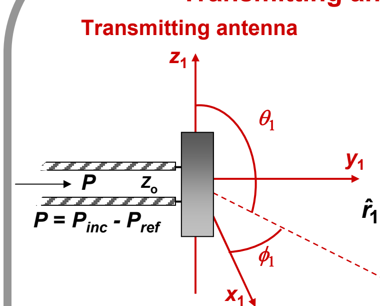
More generally one can write:

$$P_{out} = \eta_p \, A(\theta,\phi) \left\langle \vec{S}(t) \right\rangle . (-\hat{r})$$

Where: $\eta_p = \cos^2(\alpha)$

And: α is the angle between the polarization of the incident radiation and the polarization of the radiation emitted by the antenna if it were to radiate in the direction of the incident radiation

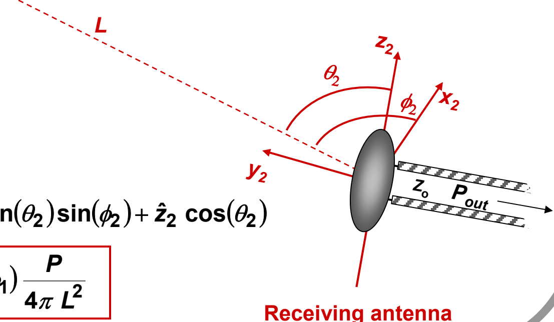
Transmitting and Receiving Antenna Systems



Power per unit area at the location of the second antenna:

$$\left\langle \vec{S}(\vec{r},t) \right\rangle . \hat{r}_1 \Big|_{r=L} = \frac{P}{4\pi L^2} G_1(\theta_1,\phi_1)$$

$$\hat{r}_1 = \hat{x}_1 \sin(\theta_1)\cos(\phi_1) + \hat{y}_1 \sin(\theta_1)\sin(\phi_1) + \hat{z}_1 \cos(\theta_1)$$



Power delivered to a matched load by the second antenna:

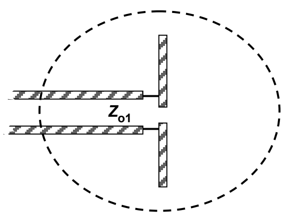
$$P_{out} = \eta_p \, A_2(\theta_2,\phi_2) \left\langle \vec{S}(t) \right\rangle . (-\hat{r}_2)$$

$$\hat{r}_2 = \hat{x}_2 \sin(\theta_2)\cos(\phi_2) + \hat{y}_2 \sin(\theta_2)\sin(\phi_2) + \hat{z}_2 \cos(\theta_2)$$

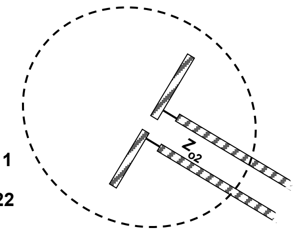
$$\Rightarrow \quad P_{out} = \eta_p \, A_2(\theta_2,\phi_2) G_1(\theta_1,\phi_1) \frac{P}{4\pi L^2}$$

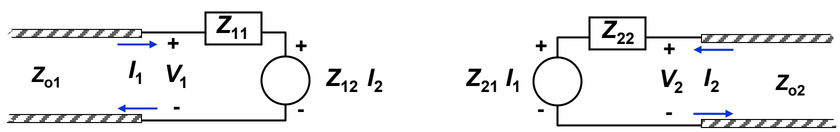
Circuit Model of Transmitting and Receiving Antenna System

Transmitting and receiving antenna | **Transmitting and receiving antenna**



$$Z_{11} = (R_{rad} + R_{diss})_{11} + jX_{11}$$
$$Z_{22} = (R_{rad} + R_{diss})_{22} + jX_{22}$$



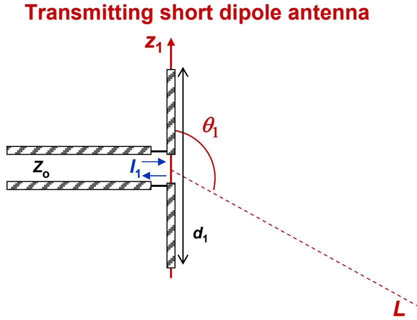


$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Reciprocity condition: $Z_{12} = Z_{21}$

Circuit Model: Short Dipole Example

Transmitting short dipole antenna



$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

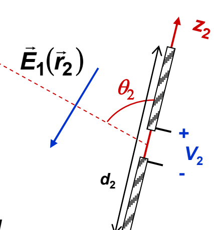
Field produced by the current of the transmitting antenna at the receiving antenna

$$\vec{E}_1(\vec{r}_2) = \hat{\theta}_1 \frac{j\eta_o k I_1 d_{eff1}}{4\pi L} \sin(\theta_1) e^{-jkL}$$

Open-circuit voltage at the receiving antenna:

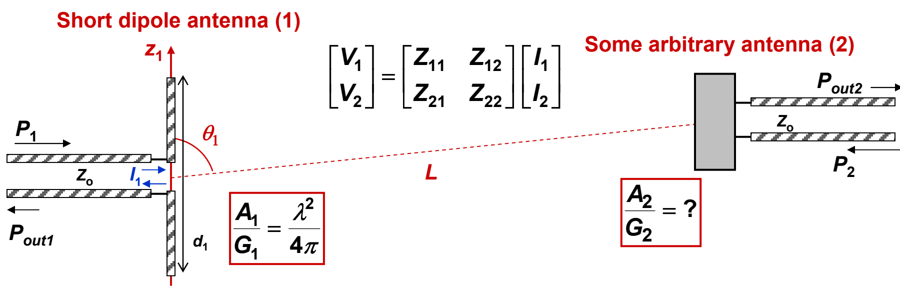
$$\begin{aligned} V_2 &= Z_{21} I_1 \\ &= \vec{E}_1(\vec{r}_2).\hat{z}_2 \ \ d_{eff2} \\ &= \eta_p \frac{j\eta_o k I_1 d_{eff1} d_{eff2}}{4\pi L} \sin(\theta_1)\sin(\theta_2) e^{-jkL} \end{aligned}$$

$$\Rightarrow \quad Z_{21} = \eta_p \frac{j\eta_o k d_{eff1} d_{eff2}}{4\pi L} \sin(\theta_1)\sin(\theta_2) e^{-jkL}$$



Receiving short dipole antenna

Antenna Theorem: More General Proof - I



$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$$\boxed{\frac{A_1}{G_1} = \frac{\lambda^2}{4\pi}} \qquad \boxed{\frac{A_2}{G_2} = ?}$$

When (1) transmits and (2) receives: $P_{out2} = \eta_p \frac{P_1}{4\pi L^2} G_1 A_2$

$$\Rightarrow \frac{P_{out2}}{P_1} = \frac{|Z_{21}|^2}{4\,\mathrm{Re}[Z_{11}]\mathrm{Re}[Z_{22}]} = \eta_p \frac{G_1 A_2}{4\pi L^2}$$

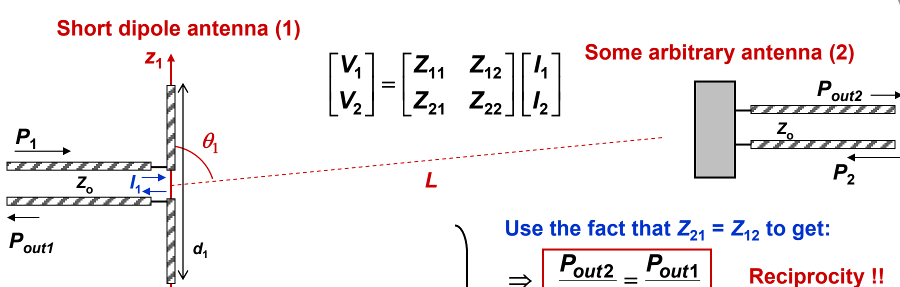
$$\begin{cases} P_1 = \frac{1}{2}|I_1|^2 \mathrm{Re}[Z_{11}] \\ P_{out2} = \frac{|Z_{21} I_1|^2}{8\,\mathrm{Re}[Z_{22}]} \end{cases}$$

When (2) transmits and (1) receives: $P_{out1} = \eta_p \frac{P_2}{4\pi L^2} G_2 A_1$

$$\Rightarrow \frac{P_{out1}}{P_2} = \frac{|Z_{12}|^2}{4\,\mathrm{Re}[Z_{11}]\mathrm{Re}[Z_{22}]} = \eta_p \frac{G_2 A_1}{4\pi L^2}$$

$$\begin{cases} P_2 = \frac{1}{2}|I_2|^2 \mathrm{Re}[Z_{22}] \\ P_{out1} = \frac{|Z_{12} I_2|^2}{8\,\mathrm{Re}[Z_{11}]} \end{cases}$$

Antenna Theorem: More General Proof - II



$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$$\Rightarrow \frac{P_{out2}}{P_1} = \frac{|Z_{21}|^2}{4\,\mathrm{Re}[Z_{11}]\mathrm{Re}[Z_{22}]} = \eta_p \frac{G_1 A_2}{4\pi L^2}$$

$$\Rightarrow \frac{P_{out1}}{P_2} = \frac{|Z_{12}|^2}{4\,\mathrm{Re}[Z_{11}]\mathrm{Re}[Z_{22}]} = \eta_p \frac{G_2 A_1}{4\pi L^2}$$

Use the fact that $Z_{21} = Z_{12}$ to get:

$$\Rightarrow \boxed{\frac{P_{out2}}{P_1} = \frac{P_{out1}}{P_2}} \qquad \text{Reciprocity !!}$$

$$\Rightarrow \eta_p \frac{G_2 A_1}{4\pi L^2} = \eta_p \frac{G_1 A_2}{4\pi L^2}$$

$$\Rightarrow \frac{A_1}{G_1} = \frac{A_2}{G_2}$$

But: $\frac{A_1}{G_1} = \frac{\lambda^2}{4\pi}$

Therefore: $\boxed{\frac{A_2}{G_2} = \frac{\lambda^2}{4\pi}}$ QED