Lecture 3

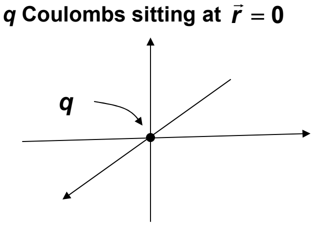
Electrostatics

In this lecture you will learn:

- **Three ways to solve problems in electrostatics:**
  - **a) Application of the Superposition Principle (SP)**
  - **b) Application of Gauss' Law in Integral Form (GLIF)**
  - **c) Application of Gauss' Law in Differential Form (GLDF)**

Field of a Point Charge (GLIF)

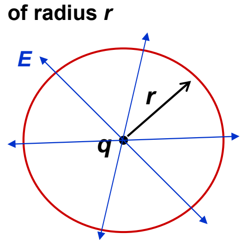
Consider a point charge of q Coulombs sitting at $\vec{r} = 0$



By symmetry, electric field can only point in the radial direction

$$\oiint \varepsilon_o \vec{E} \, . \, d\vec{a} = \iiint \rho \, dV$$

Surround the charge by a Gaussian surface in the form of a spherical shell of radius r



By symmetry, the E-field magnitude on the surface must be uniform and pointing in the radial direction

Using Gauss' Law: $\varepsilon_o E_r \left(4\pi r^2\right) = q$

$$\Rightarrow \quad E_r = \frac{q}{4\pi\varepsilon_o r^2} \quad \text{or} \quad \vec{E} = \frac{q}{4\pi\varepsilon_o r^2}\hat{r}$$

The Principle of Superposition in Electromagnetics

Maxwell's equations are LINEAR and allow for the superposition principle to hold

- Suppose for some charge and current densities, ρ_1 and $\vec{J}_1$, one has found the E and H fields, $\vec{E}_1$ and $\vec{H}_1$
- Suppose for some other charge and current densities, ρ_2 and $\vec{J}_2$, one has also found the E and H fields, $\vec{E}_2$ and $\vec{H}_2$

The superposition principle says that the sums, $\left(\vec{E}_1+\vec{E}_2\right)$ and $\left(\vec{H}_1+\vec{H}_2\right)$, are the solution for the charge and current densities, $\left(\rho_1+\rho_2\right)$ and $\left(\vec{J}_1+\vec{J}_2\right)$

A Simple Proof

$$\nabla.\,\varepsilon_o\,\vec{E}_1=\rho_1 \qquad \nabla.\,\varepsilon_o\,\vec{E}_2=\rho_2 \qquad \nabla.\,\varepsilon_o\left(\vec{E}_1+\vec{E}_2\right)=\left(\rho_1+\rho_2\right)$$

$$\nabla.\,\mu_o\,\vec{H}_1=0 \qquad \nabla.\,\mu_o\,\vec{H}_2=0 \qquad \nabla.\,\mu_o\left(\vec{H}_1+\vec{H}_2\right)=0$$

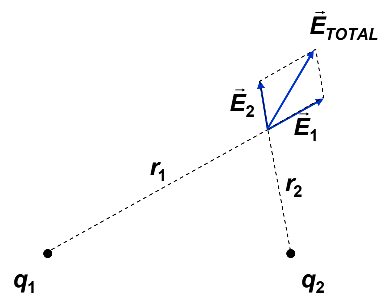
$$\nabla\times\vec{E}_1=-\frac{\partial\,\mu_o\vec{H}_1}{\partial t} \quad + \quad \nabla\times\vec{E}_2=-\frac{\partial\,\mu_o\vec{H}_2}{\partial t} \quad = \quad \nabla\times\left(\vec{E}_1+\vec{E}_2\right)=-\frac{\partial\,\mu_o\left(\vec{H}_1+\vec{H}_2\right)}{\partial t}$$

$$\nabla\times\vec{H}_1=\vec{J}_1+\frac{\partial\,\varepsilon_o\vec{E}_1}{\partial t} \qquad \nabla\times\vec{H}_2=\vec{J}_2+\frac{\partial\,\varepsilon_o\vec{E}_2}{\partial t} \qquad \nabla\times\left(\vec{H}_1+\vec{H}_2\right)=\left(\vec{J}_1+\vec{J}_2\right)+\frac{\partial\,\varepsilon_o\left(\vec{E}_1+\vec{E}_2\right)}{\partial t}$$

Two Point Charges (SP)

We know that a single charge produces a radial field given by: $\vec{E}=\frac{q}{4\pi\varepsilon_o\,r^2}\hat{r}$

Consider Two Charges



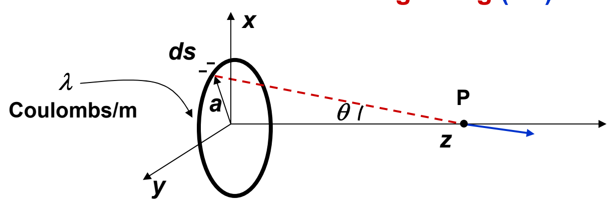
$$\vec{E}_1=\frac{q_1}{4\pi\varepsilon_o\,r_1^2}\hat{r}_1$$

$$\vec{E}_2=\frac{q_2}{4\pi\varepsilon_o\,r_2^2}\hat{r}_2$$

Total E-field at any point is the vector sum of the contributions from each charge

$$\vec{E}_{TOTAL}=\vec{E}_1+\vec{E}_2$$

A Charge Ring (SP)



Question: What is the total electric field at point P at a distance *z* from the ring center on the z-axis ?

Symmetry Argument: The total contribution to the x-component and also to the y-component of the field at point P are both zero (why?)

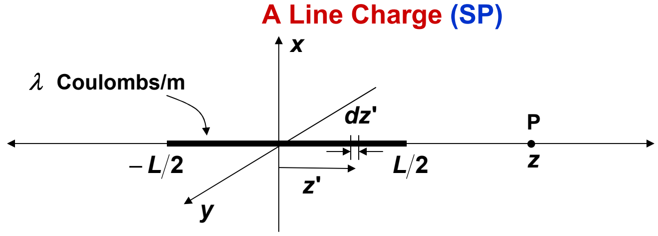
- The z-component of the contributions from each element *ds* can be added algebraically to get the final answer:

$$E_z(x=y=0,z)=\frac{\lambda 2\pi a}{4\pi\varepsilon_o\left(a^2+z^2\right)}\cos(\theta)=\frac{\lambda 2\pi a}{4\pi\varepsilon_o\left(a^2+z^2\right)}\frac{z}{\sqrt{a^2+z^2}}$$

- For distances *z* >> *a*, the field behaves like that of a point charge with total charge $\lambda 2\pi a$

$$E_z(x=y=0,z)\Big|_{z>>a}=\frac{\lambda 2\pi a}{4\pi\varepsilon_o z^2}$$

A Line Charge (SP)



Question: What is the total electric field at point P at a distance *z* from the line center on the z-axis ?

Symmetry Argument: The total contribution to the x-component and also to the y-component of the field at point P are both zero (why?)

- The z-component of the contributions from each element *dz'* can be added algebraically to get the final answer in an integral form:

$$E_z(x=y=0,z)=\int_{-L/2}^{L/2}\frac{\lambda\, dz'}{4\pi\varepsilon_o(z-z')^2}=\frac{\lambda L}{4\pi\varepsilon_o\left(z^2-(L/2)^2\right)}$$

- For distances *z* >> *L*, the field resembles that of a point charge with total charge λL

$$E_z(x=y=0,z)\Big|_{z>>L}=\frac{\lambda L}{4\pi\varepsilon_o z^2}$$

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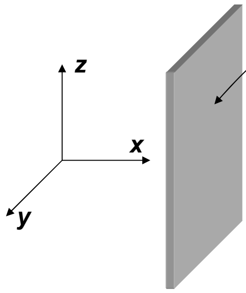
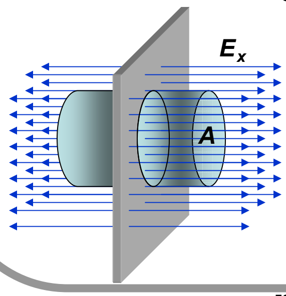
Field of an Infinite Charge Plane (GLIF)



σ = surface charge density (units: Coulombs/m^2)

Symmetry Argument: The charge distribution is symmetric w.r.t. +z and –z directions. Therefore, if at any point there is an E-field component in the +z direction, there must also be E-field component in the –z direction. Since the field cannot have a z-component pointing in both +z and –z directions at the same time, there cannot be a z-component of the field.

- Similarly, there cannot be a y-component of the E-field
- So the field can only have an x-component



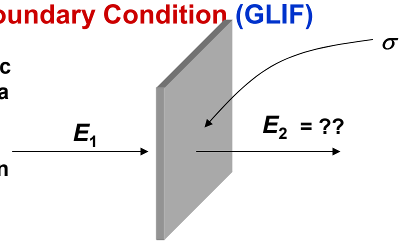
- Draw a Gaussian surface in the form of a cylinder of area *A* piercing the charge plane
- Total electric flux coming out of the surface = $2\varepsilon_o E_x A$
- Total charge enclosed by the surface = σA
- By Gauss' Law: $2\varepsilon_o E_x A = \sigma A$

$$\Rightarrow \boxed{E_x = \frac{\sigma}{2\varepsilon_o}}$$

Surface Charge Density Boundary Condition (GLIF)

Suppose we know the surface normal electric field on just one side of a charge plane with a surface charge density σ

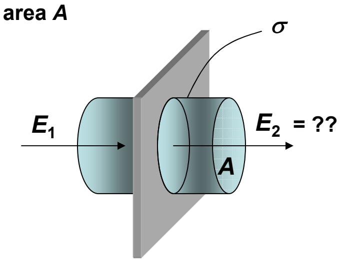
Question: What is the surface normal field on the other side of the charge plane?



Solution:

- Draw a Gaussian surface in the form of a cylinder of area *A* piercing the charge plane
- Total flux coming out of the surface = $\varepsilon_o (E_2 - E_1) A$
- Total charge enclosed by the surface = σA
- By Gauss' Law: $\varepsilon_o (E_2 - E_1) A = \sigma A$

$$\Rightarrow \varepsilon_o (E_2 - E_1) = \sigma$$

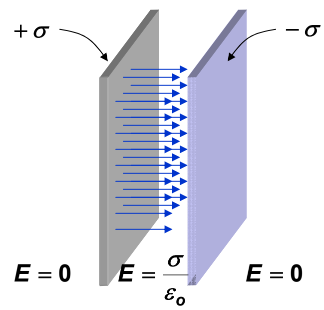


$$\boxed{\varepsilon_o (E_2 - E_1) = \sigma}$$

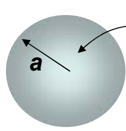
This an extremely important result that relates surface normal electric fields on the two sides of a charge plane with surface charge density σ

Two Oppositely Charged Planes (SP)

When the charge planes are brought together the field contributions from each plane outside the planes cancel each other and the fields inside the planes reinforce each other

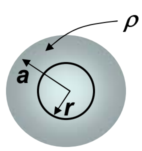


Field of a Uniformly Charged Sphere – I (GLIF)



ρ = volume charge density (units: Coulombs/m^3)

By symmetry, the E-field must only have a component in the radial direction (i.e. in the $\hat{r}$ direction)



Work in spherical co-ordinates

For $0 \le r \le a$:

- Draw a Gaussian surface in the form of a spherical shell of radius r
- Total flux coming out of the surface = $\varepsilon_o E_r \left(4\pi r^2\right)$
- Total charge enclosed by the surface = $\rho\left(\frac{4}{3}\pi r^3\right)$
- By Gauss' Law: $E_r = \dfrac{\rho}{3\,\varepsilon_o} r$

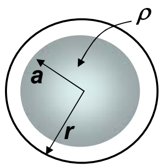
Check the Answer (does the solution satisfy Gauss' Law?):

Lets calculate the divergence of the E-field in the answer obtained

$$\nabla . \varepsilon_o \vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,\varepsilon_o\,E_r\right) = \rho \quad \checkmark$$

Using the formula for divergence in spherical coordinates

Field of a Uniformly Charged Sphere – II (GLIF)



For $a \le r \le \infty$:

- Draw a Gaussian surface in the form of a spherical shell of radius r
- Total flux coming out of the surface = $\varepsilon_o E_r \left(4\pi r^2\right)$
- Total charge enclosed by the surface = $\rho\left(\frac{4}{3}\pi a^3\right)$
- By Gauss' Law: $E_r = \dfrac{\left(\rho \frac{4}{3}\pi a^3\right)}{4\pi \varepsilon_o r^2}$

Check the Answer (does the solution satisfy Gauss' Law?):

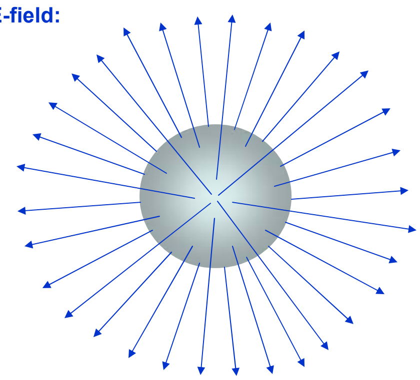
Lets calculate the divergence of the E-field in the answer obtained

$$\nabla . \varepsilon_o \vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \varepsilon_o E_r\right) = 0 \quad \checkmark$$

} Using the formula for divergence in spherical coordinates

Field of a Uniformly Charged Sphere – III

Sketch of the E-field:



Note that the non-zero field divergence is included in the sketch

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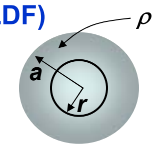
Field of a Uniformly Charged Sphere – IV (GLDF)

Using Gauss' Law in Differential Form: $\nabla . \varepsilon_o \vec{E} = \rho$



For $0 \le r \le a$:

$$\nabla . \varepsilon_o \vec{E} = \rho$$

$$\Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) = \frac{\rho}{\varepsilon_o}$$

Multiply both sides by r^2 and Integrate from 0 to r (where $r \le a$)

$$\int_0^r \frac{\partial}{\partial r}\left(r^2 E_r\right) dr = \int_0^r \frac{\rho}{\varepsilon_o} r^2 \, dr$$

$$\Rightarrow r^2 E_r(r) = \frac{\rho}{3\,\varepsilon_o} r^3$$

$$\Rightarrow E_r(r) = \frac{\rho}{3\,\varepsilon_o} r$$

same as before !

For $a \le r \le \infty$:

$$\nabla . \varepsilon_o \vec{E} = 0$$

$$\Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) = 0$$

Multiply both sides by r^2 and Integrate from a to r (where $r \ge a$)

$$\int_a^r \frac{\partial}{\partial r}\left(r^2 E_r\right) dr = 0$$

$$\Rightarrow r^2 E_r(r) - a^2 E_r(a) = 0$$

$$\Rightarrow E_r(r) = \frac{a^2}{r^2} E_r(a) = \frac{\left(\rho \frac{4}{3}\pi a^3\right)}{4\pi\,\varepsilon_o\, r^2}$$

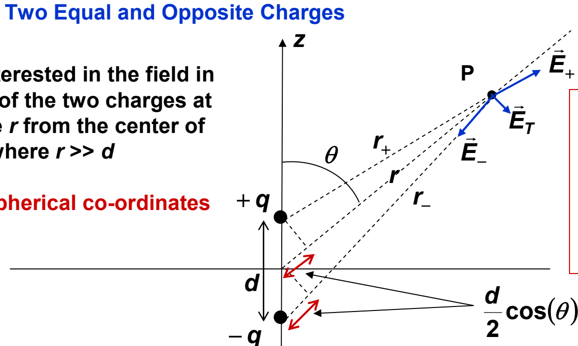
same as before !

Field of a Charge Dipole (SP)

Consider Two Equal and Opposite Charges

We are interested in the field in the plane of the two charges at a distance r from the center of the pair, where $r >> d$

Work in spherical co-ordinates



$$\vec{E}_T = \vec{E}_+ + \vec{E}_-$$

$$r_+ = r - \frac{d}{2}\cos(\theta)$$

$$r_- = r + \frac{d}{2}\cos(\theta)$$

$$\vec{E}_T . \hat{r} \approx \frac{q}{4\pi\varepsilon_o\left(r - \frac{d}{2}\cos(\theta)\right)^2} - \frac{q}{4\pi\varepsilon_o\left(r + \frac{d}{2}\cos(\theta)\right)^2} \approx \frac{q}{4\pi\varepsilon_o\, r^3} 2d\cos(\theta)$$

$$\vec{E}_T . \hat{\theta} \approx \frac{q}{4\pi\varepsilon_o\, r^2}\frac{d}{2r}\sin(\theta) + \frac{q}{4\pi\varepsilon_o\, r^2}\frac{d}{2r}\sin(\theta) = \frac{q}{4\pi\varepsilon_o\, r^3} d\sin(\theta)$$

$$\vec{E}_T \approx \frac{qd}{4\pi\varepsilon_o\, r^3}\left(2\cos(\theta)\hat{r} + \sin(\theta)\hat{\theta}\right)$$

Field of a Charge Dipole

Sketch of the E-field: